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SOLUTION OF OPTIMAL CONTROL PROBLEMS VIA HAAR WAVELETS

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Abstract: The Haar wavelet method is applied for solving constrained optimal control problems. The methodology of the proposed solution is demonstrated with the help of three examples. It follows from the computer simulation that only a small number of grid points is needed to obtain very satisfactory results.

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1. Introduction

Fundamentals of the contemporary optimal control theory can be found in textbooks from which we cite here [1], [8], [7] and [5]. This theory has also several applications, e.g. in structural dynamics, space flights, chemical engineering, economy.

The optimality conditions usually lead to solutions of two-point boundary value problems, which are difficult to solve. To overcome these difficulties various numerical methods, as shooting technics, nonlinear programming, quasilinearization, etc., have been applied.

In this paper we are interested in wavelet solutions of optimal control. The

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wavelet approach has some advantages, compared with other numerical methods. Due to the sparcity of the transform matrices and a small number of significant wavelet coefficients high accuracy of results is guaranteed already for a small number of grid points. The wavelet methods are also very convenient for soving boundary value problems, since the boundary conditions are taken into account automatically.

We found only a few papers where the wavelet method was applied for solving optimal control problems. The first of them probably belongs to Chen and Hsiao [2], who made use of the Haar wavelets. In papers [12], [6] and [13] solutions based on the Legendre wavelets are discussed. The Daubechies wavelets were applied in [3] and [4].

In most of the wavelet families an explicit expression of the wavelet function is missing and therefore analytical differentiation or integration are not possible. For many problems integrals of products of wavelets and their derivatives must be computed. This can be done by introducing the "connection coefficients", which essentially complicates the solution.

The Haar wavelets are free from these disadvantages. Due to the simplicity these wavelets have been an effective tool for solving several problems of differential and integral calculus (see [9], [10] and [11]). The Haar series are also notable for their rapid convergence compared with other methods.

The main purpose of the present paper is to demonstrate the efficiency of the Haar wavelets in optimal control. The paper is organized as follows. In Section 2 the fundamentals of the optimal control theory are shortly reviewed. In Section 3 the Haar wavelet method is described. In Sections 4-6 three problems with different constraints are solved. It should be mentioned that in most papers also the control force $u(t)$ is developed into the wavelet series but we have made use of an alternative method according to which the function $u(t)$ is calculated directly from the extremality conditions. This essentially simplifies the solution.

The selection of problems solved in Sections 4-6 is mainly conditioned from the methodological considerations: we have chosen simple problems for which mostly exact solutions are known. This allows us to estimate the exactness of the achieved results. The proposed method of solution is, of course, applicable for more complicated problems.

2. Basic Elements of the Optimal Control Theory

Let us remind some necessary conditions for the optimal control. In the following bold letters denote vectors or matrices.

Performance Index. We have to find the control history $\boldsymbol{u}(t)$ that minimizes the functional

$$I = \int_{t_0}^{T} F(t, \boldsymbol{x}, \boldsymbol{u}) dt \,. \tag{1}$$

Here t, t_0, T are scalars, the state variables $\boldsymbol{x}(t)$ and controls $\boldsymbol{u}(t)$ are n- and r-dimensional vectors, respectively. It is assumed that $\boldsymbol{x}, \boldsymbol{u}$ have as many derivatives as are needed for the theory being developed.

State Variables. They are subjected to differential constraints

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}) \,. \tag{2}$$

Adjoint Variables. If $\hat{H}$ denotes the Hamiltonian

$$\hat{H}(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\psi}) = -F + \boldsymbol{\psi} \boldsymbol{f} \,, \tag{3}$$

then the adjoint variables (Lagrange multipliers) are calculated from the equations

$$\dot{\boldsymbol{\psi}} = -\frac{\partial \hat{H}}{\partial \boldsymbol{x}} \,. \tag{4}$$

Here $\boldsymbol{f}$ and $\boldsymbol{\psi}$ are n-dimensional vectors.

Admissible Controls. If the vector $\boldsymbol{u}$ is unconstrained then optimal controls are calculated from

$$\frac{\partial \hat{H}}{\partial \boldsymbol{u}} = 0 \,. \tag{5}$$

If $\boldsymbol{u} \in U$, where U is a closed set, the maximum principle (Pontryagin's principle) holds

$$\hat{H}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\psi}(t)) = \max_{\boldsymbol{u} \in U} \hat{H}(t, \boldsymbol{x}(t), \boldsymbol{u}, \boldsymbol{\psi}(t)) \,. \tag{6}$$

For simplicity of discussion in following we consider only the case, where u is a scalar satisfying the inequality constraint $|u(t)| < u_0$.

