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LOW DEGREE SPANNED SHEAVES WITH PURE RANK 1 ON REDUCIBLE CURVES

E. Ballico

Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a reduced projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Here we describe all spanned torsion free sheaves on X with pure rank 1 and degree 1 or 2. The list depends on combinatorial properties of X (e.g. the existence of a disconnecting node of X).

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1. Introduction

It seems worthwhile to study stable curves with spanned line bundles or spanned rank 1 torsion free sheaves with very low degree, say degree 1 or degree 2. We do not say that these curves have gonality 1 or 2, because the reader will learn that their structure and the number of their moduli depend essentially only on the combinatorial data of the dual graph of the stable curve. For instance, the stable curve X has a degree 1 spanned sheaf with pure rank 1 and depth 1 which is not locally free if and only if its has a disconnecting node (see Theorem 2). Let X be a reduced and connected projective curve defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We assume that every singular point of X lying on at least two irreducible components of X is an ordinary node of X. Set $g := p_a(X)$. Let $\mathcal{B}(X)$ denote the set of the irreducible components of X.

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For every union A of some of the irreducible components of X let $A^{[c]}$ (or $A_X^{[c]}$ if we need to specify the ambient curve X) denote the closure of $X\backslash A$ in X. Thus $(A^{[c]})^{[c]} = A$. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [6], parts VII and VIII. We say that a depth 1 sheaf F on X has pure pure rank 1 if its restriction to X_{reg} is a pure rank 1 vector bundle. Let F be sheaf on X with pure rank 1 and with depth 1. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. The degree $\deg(F)$ of F may be defined by the Rieman-Roch formula $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$, even if the curve is not connected. For any integer d let $D_d(X)$ (resp. $S_d(X)$) denote the set of all spanned line bundles (resp. sheaves with pure rank 1 and depth 1) on X with degree d. Obviously, $D_d(X) = S_d(X) = \emptyset$ for all $d < 0$ and $D_0(X) = S_0(X) = \{\mathcal{O}_X\}$. Here we study low degree spanned line bundles and low degree sheaves with depth 1 and pure rank 1 on stable or semistable curves. D. Schubert introduced a compactification of $\mathcal{M}_g$ which allows cuspidal curves. To cover this case we consider the following more general set-up. Let X be a connected projective curve such that each point of X lying on at least two irreducible components is an ordinary node of X. Let $\mathrm{Sing}(X)''$ be the set of all points of X lying on at least two irreducible components of X. By assumption every point of $\mathrm{Sing}(X)''$ is an ordinary node of X. Set $g := p_a(X)$. We assume $g \geq 2$. Let $\mathfrak{S}(X)$ be the set of all $T \in \mathcal{B}(X)$ such that $T \cong \mathbb{P}^1$ and $X\backslash T$ has $\sharp(T \cap \mathrm{Sing}(X))$ connected components. Let $\mathbb{D}(X)$ denote the set of all disconnecting nodes of X, i.e. the set of all $P \in X$ such that $X\backslash\{P\}$ is not connected. Fix any $P \in \mathbb{D}(X)$. Since we assumed that each point of X lying on at least two irreducible components of X is an ordinary node of X, $X\backslash\{P\}$ has two irreducible components and their closure in X have the reduced scheme $\{P\}$ as their scheme-theoretic intersection. Let $f_P : X_P \to X$ be the partial normalization of X in which we normalize only the point P. Since P is a disconnecting node of X, X_P has 2 connected components, i.e. $h^0(X_P, \mathcal{O}_{X_P}) = 2$.

Theorem 1. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Assume $g := p_a(X) \geq 2$. $D_1(X) \neq \emptyset$ if and only if $\mathfrak{S}(X) \neq \emptyset$. There is a bijection between $\mathfrak{S}(X)$ and $D_1(X)$ constructed in the following way. Take $T \in \mathfrak{S}(X)$ and an automorphism $\phi : T \to \mathbb{P}^1$. There is a unique spanned degree 1 line bundle L_T on X such that $h^0(X, L_T) = 2$ and the morphism $h_{L_T} : X \to \mathbb{P}^1$ induces ϕ on T, while it sends the connected components $Y_1, \dots, Y_s$, $s := \sharp(T \cap \mathrm{Sing}(X))$, of $T^{[c]}$ respectively to the points $\phi(T \cap Y_1), \dots, \phi(T \cap Y_s)$. If $D_1(X) \neq \emptyset$, then for every integer $d \geq 2$ there is $L \in D_d(X)$ such that $h^0(X, L) = d + 1$.*

Theorem 2. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Assume $g := p_a(X) \geq 2$. There is a bijection between the set $\mathbb{D}(X)$ of all disconnecting nodes and $S_1(X)\backslash D_1(X)$ constructed in the following way. For each $P \in \mathbb{D}(X)$ let $f_P : X_P \to X$ be the partial normalization of X in which we only normalize the point P. Then $f_{P*}(\mathcal{O}_{X_P}) \in S_1(X)\backslash D_1(X)$. Any $F \in S_1(X)\backslash D_1(X)$ has a unique singular point, this singular point is a disconnecting node, and $h^0(X, F) = 2$.*

