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ON STABILITY (CONSTANCY) OF GOMPERTZ PARAMETER

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Abstract: The stability analysis of age-dependent Gompertz parameter α with respect to the population size N leads us to conclude the constancy of α.

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1. Introduction

Aging research has long employed demographic models of age-specific adult mortality that are based on approximately exponential increase in mortality. The best known is the Gompertz model which has been the major mortality model in gerontology for more than 65 years Carey et al [1], Comfort [2]. The Gompertz model has also served as basis for comparative measures of aging rate, including the initial mortality rate (IMR) and mortality rate doubling time (MRD). These measures are widely used to quantify the mortality and senescence characteristics of species and population Carey et al [1], Comfort [2], Witten et al [11], see [1, 2, 11], respectively. According to Gompertz law, human mortality rates double over about every eight years of adult age. Moreover, he found that at advanced ages mortality rates increase less rapidly than an exponential function. The Gompertz law of exponential increase in mortality rates is observed in many biological species.

In the presence of mortality data by age, the Gompertz parameter A and

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α have been estimated by using various statistical methods like maximum likelihood, linear regression, non-linear regression Finch [3]. In the absence of age-specific mortality data, Carey et al [1] have developed a method to estimate initial mortality rate (IMR) and mortality rate to double (MRD) from the average mortality rate (A_{av}) and maximum life span (t_m).

As alternatives to the Gompertz model, power functions such as the Weibull model are used to describe population senescence Finch et al [4]. As shown for several invertebrates Finch et al [5], the fit of the Gompertz and Weibull formula varies between populations.

When studying the physical properties of a problem, it is essential to prove the existence, uniqueness and stability of the problem. E.S. Lakshminarayanan and M. Pitchaimani have already proved the existence (asymptotic formula) and uniqueness of age-dependent parameter α in Gompertz survival model Lakshminarayanan et al [6, 7]. In this work we establish the stability of the parameter α with respect to the population size N. In view of the mortality deceleration at advanced ages, the stability of Gompertz parameter α is valid in the Gompertz segment [A, average life span], that is, from puberty to average life span.

2. Estimation of Age-Dependent Parameter α

Following Finch, Carey et al [1] the proportion of a population surviving from puberty to adult age $t, S(t)$, may be obtained from the Gompertz equation of mortality rate (see Muller et al [9], Slob et al [9]):

$$m(t) = Ae^{\alpha t}, \tag{1}$$

$$S(t) = e^{\frac{A}{\alpha}(1-e^{\alpha t})} \tag{2}$$

for a population of size N, the age at which the population has diminished to one survivor is ($S(t) = \frac{1}{N}$) approximates t_m. Thus,

$$S(t_m) \approx \frac{1}{N} = \exp\{\frac{A}{\alpha}(1 - e^{\alpha t_m})\} \tag{3}$$

or

$$t_m = \frac{\ln[1 + \frac{\alpha \ln N}{A}]}{\alpha}. \tag{4}$$

The average mortality rate of a steady-state population subject to age-specific mortality rates of equation (1) is (see Finch et al [5]):

$$A_{av} = \frac{1}{\int\limits_0^\infty S(t)dt}. \tag{5}$$

After a little algebra, equation (4) leads to

$$\frac{A}{\alpha} = \frac{\ln N}{e^{\alpha t_m} - 1} \tag{6}$$

and upon substitution of equation (2) into equation (5) we arrive at the following

$$\frac{1}{A_{av}} = \int\limits_0^\infty e^{\frac{A}{\alpha}(1-e^{\alpha t})}dt.$$

A simple substitution in the integral gives

$$\alpha = A_{av}e^{A/\alpha}\int\limits_{A/\alpha}^\infty \frac{e^{-z}}{z}dz$$

and using (6), we get

$$\alpha = A_{av}e^{\frac{\ln N}{e^{\alpha t_m}-1}}\int\limits_{\frac{\ln N}{e^{\alpha t_m}-1}}^\infty \frac{e^{-z}}{z}dz. \tag{7}$$

Muller et al [10], Witten [10] C.E. Finch et al have solved equations (6), (7) numerically for A and α, for a given A_{av}, t_m and N. The basic equation (7) is transcendental, involving exponential integral and the age-dependent parameter α is a function of $A_{av}, \ln N, t_{max}$. In this paper, we have proved a theorem on stability (constancy) of age-dependent Gompertz parameter α with respect to the population size N.