Boundary Conditions. We confine ourselves with the case where the values $x_\alpha(t_0)$, $\alpha = 1, 2, ..., \nu$ are prescribed and the remained values $x_\alpha(t_0)$, $\beta = 1, 2, ..., n - \nu$ are free. It follows from the transversality conditions that $\psi_\alpha(t_0)$ is free and $\psi_\beta(t_0) = 0$. Similar results hold also for the final time $t = T$.

Free Final Time. If the final time T is not fixed the complementary condition $\hat{H}|_{t=T} = 0$ holds. If $F = 1$ we get a minimum time problem $T = \min$.

Integral Constraint. A scalar integral constraint has the form

$$\int_{t_0}^{T} G(\boldsymbol{x}, \boldsymbol{u}) dt = K \,. \tag{7}$$

If this case it is suitable to introduce a new state variable

$$\dot{x}_{n+1} = G(\boldsymbol{x}, \boldsymbol{u}) \,, \quad x_{n+1}(t_0) = 0 \,, \quad x_{n+1}(t) = K \,. \tag{8}$$

State Inequality Constraint. Consider the case

$$g(t, \boldsymbol{x}) \leq 0 \,, \tag{9}$$

where g is a prescribed function. It is assumed that the equality $g(t, \boldsymbol{x}) = 0$ holds for $t \in [t_1, t_2]$. In this case the adjoint system gets the form

$$\dot{\boldsymbol{\psi}} = -\frac{\partial H}{\partial \boldsymbol{x}} + \boldsymbol{\mu} \bigtriangledown S \,, \quad t \in [t_1, t_2] \,, \tag{10}$$

where

$$S(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial g}{\partial \boldsymbol{x}} \boldsymbol{f} \,, \quad \bigtriangledown S = \frac{\partial S}{\partial \boldsymbol{x}} \,. \tag{11}$$

Here $\boldsymbol{\mu}(t)$ denotes the Lagrange multiplier.

It can be shown that the following conditions hold

$$\begin{aligned} \boldsymbol{\psi}(t_1 + 0) &= \boldsymbol{\psi}(t_1 - 0) \,, \\ \boldsymbol{\psi}(t_2 + 0) &= \boldsymbol{\psi}(t_2 - 0) - \boldsymbol{\mu}(t_2) \bigtriangledown S(\boldsymbol{u}(t_2), \boldsymbol{x}(t_2)) \,. \end{aligned} \tag{12}$$

Consequently, the adjoint variables are continuous going on the boundary $g = 0$ and a jump can occur coming off the boundary.

Different generalizations of these equations are possible, for them consult some text-book about optimal control (e.g. [1], [8], [7] and [5]).

3. Haar Wavelet Method

Consider the interval $t \in [A, B]$, where A and B are given constants. Define the quantity $M = 2^J$, where J is the maximal level of resolution. Distribute the interval $[A, B]$ into $2M$ subintervals of equal length: $\Delta t = (B - A)/2M$. Next two parameters are introduced: the dilation parameter $j = 0, 1, ..., J$ and the translation parameter $k = 0, 1, ..., m - 1$, where $m = 2^J$. The wavelet number is defined as $i = m + k + 1$.

Following the paper [11] we define the i-th Haar wavelet as:

$$h_i(t) = \begin{cases} 1 & \text{for} \quad t \in [\xi_1(i), \xi_2(i)]\,, \\ -1 & \text{for} \quad t \in [\xi_2(i), \xi_3(i)]\,, \\ 0 & \text{elsewhere}\,, \end{cases} \tag{13}$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta t\,, & \xi_2(i) &= A + (2k+1)\mu\Delta t\,, \\ \xi_3(i) &= A + 2(k+1)\mu\Delta t\,, & \mu &= M/m\,. \end{aligned}$$

The case $i = 1$ corresponds to the scaling function $h_1(t) = 0$ for $t \in [A, B]$.

In the following we need the integrals

$$p_i(t) = \int_A^t h_i(t)dt\,. \tag{14}$$

In view of [13] these integrals can be evaluated analytically; by doing this we find

$$p_i(t) = \begin{cases} 0 & \text{for} \quad t \leq \xi_1(i)\,, \\ t - \xi_1(i) & \text{for} \quad t \in [\xi_1(i), \xi_2(i)]\,, \\ -t - \xi_1(i) + 2\xi_2(i) & \text{for} \quad t \in [\xi_1(i), \xi_3(i)]\,, \\ 0 & \text{for} \quad t \geq \xi_3(i)\,. \end{cases} \tag{15}$$

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A$, $\xi_2 = \xi_3 = B$ and

$$p_1(t) = t - A\,. \tag{16}$$

In this paper the collocation method for solving ODEs is applied. The collocation points are

$$t_l = 0,5(\tilde{t}_{l-1} + \tilde{t}_l)\,, \quad l = 1, 2, ..., 2M\,. \tag{17}$$

The symbol $\tilde{t}_l$ denotes the l-th grid point $\tilde{t}_l = A + l\Delta t$. Equations (13)-(16) are discretisized by replacing $t \to t_l$. It is convenient to introduce the Haar matrices $H(i,l) = h_i(t_l)$, $P(i,l) = p_i(t_l)$.