For degrees $d > 1$ the situation becomes more complicated. The statements of Theorems 3 and 4 are simultaneously a description of the various cases and a recipe (starting from certain components) to construct X and $L \in D_2(X)$ or $F \in S_2(X)\backslash D_2(X)$ with the required properties. Let $\mathfrak{A}[X]_2$ be the set of all $T \in \mathcal{B}(X)$ with the following properties. If $p_a(T) \geq 2$ assume the existence of a degree 2 morphism $u : T \to \mathbb{P}^1$. Let $Y_1, \dots, Y_s$, be the connected components of $T^{[c]}$. Assume $\sharp(Y_i \cap T) \leq 2$ for all i, say $\sharp(Y_i \cap T) = 2$ for $1 \leq i \leq t$, and $\sharp(Y_i \cap T) = 1$ for $t + 1 \leq i \leq s$; here $t \in \{0, \dots, s\}$. If $t \geq 1$, then assume the existence of a degree 2 morphism $v : T \to \mathbb{P}^1$ (the given one if $p_a(T) \geq 2$) such that $\sharp(v(Y_i \cap T)) = 1$ for every integer $i \in \{1, \dots, t\}$. The integers s, t may depend on T. If X is irreducible, then we allow the same notation with $s = 0$, i.e. if X is irreducible, then either $\mathfrak{A}[X]_2 = \{X\}$ (case X hyperelliptic) or $\mathfrak{A}[X]_2 = \emptyset$ (case X not hyperelliptic). Let $\mathfrak{A}[X]_{1,1}$ be the set of all $S \subset \mathcal{B}(X)$ with $\sharp(S) = 2$, say $S = \{T_1, T_2\}$, with the following properties; $T_1 \neq T_2$; $T_1 \cap T_2 \neq \emptyset$; $T_1 \cong T_2 \cong \mathbb{P}^1$; set $C := T_1 \cup T_2$ and call $Y_1, \dots, Y_s$ the connected components of $C^{[c]}$; if $p_a(C) \geq 2$ (i.e. if $\sharp(T_1 \cap T_2) \geq 3$) assume the existence of a degree 2 morphism $u : T \to \mathbb{P}^1$; assume $\sharp(Y_i \cap C) \leq 2$ for all i, say $\sharp(Y_i \cap C) = 2$ for $1 \leq i \leq t$, and $\sharp(Y_i \cap TC) = 1$ for $t + 1 \leq i \leq s$; here $t \in \{0, \dots, s\}$; if $t \geq 1$, then assume the existence of a degree 2 morphism $v : T \to \mathbb{P}^1$ (the given one if $p_a(C) \geq 2$) such that $\sharp(v(Y_i \cap C)) = 1$ for every integer $i \in \{1, \dots, t\}$. If $X = C_1 \cup C_2$, then we allow the same notation with $s = 0$. Let $\mathfrak{A}[X]_{1,0,1}$ be the set of all $S \subset \mathcal{B}(X)$ with $\sharp(S) = 2$, say $S = \{T_1, T_2\}$, and the following properties; $T_1 \cap T_2 = \emptyset$; $T_1 \cong T_2 \cong \mathbb{P}^1$; let $A_1, \dots, A_c$ be the connected components of $(T_1 \cup T_2)^{[c]}$, with, say A_i intersecting both T_1 and T_2 if and only if $1 \leq b \leq c$; if $b \geq 4$ we also assume the existence of isomorphisms $h_1 : T_1 \to \mathbb{P}^1$ and $h_2 : T_2 \to \mathbb{P}^1$ such that the ordered b-ple of distinct points $(h_1(T_1 \cap A_1), \dots, h_1(T_1 \cap A_b))$ of $\mathbb{P}^1$ is projectively equivalent to the ordered b-ple of distinct points $(h_2(T_2 \cap A_1), \dots, h_2(T_2 \cap A_b))$.

Theorem 3. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of*

X. Assume $g := p_a(X) \geq 2$. Set $D_2(X)_1 := \{L \in D_2(X)$: there is a unique $T_L \in \mathcal{B}(X)$ such that $\deg(L|T_L) > 0\}$. Set $D_2(X)_{1,0,1} := \{L \in D_2(X)$: there are two disjoint $T_{1,L}, T_{2,L} \in \mathcal{B}(X)$ such that $\deg(L|T_{i,L}) > 0$ and $i = 1, 2\}$. Set $D_2(X)_{1,1} := \{L \in D_2(X)$: there are two irreducible components $T_{1,L}, T_{2,L} \in \mathcal{B}(X)$ such that $\deg(L|T_{i,L}) > 0$, $i = 1, 2$, and $T_{1,L} \cap T_{2,L} \neq \emptyset\}$.

(a) There is a surjection $\psi : D_2(X)_1 \to \mathfrak{A}[X]_2$ with the following properties. Fix $T \in \mathfrak{A}[X]_2$. If $p_a(T) \geq 2$, then $\sharp(\psi^{-1}(T)) = 1$; in this case $h^0(X, L) = 2$. If $p_a(X) = 1$, then there is a bijection between the fiber $\psi^{-1}(T)$ and the set of all degree 2 morphisms $v : T \to \mathbb{P}^1$ allowable by the definition of $\mathfrak{A}[X]_2$ for the fixed component T; hence $h^0(X, L) = 2$ and either $\sharp(\psi^{-1}(T)) = 1$ (case $t > 0$) or $\psi^{-1}(T)$ is isomorphic to $Pic^2(T) \cong T_{reg}$. If $p_a(X) = 0$, then $\sharp(\psi^{-1}(T)) = 1$, $2 \leq h^0(X, L) \leq 3$, and $h^0(X, L) = 3$ if and only if $t = 0$.

