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GONALITY OF REDUCED CURVES ON A SMOOTH SURFACE WITH GENERICALLY SPANNED ANTICANONICAL LINE BUNDLE

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Abstract: Here we consider reduced curves on a smooth projective surface S whose anticanonical is generically spanned (e.g. Hirzebruch surfaces) and study their gonality using an extension method due to T. Gomez.

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Our main aim is to introduce the following set-up. Let X be a reduced projective curve, F a depth 1 sheaf on X with pure rank 1 (see [6], Chapters VII and VIII) and V a non-zero linear subspace of $H^0(X,F)$. Let $G_{(F,V)} \subseteq F$ denot the image of the evaluation map $e_{F,C} : V \otimes \mathcal{O}_X \to F$. Since $V \neq \{0\}$, $G_{(F,V)} \neq 0$. Since F has depth 1, its subsheaf $G_{(F,V)}$ has depth 1. Since F has pure rank 1, the support $X_{(F,V)}$ of $G_{(F,V)}$ is a non-empty subcurve of X. An irreducible component T of X is contained in $X_{(F,V)}$ if and only if for a general $P \in T$ there is $s \in V$ such that $s(P) \neq 0$. The curve $X_{(F,V)}$ is called the *support* of the pair (F,V). The sheaf $G_{(F,V)}$ is called the *shadow* of the pair (F,V). Set $X_F := X_{(F,H^0(X,F))}$ and $G_F := F_{(F,H^0(X,F))}$. The sheaf G_F is called the shadow of F. The curve $X_{(F,V)}$ may be disconnected even if X is connected. Notice that F is spanned if and only if the natural inclusion $G_F \to F$ is surjective and

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that this is the case (for degree reasons) if and only if the sheaves G_F and F are isomorphic as $\mathcal{O}_X$-sheaves. We say that F is *generically spanned* if $X_F = X$. From now on we assume that X is Goenstein. Hence local duality gives that every depth 1 sheaf on X is reflexive. We say that F is *primitive* if both F and $\omega_X \otimes F^*$ are spanned. We say that F is *generically primitive* if both F and $\omega_X \otimes F^*$ are generically spanned. For any generically primitive F set $F_- := G_G$, where $G := (\omega_X \otimes (G_F)^*)_{\omega_X \otimes (G_F)^*}$.

Lemma 1. *Let F be a generically primitive depth 1 sheaf with pure rank 1 on the Gorenstein curve X. We have $h^0(X, G_F) = h^0(X, F)$ and $h^1(X, G_F) = h^1(X, F) + \deg(F/G_F)$. The sheaf G_F is a subsheaf of F_-, both sheaves have pure rank 1, $h^1(X, F_-) = h^1(X, G_F)$ and $h^0(X, F_-) = h^0(X, G_F) + \deg(F_-/G_F)$. Both F_- and $\omega_X \otimes F_-$ are spanned, i.e. F_- is primitive.*

Proof. The definition of G_F gives that the natural inclusion $G_F \hookrightarrow F$ induces an isomorphism of global sections. Since F is generically spanned, $\mathrm{Coker}(j)$ has finite support and G_F has pure rank 1. Since $h^0(X, G_F) = h^0(X, F)$, Riemann-Roch gives $h^1(X, G_F) = h^1(X, F) + \deg(F/G_F)$. Duality gives that $\omega_X \otimes F_-$ is the subsheaf of $\omega_X \otimes (G_F)^*$ spanned by its global sections. Hence duality gives $h^1(X, F_-) = h^1(X, G_F)$. Riemann-Roch gives $h^0(X, F_-) = h^0(X, G_F) + \deg(F_-/G_F)$. The last assertion is proved in [2], Lemma 3.3.1 (notice that in the proof of lemma he don't use the irreducibility of the curve C and that the curve has planar singularities, but only that it is Gorenstein). □

For any reduced projective curve Y let $\mathcal{B}(Y)$ denote the set of the irreducible components of Y.

The proof of [2] gives the following result.

Theorem 1. *Let S be a smooth and connected projective surface such that ω_S^* is spanned outside a proper closed subset $\mathcal{B}$ of S. Let $X \subset S$ be a reduced curve not contained in $\mathcal{B}$ and F a quasi-primitive torsion free sheaf on X with pure rank 1. Then there are an integral affine curve $\Delta \subseteq |X|$, $o \in \Delta|$, and a flat family $\{F_t\}_{t \in \Delta}$ of sheaves on the family $\{X_t\}_{t \in \Delta}$ associated to Δ with the following properties:*

(a) $X_o = X$ and $F_o \cong F$;

(b) X_t is smooth outside $\mathcal{B}$ for every $t \in \Delta \backslash \{o\}$;

(c) If X is smooth at each point of $X \backslash \mathcal{B}$, then X_t is smooth for every $t \in \Delta \backslash \{o\}$;

(d) each F_t, $t \in \Delta$, is a depth 1 sheaf on X_t with rank 1 and $h^0(X_t, F_t) \geq h^0(X, F)$.

We were at first interested in the case $\mathcal{B} = \emptyset$, but there are at least two interesting surfaces S for which ω_S^* is generically spanned, but not spanned. The first example is given by del Pezzo surfaces of degree 8 (here $\mathcal{B}$ is finite) and degree 9 (here $\mathcal{B}$ is the only curve in the anticanonical linear system of S). The second example is given by most Hirzebruch surface. Let F_e, $e \geq 0$, be a Hirzebruch surface. We recall that $\omega_{F_e}^*$ is very ample (resp. spanned) if and only if $0 \leq e \leq 1$ (resp. $0 \leq e \leq 2$). If $e \geq 3$, then the base locus $\mathcal{B}$ of $|\omega_{F_e}^*|$ is the unique curve of F_e with negative self-intersection.

Applying Theorem 1 and the smooth case of a theorem of G. Martens (see [5], theorem in the introduction), we get the following result; to state it we recall that as in [5] we take as a basis of $\mathrm{Pic}(F_e)$ a fiber F of a ruling of F_e and an irreducible curve C_0 with minimal self-intersection; thus $C_0 + eF$ is the divisor H of [3], pp. 369–385.

Theorem 2. *Let C be a reduced closed subcurve of the Hirzebruch surface F_e, C not a fiber of a ruling of F_e. Set $|kC_0 + bF| = |C|$. If $e = 0$ assume $b \geq k$. If $e = 1$ assume $b > k$. If $e \geq 3$ assume that C is smooth at each point of $C \cap C_0$. Then there is no quasi-primitive depth 1 sheaf on C with pure rank 1 and degree at most $k-1$ and the unique such sheaves with degree k are locally free and induced by a ruling of F_e.*

Obviously, the degree k spanned line bundle is not unique if and only if $e = 0$ and $b = k$.

Another case to which Theorem 1 (with $\mathcal{B} = \emptyset$) may be applied is to reducible plane curves and their gonality with respect to quasi-primitive sheaves with pure rank 1 (see [4], Theorem 2.1, and [1], Theorem 3.2.1, for much more in the case of integral plane curves).

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