For solving the boundary value problems we need the values of p_i at $t = B$. In view of (15)-(16) we find

$$p_i(B) = \begin{cases} B - A & \text{for} \quad i = 1\,, \\ 0 & \text{for} \quad i \neq 1\,. \end{cases}$$

It is convenient to introduce the matrix $\boldsymbol{R}$ with the elements

$$R(i,l) = P(i,l) - p_i(B)\,. \tag{18}$$

Next some formulae which are useful in solving some problems are presented. First we introduce the vectors

$$\boldsymbol{E} = [1, 1, ..., 1]; \quad \boldsymbol{E_1} = [1, 0, 0, ..., 0]; \quad \boldsymbol{t} = (t_l); \quad \hat{t} = \boldsymbol{t} - A\boldsymbol{E}\,.$$

It can be verified by the computer simulation that the following formulae hold

$$\boldsymbol{E_1} = \boldsymbol{E}/\boldsymbol{H}\,, \quad (\boldsymbol{E}/\boldsymbol{H})\boldsymbol{P} = \boldsymbol{E_1}\boldsymbol{P} = \boldsymbol{t} - A\boldsymbol{E} = \hat{\boldsymbol{t}}\,, \tag{19}$$

$$\frac{1}{\alpha!}(\hat{\boldsymbol{t}}^{\alpha}/\boldsymbol{H})\boldsymbol{P} = \frac{1}{(\alpha+1)!}\hat{\boldsymbol{t}}^{\alpha+1}\,. \tag{20}$$

If we have to integrate by the Haar wavelet method the equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, u)\,, \quad \boldsymbol{x} = (x_i), \tag{21}$$

then the solution is sought in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{a}\boldsymbol{H}\,. \tag{22}$$

By integrating (21) we find

$$\boldsymbol{x} = \boldsymbol{a}\boldsymbol{P} + \boldsymbol{k}\,. \tag{23}$$

Here $\boldsymbol{k}$ stands for the vector of integration constants; these constants can be calculated from the initial or boundary conditions. Replacing (22)-(23) into (21) we obtain a system of $2M$ equations for evaluating the wavelet coefficients $\boldsymbol{a} = (a_i)$.

In this paper computer simulations were carried out with the aid of *Matlab* programs for which the matrix representation is very effective. It is expedient to put together subprograms for calculating the matrices $\boldsymbol{H}, \boldsymbol{P}$, which can be used without changes for solving arbitrary problems.

4. Example 1: Optimal Control with an Integral Constraint

Consider the problem

$$\int_0^2 x\,dt \to \inf\,, \quad \int_0^2 \ddot{x}^2 dt = 1\,, \qquad x(0) = \dot{x}(0) = 0\,, \quad \ddot{x}(2) = 0\,. \tag{24}$$

We interprete the acceleration $\ddot{x}$ as a control and introduce the state variables

$$x_1 = x\,, \quad x_2 = \dot{x}\,, \quad \dot{x}_3 = \ddot{x} = u\,, \quad x_4 = \int_0^t u^2 dt\,.$$

The state equations are

$$\dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{x}_{\boldsymbol{2}}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{2}} = \boldsymbol{x}_{\boldsymbol{3}}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{3}} = \boldsymbol{u}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{4}} = \boldsymbol{u}^2, \tag{25}$$

with the boundary conditions $x_1(0) = x_2(0) = x_3(2) = x_4(0) = 0$, $x_4(2) = 1$.

The Hamiltonian is

$$\hat{H} = -\boldsymbol{x}_{\boldsymbol{1}} + \boldsymbol{\psi}_{\boldsymbol{1}}\boldsymbol{x}_{\boldsymbol{2}} + \boldsymbol{\psi}_{\boldsymbol{2}}\boldsymbol{x}_{\boldsymbol{3}} + \boldsymbol{\psi}_{\boldsymbol{3}}\boldsymbol{u} + \boldsymbol{\psi}_{\boldsymbol{4}}\boldsymbol{u}^2\,. \tag{26}$$

The adjoint system has the form

$$\dot{\boldsymbol{\psi}}_{\boldsymbol{1}} = \boldsymbol{E}\,, \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{2}} = -\boldsymbol{\psi}_{\boldsymbol{1}}\,, \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{3}} = -\boldsymbol{\psi}_{\boldsymbol{2}}\,, \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{4}} = 0\,. \tag{27}$$

According to the transversality conditions $\psi_1(2) = \psi_2(2) = \psi_3(2) = 0$.