(b) There is a surjection $\psi' : D_2(X)_{1,1} \to \mathfrak{A}[X]_{1,1}$ with the following properties. Set $C := T_{1,L} \cup T_{2,L}$. The fibers of ψ' are as the fibers of ψ taking C instead of T, i.e. analysing separately the cases $p_a(C) \geq 2$ (i.e. $\sharp(T_{1,L} \cap T_{2,L}) \geq 3$), $p_a(C) = 1$ (i.e. $\sharp(T_{1,L} \cap T_{2,L}) = 2$), and $p_a(C) = 0$ (i.e. $\sharp(T_{1,L} \cap T_{2,L}) = 1$).

(c) All pairs (X, L) are described in the following way. Fix integers $c \geq d \geq b \geq 1$, two abstract smooth rational curves $T_{1,L}, T_{2,L}$, isomorphisms $h_1 : T_{1,L} \to \mathbb{P}^1$, $h_2 : T_{2,L} \to \mathbb{P}^1$, and an ordered b-ple $P_1, \dots, P_b$ of b distinct points of $\mathbb{P}^1$. Set $Q_{j,i} := h_j^{-1}(P_i)$, $j = 1, 2$, $1 \leq i \leq b$. Let A_i, $1 \leq i \leq c$, be non-empty reduced and connected curves such that each point of some A_i lying on at least two irreducible components of A_i is an ordinary node of A_i. X is obtained from the disjoint uion $W := T_{1,L} \sqcup T_{2,L} \sqcup \bigsqcup_{i=1}^{c} A_i$ gluing each A_i, $1 \leq i \leq b$, at the point $Q_{j,i}$, $j = 1, 2$, $1 \leq i \leq b$, A_i intersecting $T_{1,L}$, but not $T_{2,L}$, if $b + 1 \leq i \leq d$, and A_i intersecting $T_{2,L}$, but not $T_{1,L}$ if $d + 1 \leq i \leq c$. We also impose that the gluing is such that every singular point of X lying on two irreducible components of X is an ordinary node. $h^0(X, L) = 3$ if and only if $b = 1$.

Part (a) of Theorem 3 implies that $D_2(X)_1 = \emptyset$ if and only if $\mathfrak{A}[X]_2 = \emptyset$. To state the next result we need the following example due to D. Eisenbud, J. Koh, M. Stillman and J. Harris (see [1], p. 353, [3], Lemma 3.3 and Theorem 3.4).

Example 1. Fix an integer $q \geq 2$. Fix $P \in \mathbb{P}^{q+1}$, a hyperplane H of $\mathbb{P}^{q+1}$ and a rational normal curve $D \subset H$. Let S be the cone with vertex P and base D. There is are many integral curves $C \subset S$ such that $p_a(C) = q$, $\deg(C) = 2q + 1$, and P is a point with multiplicity $q + 1$ of C (see [3], Lemma 3.3 and Theorem 3.4). Each of them has a degree 2 spanned torsion free sheaf such that $\mathrm{Sing}(F_C) = \{P\}$ and $h^0(C, F_C) = 2$. C is not Gorenstein.

Theorem 4. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Assume $g := p_a(X) \geq 2$.*

(a) There is a bijection between the set of all $F \in S_2(X)$ such that $Sing(F) \subseteq Sing(X)''$ and $\sharp(Sing(F)) = 2$ and the pairs $\{P_1, P_2\}$ of points of $Sing(X)''$ such that $X\backslash\{P_1, P_2\}$ is not connected and either both or none of the points of X are disconnecting nodes of X. If the points $\{P_1, P_2\}$ are (resp. are not) disconnecting nodes, then $h^0(X, F) = 3$ (resp. $h^0(X, F) = 2$).

(b) There is no $F \in S_2(X)$ such that $\sharp(Sing(F) \cap Sing(X)'') \geq 3$.

(c) There is no $F \in S_2(X)$ such that $Sing(F) \cap Sing(X)'' \neq \emptyset$ and $Sing(F) \cap (Sing(X)\backslash Sing(X)'') \neq \emptyset$.