3. Stability of α with Respect to the Population Size N

3.1. Necessary Condition for Stability

Let α_1, α_2 be age dependent parameters with population size N_1, N_2 respectively, then

$$\alpha_i = A_{av}e^{\frac{\ln N_i}{e^{\alpha_i t_m}-1}}\int\limits_{\frac{\ln N_i}{e^{\alpha_i t_m}-1}}^\infty \frac{e^{-z}}{z}dz, \quad i = 1, 2.$$

Consider

$$\alpha_1 - \alpha_2 = A_{av}\left[e^{x_1}\int_{x_1}^{\infty}\frac{e^{-z}}{z}dz - e^{x_2}\int_{x_2}^{\infty}\frac{e^{-z}}{z}dz\right],$$

where

$$x_i = \frac{\ln N_i}{e^{\alpha_i t_m} - 1}. \tag{8}$$

A simple substitution in the above equation gives

$$\alpha_1 - \alpha_2 = A_{av}\int_0^{\infty} e^{-u}\left[\frac{1}{u + x_1} - \frac{1}{u + x_2}\right]du.$$

Set

$$x_1 = \frac{x_1 + x_2}{2} + \frac{x_1 - x_2}{2}, \;\; x_2 = \frac{x_1 + x_2}{2} - \frac{x_1 - x_2}{2}. \tag{9}$$

Note that $\frac{x_1+x_2}{2}$ is the arithmetic mean and $\frac{x_1-x_2}{2}$ is the perturbation term of x_1, x_2.

Substitution of (9) into the above equation results in

$$\alpha_1 - \alpha_2 = A_{av}e^{\frac{x_1+x_2}{2}}\int_{\frac{x_1+x_2}{2}}^{\infty} e^{-y}\left[\frac{1}{y + \frac{x_1-x_2}{2}} - \frac{1}{y - \frac{x_1-x_2}{2}}\right]dy, \tag{10}$$

where $y = u + \frac{x_1+x_2}{2}$.

On the RHS of (10), the expression

$$\frac{1}{y + \frac{x_1-x_2}{2}} - \frac{1}{y - \frac{x_1-x_2}{2}} = \frac{1}{y}\left(1 - \frac{x_1 - x_2}{2y} + ...\right) - \frac{1}{y}\left(1 + \frac{x_1 - x_2}{2y} + ...\right) \tag{11}$$

can be approximated to $-(\frac{x_1-x_2}{y^2})$ by neglecting higher order perturbation terms in each expression on the RHS of (11), since $\left|\frac{x_1-x_2}{x_1+x_2}\right| < 1$.

On account of (11), (10) becomes

$$\alpha_1 - \alpha_2 = A_{av}e^{\frac{x_1+x_2}{2}}\int_{\frac{x_1+x_2}{2}}^{\infty} -\left(\frac{x_1 - x_2}{y^2}\right)e^{-y}dy.$$

Further,

$$|\alpha_1 - \alpha_2| \;\;\leq\;\; A_{av}e^{\frac{x_1+x_2}{2}}\int_{\frac{x_1+x_2}{2}}^{\infty}\frac{|x_1 - x_2|}{y^2}\,e^{-y}dy$$

$$\leq \quad A_{av}|x_1 - x_2| e^{\frac{x_1+x_2}{2}} e^{-\frac{x_1+x_2}{2}} \int_{\frac{x_1+x_2}{2}}^{\infty} \frac{dy}{y^2}.$$

Upon integration we get

$$|\alpha_1 - \alpha_2| \leq A_{av} \left| \frac{x_1 - x_2}{\frac{x_1+x_2}{2}} \right|. \tag{12}$$

Retrieving x_i from (8) and substituting into equation (12), we get

$$|\alpha_1 - \alpha_2| \leq A_{av} \left| \frac{\ln N_1 e^{\alpha_2 t_m} - \ln N_2 e^{\alpha_1 t_m} - (\ln N_1 - \ln N_2)}{\frac{\ln N_1 e^{\alpha_2 t_m} + \ln N_2 e^{\alpha_1 t_m}}{2} - \frac{(\ln N_1 + \ln N_2)}{2}} \right|. \tag{13}$$

We represent Gompertz parameter α as a sum of mean and perturbation as follows:

$$\alpha_1 = \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_1 - \alpha_2}{2}, \;\; \alpha_2 = \frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_1 - \alpha_2}{2}. \tag{14}$$

Substitution of (14) into (13) and after a little algebra we obtain

$$|\alpha_1 - \alpha_2| \leq A_{av} \left| \frac{\frac{2 \ln N_1}{\ln N_1 + \ln N_2} P - \frac{2 \ln N_2}{\ln N_1 + \ln N_2} Q - \frac{\ln N_1 - \ln N_2}{\frac{\ln N_1 + \ln N_2}{2}} R}{\frac{\ln N_1}{\ln N_1 + \ln N_2} P + \frac{\ln N_2}{\ln N_1 + \ln N_2} Q - R} \right|, \tag{15}$$

where we have denoted $P = e^{-\frac{\alpha_1 - \alpha_2}{2} t_m}$, $Q = e^{\frac{\alpha_1 - \alpha_2}{2} t_m}$ and $R = e^{-\frac{\alpha_1 + \alpha_2}{2} t_m}$.