To begin with we integrate (27) by assuming

$$\begin{aligned} &\dot{\boldsymbol{\psi}}_{\boldsymbol{1}} = \boldsymbol{b}_{\boldsymbol{1}}\boldsymbol{H}\,, \quad \boldsymbol{\psi}_{\boldsymbol{1}} = \boldsymbol{b}_{\boldsymbol{1}}\boldsymbol{R}\,, \\ &\dot{\boldsymbol{\psi}}_{\boldsymbol{2}} = \boldsymbol{b}_{\boldsymbol{2}}\boldsymbol{H}\,, \quad \boldsymbol{\psi}_{\boldsymbol{2}} = \boldsymbol{b}_{\boldsymbol{2}}\boldsymbol{R}\,, \\ &\dot{\boldsymbol{\psi}}_{\boldsymbol{3}} = \boldsymbol{b}_{\boldsymbol{3}}\boldsymbol{H}\,, \quad \boldsymbol{\psi}_{\boldsymbol{3}} = \boldsymbol{b}_{\boldsymbol{3}}\boldsymbol{R}\,, \quad \boldsymbol{\psi}_{\boldsymbol{4}} = \lambda = \text{const}\,. \end{aligned} \tag{28}$$

The matrix $\boldsymbol{R}$ is calculated from (18). Substituting (28) into (27) we obtain

$$\boldsymbol{b}_{\boldsymbol{1}}\boldsymbol{H} = \boldsymbol{E}\,, \quad \boldsymbol{b}_{\boldsymbol{1}}\boldsymbol{R} + \boldsymbol{b}_{\boldsymbol{2}}\boldsymbol{H} = 0\,, \quad \boldsymbol{b}_{\boldsymbol{2}}\boldsymbol{R} + \boldsymbol{b}_{\boldsymbol{3}}\boldsymbol{H} = 0\,. \tag{29}$$

From here we calculate the wavelet coefficients $\boldsymbol{b}_{\boldsymbol{1}}$, $\boldsymbol{b}_{\boldsymbol{2}}$, $\boldsymbol{b}_{\boldsymbol{3}}$ and from (28) the adjoint variables $\boldsymbol{\psi}_{\boldsymbol{1}}$, $\boldsymbol{\psi}_{\boldsymbol{2}}$, $\boldsymbol{\psi}_{\boldsymbol{3}}$.

Evaluating the optimal control from (5) we find

$$\boldsymbol{u} = -\frac{\boldsymbol{\psi}_{\boldsymbol{3}}}{2\psi_4} = -\frac{\boldsymbol{\psi}_{\boldsymbol{3}}}{2\lambda}\,. \tag{30}$$

Since $\boldsymbol{\psi}_{\boldsymbol{3}}$ is already known we can calculate the auxiliary variable

$$\tilde{\boldsymbol{u}} = \lambda\boldsymbol{u} = -\boldsymbol{\psi}_{\boldsymbol{3}}/2\,. \tag{31}$$

Next the state variables are developed into the Haar series:

$$\begin{aligned} &\dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{a}_{\boldsymbol{1}}\boldsymbol{H}\,, \quad \boldsymbol{x}_{\boldsymbol{1}} = \boldsymbol{a}_{\boldsymbol{1}}\boldsymbol{P}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{2}} = \boldsymbol{a}_{\boldsymbol{2}}\boldsymbol{H}\,, \quad \boldsymbol{x}_{\boldsymbol{2}} = \boldsymbol{a}_{\boldsymbol{2}}\boldsymbol{P}\,, \\ &\dot{\boldsymbol{x}}_{\boldsymbol{3}} = \boldsymbol{a}_{\boldsymbol{3}}\boldsymbol{H}\,, \quad \boldsymbol{x}_{\boldsymbol{3}} = \boldsymbol{a}_{\boldsymbol{3}}\boldsymbol{P}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{4}} = \boldsymbol{a}_{\boldsymbol{4}}\boldsymbol{H}\,, \quad \boldsymbol{x}_{\boldsymbol{4}} = \boldsymbol{a}_{\boldsymbol{4}}\boldsymbol{P}\,. \end{aligned} \tag{32}$$

Replacing these results into (25) we get

$$\begin{aligned} &\boldsymbol{a}_{\boldsymbol{1}}\boldsymbol{H} - \boldsymbol{a}_{\boldsymbol{2}}\boldsymbol{P} = 0\,, \quad \boldsymbol{a}_{\boldsymbol{2}}\boldsymbol{H} - \boldsymbol{a}_{\boldsymbol{3}}\boldsymbol{P} = 0\,, \\ &\boldsymbol{a}_{\boldsymbol{3}}\boldsymbol{H} = \tilde{\boldsymbol{u}}/\lambda\,, \quad \boldsymbol{a}_{\boldsymbol{4}}\boldsymbol{H} = \tilde{\boldsymbol{u}}^2/\lambda^2\,. \end{aligned} \tag{33}$$