(d) Let $\eta_2(X)$ be the set of all $F \in S_2(X)$ such that $\sharp(Sing(F)) = 1$ and $Sing(F) \subseteq Sing(X)''$. Every $F \in \eta_2(X)$ is constructed uniquely in the following way. There is $C \in \mathcal{B}(X)$ such that $C \cong \mathbb{P}^1$. Let $W_1, \dots, W_t$, $t \geq 1$, be the connected components of $C^{[c]}$. Let $f : Y \to X$ be the partial normalization of X in which we normalize only P. Set $M := f^*(F)/Tors(f^*(F))$ and let $D \subset Y$ be the irreducible component of Y such that f maps isomorphically D onto C. Let $Y_1, \dots, Y_s$ be the connected components of $D_Y^{[c]}$. Two cases may arise:

(i) P is a disconnecting node of X; in this case Y has two connected components, say A, B, and, up to a renaming of the components $Y_1, \dots, Y_s$, $s = t+1$, $B = Y_s$, $A = \cup_{i=1}^{s-1} A_i$, $f^{-1}(P) \subset Y_{s-1} \cup Y_s$, f maps isomorphically Y_i onto W_i for $1 \leq i \leq s-1$, while $f(Y_s \cup Y_{s-1}) = W_t$; in this case $h^0(X, F) = 3$;

(ii) P is not a disconnecting node of X; now $s = t$, and up to a renaming of the components W_i, Y_j, $f^{-1}(P) \subset D \cup Y_s$ and $P \in C \cap W_s$; f maps Y_i isomorphically onto W_i if $i \leq s-1$; $f(Y_s) = W_s$; $\sharp(Y_i \cap D) = 1$ for all $1 \leq i \leq s$; $\sharp(W_i \cap C) = 1$ if $1 \leq i \leq s-1$, while $\sharp(W_s \cap C) = 2$; in this case $h^0(X, F) = 2$.

In both cases M is the only spanned line bundle on Y such that $\deg(M|D) = 1$ and M is trivial in a neighborhood of $D^{[c]}$. In both cases $F \cong f_*(M)$.

(e) Let $\Gamma_2(X)$ be the set of all $F \in S_2(X)\backslash D_2(X)$ such that $Sing(F) \cap Sing(X)'' = \emptyset$. $\Gamma_2(X) \neq \emptyset$ if and only if there are an integer q such that $2 \leq q \leq g$ and $C \in \mathcal{B}(X)$ described as above. For any such C the unique $F \in \Gamma_2(F)$ such that $F|C \neq \mathcal{O}_C$ is the only sheaf F on X which is trivial in a neighborhood of $C^{[c]}$, while $F|C$ is isomorphic to the degree 2 spanned sheaf F_C described in Example 1. We have $h^0(X, F) = h^0(C, F_C) = 2$ and $\sharp(Sing(F)) = \sharp(Sing(F_C)) = 1$.

Notice that $\Gamma_2(X) = \emptyset$ if X is Gorenstein. Hence $\Gamma_2(X) = \emptyset$ if X is semistable or if X is in Schubert's compactification of $\mathcal{M}_g$. If $g \geq 4$ there are curves X with different components C_i, $1 \leq i \leq s$, as in Example 1 with respect to integers $q_i \geq 2$, $1 \leq i \leq s$, and hence for large g there are curves X with large $\sharp(\Gamma_2(X))$. However, $\Gamma_2(X)$ is always finite.

As a by-product of the proof of Theorem 3 we give the following (certainly well-known) result.

Proposition 1. *Let X be a stable curve of genus 2. Then there is a spanned $L \in Pic(X)$ such that* $\deg(L) = 2$.

In the proof of Proposition 1 we will also give a description of all spanned $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 2$.

2. The Proofs

Remark 1. Let X be a reduced and quasi-projective curve, $P \in X$, and F a sheaf on X with pure rank 1 and depth 1. The germ F_P of F at P is a torsion free $\mathcal{O}_{X,P}$-module with rank 1. Hence there exists an inclusion $j : F_P \hookrightarrow M$ with M a free $\mathcal{O}_{X,P}$-module with rank 1. The minimal integer $\dim_{\mathbb{K}}(M/F_P)$ for all such pairs (j, M) is an important invariant of the germ F_P. Call $\ell(F, P)$ this integer. We have $\ell(F, P) \geq 0$ and $\ell(F, P) = 0$ if and only if F_P is a free $\mathcal{O}_{X,P}$-module. This invariant may be computed on the formal completion of $\mathcal{O}_{X,P}$. Let $m_{X,P}$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Notice that $m_{X,P}$ is a free $\mathcal{O}_{X,P}$-module if and only if $P \in X_{reg}$. Hence if $P \in \mathrm{Sing}(X)$ and $F_P \cong m_{X,P}$, then $\ell(F, P) = 1$. Now assume that X is projective. Fix a finite set $S \subseteq \mathrm{Sing}(X)$ and let $f : C \to X$ be the partial normalization of X in which we normalize only the points of S. The torsion of $f^*(F)$ is supported on the finite set $f^{-1}(S)$. Set $G := f^*(F)/\mathrm{Tors}(f^*(F))$. G is a coherent sheaf on C with depth 1 and pure rank 1. Since X and C are projective, the integers $\deg(F)$ and $\deg(G)$ are well-defined and satisfy the Riemmann-Roch formulas $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$, $\chi(G) = \deg(G) + \chi(\mathcal{O}_C)$ even if X or C are not connected. We have

$$\deg(G) = \deg(F) - \sum_{P \in S} \ell(F, P). \tag{1}$$

We need this formula only when each point of S is an ordinary node of X. In this case we may decompose f into $\sharp(S)$ partial normalizations of a single node. Hence for nodes it is sufficient to prove it when $\sharp(S) = 1$, say $S = \{P\}$. In this case (1) is obviously true if F_P is free. If $F = \mathcal{I}_P$, then (1) holds, because

$\ell(\mathcal{I}_P, P) = 1$ and G is the ideal sheaf of the two points $f^{-1}(P)$. For an arbitrary F_P use the next result.