Since

$$e^{\pm \frac{\alpha_1 - \alpha_2}{2} t_m} \approx 1 \pm (\frac{\alpha_1 - \alpha_2}{2}) t_m \tag{16}$$

(by neglecting higher order perturbation terms in α_1, α_2) substituting (16) into equation (15) and simplifying we obtain

$$|\alpha_1 - \alpha_2| \leq A_{av} \left| \frac{\frac{\ln N_1 - \ln N_2}{\frac{\ln N_1 + \ln N_2}{2}} (1 - e^{-\frac{\alpha_1 + \alpha_2}{2} t_m}) - 2(\frac{\alpha_1 - \alpha_2}{2}) t_m}{(1 - e^{-\frac{\alpha_1 + \alpha_2}{2} t_m}) + (\frac{\alpha_1 - \alpha_2}{2}) t_m (\frac{\ln N_2 - \ln N_1}{\ln N_1 + \ln N_2})} \right|. \tag{17}$$

In (17) the last term in the denominator is a product of two perturbation terms. We neglect this higher order term to get

$$|\alpha_1 - \alpha_2| \leq A_{av} \left| \frac{\ln N_1 - \ln N_2}{\frac{\ln N_1 + \ln N_2}{2}} \right| + A_{av} t_m \left| \frac{\alpha_2 - \alpha_1}{1 - e^{-\frac{\alpha_1 + \alpha_2}{2} t_m}} \right|. \tag{18}$$

Let

$$\frac{A_{av} t_m}{1 - e^{-\frac{\alpha_1 + \alpha_2}{2} t_m}} < 1. \tag{19}$$

Then $0 < e^{-\frac{\alpha_1 + \alpha_2}{2} t_m} < 1 - A_{av} t_m$, which is true when $A_{av} t_m < 1$.

$A_{av}/year$	IMR/year	MRD/year	$t_m(years)$
Herring gull			
0.34			
N=(652)	0.18	2.82	11.3
N=10^3			11.5
N=10^9			15.7
Human			
0.015			
N=10^3	0.0002	7.967	105
N=10^5			110
N=10^7			114
N=(80,750,000)			115
N=10^9			117
N=10^{11}			120
Mouse			
0.74			
N=25	0.049	0.27	2.2
N=50			2.3
N=100			2.3
N=(738)			2.5
N=10^3			2.5
N=10^9			2.9
Rat			
0.64			
N=(250)	0.025	0.26	2.6
N=10^3			2.7
N=10^9			3.1
Japanese quail			
0.35			
N=(29)	0.091	1.163	5.8
N=10^3			6.9
N=10^9			8.8

Table 1: Reprinted from [6]

If $A_{av}t_m < 1$, further we have

$$-\left(\frac{\alpha_1 + \alpha_2}{2}\right) t_m < ln(1 - A_{av}t_m)$$

x	$1/1-e^{-x}$
0.01	100.501
0.02	50.5017
0.03	33.8358
0.04	25.5033
0.05	20.5042
0.15	7.17916
0.5	2.54149
1.0	1.58198
2.5	1.08943
4.0	1.01866
7.5	1.00055
10.5	1.00003
12.5	1
25	1
52.5	1
70.6	1
100	1

Table 2:

which gives

$$\frac{\alpha_1+\alpha_2}{2} > \frac{1}{t_m} ln\left(\frac{1}{1-A_{av}t_m}\right).$$

Note that the above estimation is independent of the population size N. Clearly, the term $\frac{\alpha_1+\alpha_2}{2}$ $-$ *mean Gompertz parameter*.

Theorem 1. *The age dependent Gompertz parameter α is stable for any population size N, provided $A_{av}t_m < 1$.*

Proof. When (19) holds, from (18) we get

$$|\alpha_1-\alpha_2| \leq C\ A_{av}\left|\frac{\ln N_1 - \ln N_2}{\frac{\ln N_1+\ln N_2}{2}}\right|, \tag{20}$$

where $C = \dfrac{1}{1-\left(\frac{A_{av}t_m}{1-e^{-\frac{\alpha_1+\alpha_2}{2}t_m}}\right)} > 0$. Hence it follows from (20) that α is stable for any N. □

Remark. There were no recorded samples for $A_{av}t_m < 1$ to compare with (19).

When $A_{av}t_m \geq 1$ (see Table 1), we get

$$\frac{A_{av}t_m}{(1 - e^{-\frac{\alpha_1+\alpha_2}{2}t_m})} \geq 1, \tag{21}$$

since the function $\frac{1}{1-e^{-x}} > 1$ for every positive x and asymptotically goes to 1 (see Table 2).

When (21) holds, it follows from (18) that:

Theorem 2. *The age dependent Gompertz parameter α to be stable with respect to the population size N, it is necessary that α be a constant, that is, $MRD = \frac{ln2}{\alpha}$ be a constant.*

Remark. Observe that the experimental data given in Table 1 confirms the theoretical approach of Theorem 2.

Corollary. *Since α is a constant, it is clear that α is stable also with respect to the maximum life span t_m.*

Conclusion. The age-dependent Gompertz parameter α is stable when t_{max} is small, otherwise it is necessary that α be a constant.

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