The Lagrange multiplier λ is calculated in the following way. From $(33)_4$ we find

$$\tilde{\boldsymbol{a}}_{\boldsymbol{4}} = \lambda^2\boldsymbol{a}_{\boldsymbol{4}} = \tilde{\boldsymbol{u}}^2/\boldsymbol{H}\,.$$

Satisfying the boundary condition $x_4(2) = Q1$ we get from (32)

$$x_4|_{t=2} = \boldsymbol{a}_{\boldsymbol{4}}\boldsymbol{P}|_{t=2} = 2a_4(1) = 1\,.$$

J	δx	δu	$\delta\lambda$
4	6.5E-4	6.3E-4	3.8E-7
5	1.6E-4	1.6E-4	3.8E-7
6	4.0E-5	4.0E-5	3.8E-7

Table 1: Error estimates for the problem (24)

From these two equations we find

$$\lambda = \sqrt{2\tilde{a}_4(1)}\,. \tag{34}$$

To end up with the solution the state variables x_1, x_2 are calculated from (32)-(33). The exact solution is

$$\begin{aligned} x_{ex} &= -\frac{1}{12\lambda}\left[\frac{1}{120}(t-2)^6 + \frac{4}{3}(t-2)^3 - \frac{72}{5}t + \frac{152}{15}\right], \\ u_{ex} &= -\frac{1}{48\lambda}(t-2)\left[(t-2)^3 + 32\right], \quad \lambda_{ex} = 0,7559\,. \end{aligned} \tag{35}$$

For estimating the accuracy of our results the error estimates

$$\begin{aligned} \delta x &= \max_i |x_{ex}(t_i) - x(t_i)|\,, \\ \delta u &= \max_i |u_{ex}(t_i) - u(t_i)|\,, \\ \delta\lambda &= |\lambda_{ex} - \lambda| \end{aligned} \tag{36}$$

are introduced. Results of the computer simulation are presented in Table 1.

It follows from this table that already a small number of points ($J = 4$; 32 grid points) guarantees sufficient accuracy.

5. Example 2: Optimal Control with a State Inequality Constraint

Bryson and Ho [1] discussed the problem

$$\begin{aligned} &I = \frac{1}{2}\int_0^1 u^2 dt \to \min\,, \quad x_1 \le l, \\ &\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u\,, \quad x_1(0) = x_1(1) = 0\,, \quad x_2(0) = -x_2(1) = 1\,, \end{aligned} \tag{37}$$

where $l > 0$ is a given constant.

Consider the case where the equality $x_1(t) = l$ holds in the subinterval $t \in [t_1, t_2]$. This subinterval cannot be near the boundaries $t = 0$ or $t = 1$, since in this case the prescribed boundary conditions cannot be satisfied.

Due to symmetry we can solve the problem (37) only for the half-interval

$t \in [0, 0.5]$. It is reasonable to assume that $x_1(t) < l$ for some interval $t \in [0, t_1]$ and $x_1(t) = l$ for $t \in [t_1, 0.5]$.

The Hamiltonian is

$$\hat{H} = -\frac{1}{2}u^2 + \psi_1 x_2 + \psi_2 u \,. \tag{38}$$

It follows from the optimality condition (5) that $\psi = u$.

Next we put together the adjoint system (10). Since in the present case $g = x_1 - l$, $S = x_2$, $\bigtriangledown S = (0, 1)$ we find

$$\dot{\psi}_1 = 0 \,, \quad \dot{\psi}_2 = -\psi_1 + \mu \,, \tag{39}$$

whereas $\mu(t) = 0$ for $t \in [0, t_1]$.

To begin with we consider the subinterval $t \in [t_1, 0.5]$. Since $x_1 = l$ it follows from the state equations that $x_2 = u = 0$ and, consequently, $\psi_2 = 0$. Integrating (39) we find $\psi_1 = C_1$, $\mu(t) = C_1$, where C_1 denotes the integration constant.

Now the problem (37) can be paraphased as

$$\begin{aligned} & I = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min \,, \\ & \dot{x}_1 = x_2 \,, \quad \dot{x}_2 = u \,, \quad x_1(0) = 0 \,, \quad x_2(0) = 1 \,, \\ & x_1(t_1) = l \,, \quad x_2(t_1) = 0 \,. \end{aligned} \tag{40}$$

The wavelet solution is sought in the matrix form

$$\begin{aligned} & \dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{a_1 H} \,, \quad \boldsymbol{x_1} = \boldsymbol{a_1 P} \,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{2}} = \boldsymbol{a_2 H} \,, \quad \boldsymbol{x_2} = \boldsymbol{a_2 P} + \boldsymbol{E} \,, \\ & \dot{\boldsymbol{\psi}}_{\boldsymbol{1}} = \boldsymbol{b_1 H} \,, \quad \boldsymbol{\psi_1} = \boldsymbol{b_1 P} + C_1 \boldsymbol{E} \,, \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{2}} = \boldsymbol{b_2 H} \,, \quad \boldsymbol{\psi_2} = \boldsymbol{b_2 P} + C_2 \boldsymbol{E} \,. \end{aligned} \tag{41}$$

The matrices $\boldsymbol{H}$ and $\boldsymbol{P}$ can be calculated according to (13), (15), (16) assuming that $A = 0$, $B = t_1$.