Remark 2. Take the set-up of the first part of Remark 1. Assume that P is either an ordinary node or an ordinary cusp of X. Assume $P \in \mathrm{Sing}(F)$. By the classification of torsion free modules on simple curves singularities (see [2], or, for nodes, [6], pp. 163–166) the germ of F at each P is formally equivalent to the maximal ideal $m_{X,P}$ of the local ring $\mathcal{O}_{X,P}$. Hence Remark 1 gives $\ell(F, P) = 1$.

Remarks 1 and 2 immediately give the following result.

Corollary 1. *Let X be a reduced projective curve and F a coherent sheaf on X with depth 1 and pure rank 1. Fix $S \subseteq Sing(F)$. Assume that each point of S is an ordinary node or an ordinary cusp of X. Let $h : D \to X$ (resp. $f : C \to X$) be the partial normalization of X in which we normalize only the points of S (resp. the points of S and the singular points of X at which F is locally free). Set $L := f^*(F)/Tors(f^*(F))$ and $R := h^*(F)/Tors(h^*(F))$. Then $\deg(L) = deg(R) = \deg(F) - \sharp(S)$.*

Lemma 1. *Let X, Y reduced, projective curves and $f : Y \to X$ a surjective morphisms which is . Let A be a coherent sheaf on Y. Then $\deg(u_*(A)) = \deg(A) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y)$.*

Proof. Obviously, $h^0(X, f_*(A)) = h^0(Y, A)$. Since f is finite, $R^1f_*(A) = 0$. Hence the Leray spectral sequence of f gives $h^1(X, f_*(A)) = h^1(Y, A)$. Thus $\chi(A) = \chi(f_*(A))$. Since $\chi(A) = \deg(A) + \chi(\mathcal{O}_Y)$ and $\chi(f_*(A)) = \deg(f_*(A)) + \chi(\mathcal{O}_X)$, we are done. □

Remark 3. For any X (even not connected) and any $L \in \mathrm{Pic}(X)$ we have

$$\sum_{T \in \mathcal{B}(X)} \deg(L|T) = \deg(L). \tag{2}$$

Notice that (2) is true for non-locally free L if we only assume that L is locally free at each point of X lying on at least two irreducible components.

Lemma 2. *Let F be a coherent sheaf on X with pure rank 1 and depth 1. Assume that each point of $\mathrm{Sing}(F)$ lying on at least two irreducible components of X is an ordinary node of X. Then*

$$\sum_{T \in \mathcal{B}(X)} \deg((F|T)/Tors(F|T)) = \deg(F) - \sharp(Sing(F)). \tag{3}$$

Proof. Let $u : U \to X$ be the partial normalzation of X in which we normalize only the points of $\mathrm{Sing}(X)$. Remarks 1 and 2 and the last sentence of

Remark 3 show that the left hand side of (3) is equal to $\deg(u^*(F))/\mathrm{Tors}(u^*(F))$. Apply b times Lemma 1. □

Proof of Theorem 1. Fix any $L \in D_1(X)$. Since $\deg(L|C) \geq 0$ for all $C \in \mathcal{B}(X)$, there is $T_L \in \mathcal{B}(X)$ such that $\deg(L|T_L) = 1$, while the morphism $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) - 1$, contracts to points all other components. Let $Y_1, \dots, Y_s$ be the closures in X of the connected components of $X \backslash T_L$. Since $L|T_L$ is spanned, $T_L \cong \mathbb{P}^1$, and $h_L|T_L$ is bijective. This implies that $h_L(Y_i) = h_L(Y_i \cap T)$ for all T. The second part of the statement of Theorem 1 shows how to construct from any $T \in \mathfrak{S}(X)$ a morphism $h_{L_T} : X \to \mathbb{P}^1$ such that the spanned line bundle $h_{L_T}^*(\mathcal{O}_{\mathbb{P}^1}(1))$ has degree 1. Obviously, the last (resp. first) constructios is the inverse of the first (resp. last) one. To check the last assertion take L constructed in a similar way taking the degree d Veronese embedding $T \hookrightarrow \mathbb{P}^d$ of $T \cong \mathbb{P}^1$, instead of the isomorphism ϕ. □

Lemma 3. *Let T be an integral projective curve. There is no spanned rank 1 torsion free sheaf F on T such that $\mathrm{Sing}(F) \neq \emptyset$ and $\deg(F) = 1$.*

Proof. Assume the existence of such a sheaf F. Since $\mathrm{Sing}(F) \neq \emptyset$, T is singular. Hence $p_a(T) \geq 1 = \deg(F)$. Since $F \neq \mathcal{O}_X$ and F is spanned, $h^0(X, F) \geq 2$. The contradiction comes from Clifford's Inequality (see [1], Theorem A at p. 532). □