Replacing (41) into the state equations (40) and into the adjoint system (39) we obtain

$$\begin{aligned} & \boldsymbol{a_1 H} = \boldsymbol{a_2 P} + \boldsymbol{E} \,, \quad \boldsymbol{a_2 H} = \boldsymbol{b_2 P} + C_2 \boldsymbol{E} \,, \\ & \boldsymbol{b_1 H} = 0 \,, \quad \boldsymbol{b_2 H} = -\boldsymbol{b_1 P} - C_1 \boldsymbol{E} \,. \end{aligned} \tag{42}$$

It follows from the third and fourth equation that $\boldsymbol{b_1} \equiv 0$, $\boldsymbol{b_2} = -C_1 \boldsymbol{E}/\boldsymbol{H} = -C_1 \boldsymbol{E_1}$, $C_2 = C_1 t_1$. Due to the continuity $\psi_2(t_1) = 0$ and we obtain

$$\boldsymbol{\psi_2} = C_1(t_1 \boldsymbol{E} - \boldsymbol{t}) = \boldsymbol{u} \,. \tag{43}$$

Integration of the served equation of (41) gives

$$\begin{aligned} & \boldsymbol{a_2 H} = -C_1 \boldsymbol{E_1 P} + C_2 \boldsymbol{E} = -C_1 \boldsymbol{t} + C_2 \boldsymbol{E} \,, \\ & \boldsymbol{x_2} = -C_1(\boldsymbol{t}/\boldsymbol{H})\boldsymbol{P} + (\boldsymbol{E}/\boldsymbol{H})\boldsymbol{P} + \boldsymbol{E} \,. \end{aligned}$$

In view of (19)-(20) this result can be put into the form

$$x_2 = C_1 t(t_1 - t/2) + 1\,. \tag{44}$$

Since

$$a_2 P = -C_1(t/H)P + C_1 t_1 (E/H)P = -C_1 t^2/2 + C_1 t_1 t\,,$$

then

$$a_1 = (a_2 \boldsymbol{P} + \boldsymbol{E})/\boldsymbol{H} = -\frac{1}{2} C_1 \boldsymbol{t}^2/\boldsymbol{H} + C_1 t_1 \boldsymbol{t}/\boldsymbol{H} + \boldsymbol{E}/\boldsymbol{H}$$

and

$$\boldsymbol{x}_1 = -\frac{1}{2} C_1 (\boldsymbol{t}^2/\boldsymbol{H})\boldsymbol{P} + C_1 t_1 (\boldsymbol{t}/\boldsymbol{H})\boldsymbol{P} + (\boldsymbol{E}/\boldsymbol{H})\boldsymbol{P}\,.$$

In view of (19)-(20) this result can be rewritten in the form

$$x_1 = -C_1 t^2 \left(\frac{t}{6} - \frac{1}{2} t_1 \right) + t\,. \tag{45}$$

The constants C_1, t_* are calculated from the boundary conditions $x_1(t_*) = l$, $x_2(t_*) = 0$. Satisfying this conditions we find $t_1 = 3l$, $C_1 = -2/t_1^2$ and

$$x_1 = l\tau(\tau^2 - 3\tau + 3)\,, \quad \tau = t/t_1\,, \quad x_2 = (1-\tau)^2\,, \quad u = -\frac{2}{3l}(1-\tau)\,. \tag{46}$$

The same results were obtained in [1].

So we see that the Haar wavelet method enabled us to find the exact analytical solution of the problem.

6. Example 3: Optimal Control with a Control Inequality Constraint

Let us solve the problem

$$\begin{gathered} I = \int_0^1 (x_1^2 + x_2^2 + \alpha u^2)dt \to \min\,, \quad |u| \le u_0\,, \\ \dot{x}_1 = x_2\,, \quad \dot{x}_2 = -x_2 + u\,, \quad x_1(0) = 0\,, \quad x_2(0) = -1\,. \end{gathered} \tag{47}$$

It is assumed that the control is smooth and thus the function $u(t)$ must be continuous.