Proof of Theorem 2. Assume the existence of $F \in S_1(X) \backslash D_1(X)$ and set $c := \sharp(\mathrm{Sing}(F))$ and $b := \sharp(\mathrm{Sing}(F) \cap \mathrm{Sing}(X)'')$. By assumption $c > 0$. If $b = 0$, then Lemma 3 and the last sentence of Remark 3 gives a contradiction. Hence we may assume $b > 0$. Let $h : D \to X$ be the partial normalization of X in which we normalize only the points of $\mathrm{Sing}(F) \cap \mathrm{Sing}(X)''$. Set $R := h^*(F)/\mathrm{Tors}(h^*(F))$. Lemma 1 gives $\deg(R) = 1 - b \leq 0$. R is a spanned sheaf with pure rank 1 and depth 1. Hence $b = 1$ and $R \cong \mathcal{O}_D$. Hence $c = b$ and F has a unique singular point. Since F has no torsion, the natural map $h^* : H^0(X, F) \to H^0(D, R)$ is injective. Hence $h^0(X, R) \geq 2$. Thus the only singular point, P, of $\mathrm{Sing}(F)$ is a disconnecting node of X. Hence $D = X_P$ and $h = f_P$. Since X_P has two connected components, we get $h^0(X, F) = 2$. Hence to prove all the assertions of Theorem 2 it is sufficient to check that $A := f_{P*}(\mathcal{O}_{X_P}) \in S_1(X) \backslash D_1(X)$. Lemma 1 gives $\deg(A) = 1$. Obviously, A is not locally free at P. We have $h^0(X, A) = h^0(X_P, \mathcal{O}_{X_P}) = 2$. Let A' be the subsheaf of A spanned by $H^0(X, A)$. If $A' = A$, then we are done. Assume $A' \neq A$. Hence $\deg(A') \leq \deg(A) - 1 \leq 0$. Since X is connected, we get $h^0(X, A') \leq 1$. Since $h^0(X, A') = h^0(X, A) = 2$, we get a contradiction. □

Proof of Theorem 3. If X is irreducible, then Theorem 3 just says that

X is hyperelliptic. Hence we may assume $\sharp(B(X)) \ge 2$. Fix $L \in D_2(X)$. Let $\mu : X \to \mathbb{P}^r$, $r := h^0(X, L) = 1$, be the morphism associated to L. Let $S := \{T \in \mathcal{B}(X)) : \deg(L|T) > 0\}$. Since L is spanned, $\deg(L|T) \ge 0$ for all T. Moreover, $L|T \cong \mathcal{O}_X$ and $\mu|T$ is constant if $\deg(L) = 0$. Hence (2) shows that either $\sharp(S) = 1$, say $S = \{T_L\}$, or $\sharp(S) = 2$, say $S = \{T_{1,L}, T_{2,L}\}$, and $\deg(L|T_{i,L}) = 1$ for all i. Since L is spanned, in the latter case we have $T_{i,L} \cong \mathbb{P}^1$ for all i. The descriptions given in parts (a), (b) and (c) below will give immediately from the data the existence of $L \in D_2(X)$.

(a) Here we assume $S = \{T_L\}$. Since $\mu|T_L$ is a degree 2 morphism, $T_L \in \mathfrak{A}[X]_2$. Let $Y_1, \dots, Y_s$ be the connected components of $T_L^{[c]}$ with $\sharp(Y_i \cap T) = 2$ if and only if $1 \le i \le t$. Since $\sharp(B(X)) \ge 2$, $s > 0$. Now assume $p_a(T_L) = 1$. Use that $\mathrm{Pic}^2(T_L) \cong \mathrm{Pic}^0(T_L) \cong (T_L)_{reg}$ and that each degree 2 line bundle on T_L is spanned and non-special. For the case $p_a(T_L) = 0$ of part (a) of Theorem 3 use that the unique degree 2 line bundle on $\mathbb{P}^1$ has $h^0 = 3$ and it is very ample.

(b) Here we assume $S = \{T_{1,L}, T_{2,L}\}$, $T_{1,L} \ne T_{2,L}$ and $T_{1,L} \cap T_{2,L} \ne \emptyset$. Set $C := T_{1,L} \cup T_{2,L}$ and take C instead of T_L in the proof of part (a).

(c) Here we assume $S = \{T_{1,L}, T_{2,L}\}$ and $T_{1,L} \cap T_{2,L} = \emptyset$. Let $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) - 1$, be the morphism associated to L. Let $A_1, \dots, A_c$ be the connected components of $(T_{1,L} \cup T_{2,L})^{[c]}$, with, say A_i intersecting both $T_{1,L}$ and $T_{2,L}$ if and only if $1 \le i \le b$. Since X is connected, $b \ge 1$. Since every point on two distinct irreducible components must be ordinary nodes, all points $T_j \cap A_i$, $j = 1, 2$, $1 \le i \le c$, are distinct. Notice that $h_L(T_j \cap A_i)$, $1 \le i \le c$, are c distinct points and that $h_L(T_1 \cap A_i) = h_L(T_2 \cap A_i$ if $1 \le i \le b$. Hence we see that $h^0(X, L) = 3$ if and only if $b = 1$. Here is the recipe to get all (X, L) in this case. We use h_L to identify $T_{1,L}$ and $T_{2,L}$ with $\mathbb{P}^1$. Fix an integer $b \ge 1$ and choose an ordered set of b distinct points on $\mathbb{P}^1$. Take their counterimages in $T_{j,L}$, $j = 1, 2$. Take integers $c \ge d \ge b$ and c connected disjoint curves $A_1, \dots, A_c$ such that each point of some A_i lying on at least two irreducible components of A_i is an ordinary nodes of A_i. Take X gluing together $T_{1,L}, T_{2,L}, A_1, \dots, A_c$ so that if $1 \le i \le b$, then A_i intersects both $T_{1,L}$ and $T_{2,L}$, if $b+1 \le i \le d$, then A_i intersects only $T_{1,L}$ and if $d+1 \le i \le c$, then A_i intersects only $T_{2,L}$. We also assume that each point $T_{j,L} \cap A_i$ is an ordinary node of X. □