Introducing the Hamiltonian

$$\hat{H} = -(x_1^2 + x_2^2 + \alpha u^2) + \psi_1 x_2 + \psi_2(-x_2 + u) \tag{48}$$

we put together the adjoint system

$$\dot{\psi}_1 = -\frac{\partial \hat{H}}{\partial x_1} = 2x_1\,, \quad \dot{\psi}_2 = -\frac{\partial \hat{H}}{\partial x_2} = 2x_2 - \psi_1 + \psi_2\,. \tag{49}$$

According to the transversaly conditions we have $\psi_1(1) = \psi_2(1) = 0$. In the regions where $|u| < u_0$ it follows from the extremum condition $\partial\hat{H}/\partial u = 0$ that $\psi_2 = 2\alpha u$.

Solution for $u = 0$ shows that $\psi_2(t)$ is a decreasing function. Therefore it is reasonable to assume that $u = u_0$ for $t \in [0, t_1]$ and $u < u_0$ for $t \in [t_1, 1]$. The value of t_1 is for the present unknown and will be calculated in the course of the solution.

Let us assign some value to t_1 and integrate the state equations for $t \in [0, t_1]$. According to the wavelet method we take (the matrices H and P are calculated for $A = 0$, $B = t_1$):

$$\begin{aligned} \dot{\boldsymbol{x}}_{\boldsymbol{1}} &= \boldsymbol{a_1 H}\,, \quad \boldsymbol{x_1} = \boldsymbol{a_1 P}\,, \\ \dot{\boldsymbol{x}}_{\boldsymbol{2}} &= \boldsymbol{a_2 H}\,, \quad \boldsymbol{x_2} = \boldsymbol{a_2 P} - \boldsymbol{E}\,. \end{aligned} \tag{50}$$

Replacing these results into the state equations $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2 + u_0$ we find

$$\begin{aligned} &\boldsymbol{a_1 H} - \boldsymbol{a_2 P} = -\boldsymbol{E}\,, \\ &\boldsymbol{a_2 (H + P)} = (1 + u_0)\boldsymbol{E}\,. \end{aligned} \tag{51}$$

Solving this system we evaluate the wavelet coefficients $\boldsymbol{a_1}, \boldsymbol{a_2}$ and calculate the functions $\boldsymbol{x_1}, \boldsymbol{x_2}$ according to (50). In the following we need the values $x_1 = x_1(t_1)$, $x_2 = x_2(t_1)$. It follows from (15) that $p_1(t_1) = t_1$ and $p_i(t_1) = 0$ for $i \neq 1$. In view of (50) we find

$$x_1^* = a_1(1)t_1\,, \quad x_2^* = a_2(1)t_1 - 1\,. \tag{52}$$

Now let us consider the subinterval $t \in [t_1, 1]$. Again we divide this interval into $2M$ equal parts and calculate the matrices H, P and R from (13)-(15), (18) assuming $A = t_1$, $B = 1 - t_1$.

The solution is sought in the form

$$\begin{aligned} &\dot{\boldsymbol{x}}_{\boldsymbol{1}} = \hat{\boldsymbol{a}}_{\boldsymbol{1}}\boldsymbol{H}\,, \quad \boldsymbol{x_1} = \hat{\boldsymbol{a}}_{\boldsymbol{1}}\boldsymbol{P} + x_1\boldsymbol{E}\,, \quad \dot{\boldsymbol{x}}_{\boldsymbol{2}} = \hat{\boldsymbol{a}}_{\boldsymbol{2}}\boldsymbol{H}\,, \quad \boldsymbol{x_2} = \hat{\boldsymbol{a}}_{\boldsymbol{2}}\boldsymbol{P} + x_2^*\boldsymbol{E}\,, \\ &\dot{\boldsymbol{\psi}}_{\boldsymbol{1}} = \hat{\boldsymbol{b}}_{\boldsymbol{1}}\boldsymbol{H}\,, \quad \boldsymbol{\psi_1} = \hat{\boldsymbol{b}}_{\boldsymbol{1}}\boldsymbol{R}\,, \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{2}} = \hat{\boldsymbol{b}}_{\boldsymbol{2}}\boldsymbol{H}\,, \quad \boldsymbol{\psi_2} = \hat{\boldsymbol{b}}_{\boldsymbol{2}}\boldsymbol{R}\,. \end{aligned} \tag{53}$$

Here $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, $\hat{b}_2$ denote wavelet coefficients for the subinterval $t \in [t_1, 1]$. The matrix R is calculated according to (18).

Substituting (53) into (47), (49) and taking into account that $\psi_2 = 2\alpha u$ we

J	t_1	$x_1(1)$	$x_2(1)$
4	0.338569	-0.48679	-0.20527
5	0.338570	-0.48676	-0.20535
6	0.338574	-0.48675	-0.20536

Table 2: Parameter t_1 and boundary values of the problem (47)

get the system

$$\begin{aligned} &\hat{\boldsymbol{a}}_{\mathbf{1}}\boldsymbol{H} - \hat{\boldsymbol{a}}_{\mathbf{2}}\boldsymbol{P} = x_2^*\boldsymbol{E}\,, \quad \hat{\boldsymbol{a}}_{\mathbf{2}}(\boldsymbol{H}+\boldsymbol{P}) = \frac{1}{2\alpha}\hat{\boldsymbol{b}}_{\mathbf{2}}\boldsymbol{R} = -x_2^*\boldsymbol{E}\,, \\ &-2\hat{\boldsymbol{a}}_{\mathbf{1}}\boldsymbol{P} + \hat{\boldsymbol{b}}_{\mathbf{1}}\boldsymbol{H} = 2x_1^*\boldsymbol{E}\,, \quad -2\hat{\boldsymbol{a}}_{\mathbf{2}}\boldsymbol{P} + \hat{\boldsymbol{b}}_{\mathbf{1}}\boldsymbol{R} + \hat{\boldsymbol{b}}_{\mathbf{2}}(\boldsymbol{H}-\boldsymbol{R}) = 2x_2^*\boldsymbol{E}\,, \end{aligned} \tag{54}$$

which can be solved numerically with the aid of *Matlab* programs.

The control $u(t)$ must be continuous at $t = t_*$. In the case of an arbitrary chosen value t_* such a requirement is not fulfilled. This discrepancy can be estimated by the function $\Delta = u(t_1 - 0) - u(t_1 + 0)$. Since

$$u(t_1 - 0) = u_0\,, \quad u(t_1 + 0) = \frac{1}{2\alpha}\psi_2(t_1 + 0) = \frac{1}{2\alpha}\hat{b}_2 R|_{t=t_1} = -\frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1)$$

we obtain

$$\Delta = \frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1) + u_0\,. \tag{55}$$

We shall vary t_* until the condition $\Delta = 0$ is fulfilled with the necessary exactness.

Computer simulation was carried out for $u_0 = 0.5$, $\alpha = 0.5$; the results are plotted in Figure 1. Since in the present case we do not know the exact solution, the error estimates (36) are unusuable. Therefore we estimate the exactness of solution by calculating the values of t_1, $x_1(1)$, $x_2(1)$ at different levels of resolution J. These results are presented in Table 2.

7. Conclusions

Efficiency of the Haar wavelet method for solving optimal control problems is demonstrated. Numerical solutions for three test problems with different equality and inequality constrains are presented. Very satisfactory exactness of the results even for a low number of collocation points is stated. In the case of some simple problems the Haar wavelet method allows to obtain analytical (exact) results (Example 2).

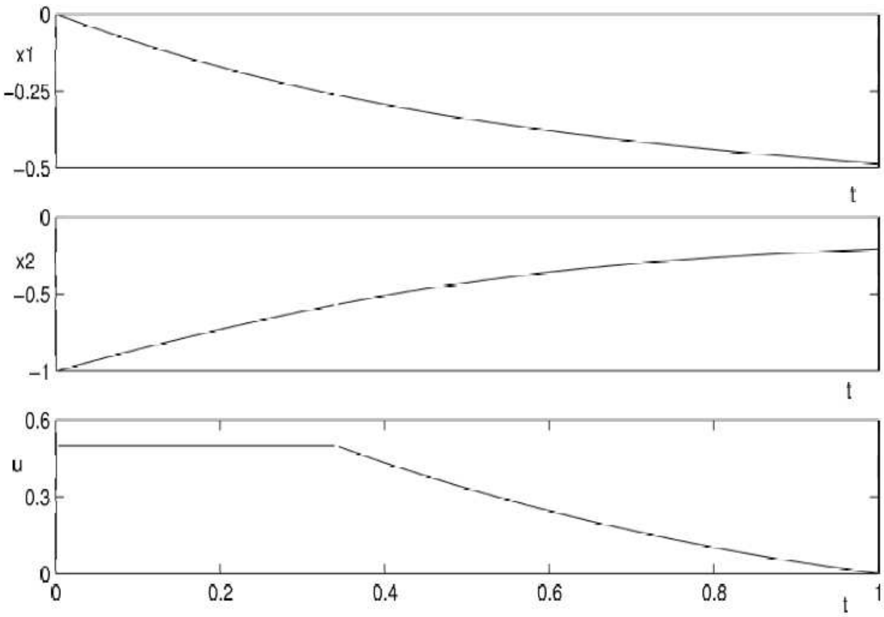

Figure 1: Solution of the problem (47) for $u_0 = 0.5$, $\alpha = 0.5$

For simplicity of discussion only linear problems were considered, but the recommended method of solution is applicable also for nonlinear systems. In this case the wavelet coefficients must be calculated by some numerical technique, e.g. by the Newton method (see [9]).

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