Proof of Theorem 4. First assume $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)''$ and $\sharp(\mathrm{Sing}(F)) = 2$, say $\mathrm{Sing}(F) = \{P_1, P_2\}$. Let $h : D \to X$ be the partial normalization of X in which we normalize only the points of $\mathrm{Sing}(F)$. Since each point of $\mathrm{Sing}(X)''$ is an ordinary node of X, D has $1 + s$ connected components, $0 \le s \le 2$. Set $R := h^*(F)/\mathrm{Tors}(h^*(F))$. Lemma 1 gives $\deg(R) = 2 - 2 = 0$. Since R is a spanned line bundle, we get $R \cong \mathcal{O}_D$. Since $\deg(h_*(\mathcal{O}_D)) = \deg(\mathcal{O}_D) + \chi(\mathcal{O}_D) -$

$\chi(\mathcal{O}_X) = 2$, and F is a subsheaf of $h_*(\mathcal{O}_D)$, we get $F = h_*(\mathcal{O}_D)$. Since F is non-trival and spanned, we get $s > 0$. Assume that $h_*(\mathcal{O}_D)$ is not spanned. Let G be the subsheaf of $h_*(\mathcal{O}_D)$ spanned by $H^0(X, h_*(\mathcal{O}_D))$. Since $G \neq h_*(\mathcal{O}_D)$, $\deg(G) \leq 1$. Since $h^0(X, h_*(\mathcal{O}_D)) \geq 2$, $\deg(G) > 0$. Hence Theorems 1 and 2 gives that exactly one of the points P_1, P_2 is a disconnecting node of X and G arises from this disconnecting node by the recipe describe in Theorem 2. We also see that $s = 2$ if and only if both P_1, P_2 are disconnecting nodes of X, concluding the proof of part (a). In the set-up of part (b) we get $\deg(R) = 2 - \sharp(\mathrm{Sing}(F) \cap \mathrm{Sing}(X)'') < 0$, contradicting the spannedness of R. Now assume that F is a counterexample to part (c). Let $m : A \to X$ be the partial normalization in which we normalize only the points of $\mathrm{Sing}(F) \cap \mathrm{Sing}(X)''$. Set $M := m^*(F)/\mathrm{Tors}(m^*(F))$. Lemma 1 gives $\deg(M) = 2 - \sharp(\mathrm{Sing}(F) \cap \mathrm{Sing}(X)'') \leq 1$. Since M is spanned, but not locally free, Lemma 3 gives a contradiction.

(i) Here we will prove part (d). Fix $F \in \eta_2(X)$ and set $\{P\} := \mathrm{Sing}(F)$. Let $f : Y \to X$ be the partial normalization of X in which we normalize only the point P. Set $M := f^*(F)/\mathrm{Tors}(f^*(F))$. Notice that M is a spanned line bundle. Corollary 1 gives $\deg(M) = 1$. Hence the spannedness of M and (1) give the existence of a unique $D \in \mathcal{B}(Y)$ such that $\deg(M|D) > 0$, while M is trivial in a neighborhood of $D^{[c]}$; here $D^{[c]}$ is the closure of $Y \backslash D$ in Y. Since $M|D$ is spanned and $\deg(M|D) = 1$, $D \cong \mathbb{P}^1$. Since M has depth 1, the natural map $F \to f_*f^*(F)$ induces an injective map $j : F \to f_*(M)$. Since $\deg(f_*(M)) = 2$ (Lemma 1), j is an isomorphism. Hence $f_*(M)$ is spanned by $f_*(H^0(Y, M))$. Y has one or two connected components. Exactly one of these components, say A, has the property that $\deg(M|A)$; since M is spanned if Y is not connected, then $M|(Y \backslash A) \cong \mathcal{O}_{Y \backslash A}$. Theorem 1 applied to $M|A$ gives a description of A. Cases (i) or (ii) are the two possibilities for X and P giving A which satisfies the thesis of Theorem 1. Notice that $h^0(Y, M) = 3$ in case (i) and $h^0(Y, M) = 2$ in case (ii). Hence $h^0(X, f_*(M)) = 3$ in case (i) and $h^0(X, f_*(M)) = 2$ in case (ii). To conclude it is sufficient to prove that if either (i) or (ii) holds, then $f_*(M)$ is spanned. Assumed that $f_*(M)$ is not spanned and call G the subsheaf of $f_*(M)$ spanned by $H^0(X, f_*(M))$. Since $G \neq f_*(M)$, $\deg(G) \leq 1$. Since $h^0(X, f_*(M)) \geq 2$, $\deg(G) > 0$. If G is locally free, then Theorem 1 applied to (Y, G) gives a contradiction, because either $h^0(X, G) = 3$ or there is a connected component of $C_X^{[c]}$ meeting C in two points. If G is not locally free then, Theorem 2 gives a conntradiction, because either P is not a disconnecting node or $h^0(X, G) = 3$.

(ii) Here we will prove part (e). Take $F \in \Gamma_2(X)$. Since $\mathrm{Sing}(F) \neq \emptyset$ and

F is locally free at all points of X lying on at least two irreducible components of X, there are $C \in \mathcal{B}(X)$, and $P \in \mathrm{Sing}(C)\backslash\mathrm{Sing}(X)''$ such that $F|C$ is a spanned torsion free sheaf on C singular at P and with either degree 1 or degree 2. Lemma 2 gives $\deg(F|C) = 2$. Hence C is described by Example 1 and $F|T \cong F_C$. Since F is spanned and $\deg(F) = 2$, Remark 3 gives that F is trivial in a neighborhood of $C^{[c]}$. The sheaf F is uniquely determined by these informations. Obviously $\sharp(\mathrm{Sing}(F)) = 1$. A Mayer-Vietoris exact sequence gives $h^0(X, F) = h^0(C, F_C)$, concluding the proof of part (e). □

The proof of part (a) of Theorem 4 gives the following result.

Proposition 2. *Fix an integer $t > 0$. There is a bijection between the following two sets:*

(i) the set of all spanned sheaves F with depth 1, pure rank 1 such that $Sing(F) \subseteq Sing(X)''$, and $\deg(F) = \sharp(Sing(F)) = t$;

(ii) a set $S \subseteq Sing(X)''$ such that $\sharp(S) = t$ and $X\backslash S$ has more connected components than $X\backslash S'$ for all $S' \subsetneqq S$.

In this correspondence $h^0(X, F)$ is the number of connected component of X.

Proof. Start with F as in (i) and set $S := \mathrm{Sing}(F)$. Let $h : D \to X$ be the partial normalization of X in which we normalize exactly S. Set $R := h^*(F)/\mathrm{Tors}(h^*(F))$. Lemma 1 gives $\deg(R) = 0$. Since R is a spanned line bundle, we get $R \cong \mathcal{O}_D$. F is a subsheaf of $h_*(\mathcal{O}_D)$. Lemma 1 gives $\deg(h_*(\mathcal{O}_D)) = t$. Hence $F = h_*(\mathcal{O}_D)$. Since $h^0(D, \mathcal{O}_D)$ is the number of connected components of $X\backslash S$, we see that $h_*(\mathcal{O}_D)$ is spanned if and only for nevery $S' \subsetneqq S$, $h^0(X, h_{S'*}(\mathcal{O}_{D_{S'}}) < h^0(X, h_*(\mathcal{O}_D))$, where $h_{S'} : D_{S'} \to X$ is the partial normalization of X in which we only normalize the points of S'. Since D and $D_{S'}$ are reduced and complete, the latter condition is equivalent to require that $X\backslash S$ has more connected components than $X\backslash S'$. Hence (ii) gives (i) with $F := h_*(\mathcal{O}_D)$. □

Proof of Proposition 1. Since X is stable, any smooth and rational component of X (if any) intersects the other components in at least 3 points. Since $p_a(X) = 2$, we get $\sharp(\mathcal{B}(X)) \leq 2$. If X is irreducible, then ω_X is spanned by a theorem of Rosenlicht (see [4], p. 187, or [1], p. 536, or many other authors quoted in [3], p. 4). By Clifford's Theorem (see [1], Theorem A at p. 532), if X is integral, then ω_X is the only degree 2 spanned line bundle on X. From now on we assume $\sharp(\mathcal{B}(X)) = 2$, say $X = C_1 \cup C_2$ with C_1, C_2 integral.

(a) Here we assume $\sharp(C_1 \cap C_2) = 1$. Since X is stable, $p_a(C_i) = 1$, $i = 1, 2$. The point $P := C_1 \cap C_2$ is a base point of $\omega_X|C_i \cong \omega_{C_i}(P)$ and hence ω_X is not spanned. As in the statement of Theorem 3 we see that all spanned degree 2 line bundles on X are obtained in the following way. Fix $i \in \{1, 2\}$ and take a degree 2 line bundle on C_i. It induces a degree 2 morphism $v : C_i \to \mathbb{P}^1$. The morphism v may be extended in a unique to morphism $u : X \to \mathbb{P}^1$ associated to a degree 2 line bundle, just prescribing $u|C_i = v$ and $u(C_{2-i}) = v(P)$.

(b) Since X is stable and with genus 2, $\sharp(C_1 \cap C_2) \neq 2$. Now we assume $\sharp(C_1 \cap C_2) \geq 3$. Since $g = 2$, we get $\sharp(C_1 \cap C_2) = 3$ and $C_1 \cong C_2 \cong \mathbb{P}^1$. There is a unique spanned $L \in \mathrm{Pic}(X)$ such that $\deg(L) = 2$. L is ample and the associated morphism $h_L : X \to \mathbb{P}^1$ maps isomorphically each component C_i. □

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