International Journal of Pure and Applied Mathematics
Volume 55 No. 1 2009, 131-136

PROPERTY N_p FOR BALANCED LINE BUNDLES ON A STABLE CURVE $C \cup D$ WITH C, D INTEGRAL, $\sharp(C \cap D) = 1$ and $p_a(D) = 1$

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Abstract: Let $X = C \cup D$ be a stable curve of genus $g \geq 4$ with C, D irreducible, $\sharp(C \cap D) = 1$ and $p_a(D) = 1$. Here we describe the very ample line bundles L on X and give for which integer p, L has Property N_p.

AMS Subject Classification: 14H51
Key Words: stable curve, balanced line bundle, Property N_p, Koszul cohomology

1. Introduction

Let X be a stable curve. The Brill-Noether theory of X consider depth 1 coherent sheaves on X with pure rank 1 (see [6]) which are ω_X-semistable, or, equivalently (see [5], Theorem 10.3.1) balanced line bundles of all quasistable curves which have X as their stable model. We recall the latter definition (see [1], [2]). Let Y be a quasistable curve. Set $g := p_a(Y)$. Fix $L \in \mathrm{Pic}(Y)$ and set $d := \deg(L)$. For any subcurve $Z \subseteq Y$, $Z \neq \emptyset$, set $w_Z := \deg(\omega_X|Z)$ and $d_Z := \deg(L|Z)$. L is called *semibalanced* if

$$|w_Y d_Z - w_Z d| \leq \sharp(Z \cap \overline{Y \backslash Z}) \cdot (g-1) \tag{1}$$

for every proper subcurve Z of Y. It is sufficient to test (1) for all proper connected subcurves of X. The line bundle L is called *balanced* if it is semibalanced

Received: July 14, 2009

and $d_E = 1$ for every exceptional curve $E \subset Y$.

Theorem 1. *Fix integers $g \geq 4$, $a \geq 3$ and $d \geq 5(g-1)$. Let X be a projective curve with 2 irreducible components C, D. Assume $p_a(C) = g-1$, $p_a(D) = 1$, and that $C \cap D$ is a unique point, P, which is an ordinary node of X. Fix $L \in Pic(X)$ such that $\deg(L|D) = a$ and $\deg(L|C) = d - a$. Then $h^1(X, L) = 0$, L is very ample, it has property N_p for $p \leq a-2$, but it has not Property N_{a-1}.*

Theorem 2. *Let X be a stable curve with 2 irreducible components C, D. Assume $p_a(C) = g-1 \geq 3$, $p_a(D) = 1$, and $t\sharp(C \cap D) = 1$. Fix $L \in Pic(X)$ such that L is very ample and semibalanced. Set $a := \deg(L|D) = a$ and $d := \deg(L)$. Then $a \geq 3$, $(2a-1)(g-1) \leq d \leq (2a+1)(g-1)$ and $h^1(X, L) = 0$. The line bundle L has property N_p if and only if $p \leq a-2$.*

Then we consider the line bundles on the quasistable curve associated to depth 1 sheaves on X with pure rank 1 and not locally free at P (see Example 4 and Theorem 3.).

2. The Proofs

Let X be a stable curve. For any depth 1 sheaf F on X set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at P. Let $u_F : X_F \to X$ be the quasistable curve such that u_F induces a bijection $E \mapsto u_F(E)$ of the set of all exceptional components of X_F and $\mathrm{Sing}(F)$. There is a unique line bundle L_F on X such that $u_{F*}(L_F) \cong F$ and $\deg(F|E) = 1$ for every exceptional component E of X_F. F is ω_X-semistable if and only if L_F is balanced (see [5], Theorem 10.3.1), $\deg(L_F) = \deg(F)$ and $h^0(X, F) = h^0(X_F, L_F)$.

Example 1. Let X be the stable curve with 2 irreducible components, C, D such that $p_A(D) = 1$, $p_a(C) = g-1 \geq 4$ and $\sharp(D \cap C) = 1$. Hence $p_a(X) = 1$ and $P := C \cap D$ is a disconnecting node of X. Fix integers d, a such that $d \geq a \geq 0$. Let $L_{d,a}$ be any line bundle on X such that $\deg(L_{d,a}) = d$ and $\deg(L_{d,a}|D) = a$. Thus $\deg(L|C) = d - a$, $w_C = 2g-3$, $w_D = 1$ and $k_C = k_D = 1$. $L_{d,a}$ is balanced if and and only if it is semibalanced if and only if

$$|(2g-2)a - d| \leq g-1\,. \tag{2}$$

Thus if $L_{d,a}$ is balanced for some $a < 0$, then $d < 0$ and $h^0(Y, L) = 0$. The line bundle $L_{d,0}$ is balanced if and only if $|d| \leq g-1$, while $L_{d,1}$ is balanced if and only if $g-1 \leq d \leq 3g-3$. If $a \geq 2$ and $L_{d,a}$ is balanced, then $d \geq 3g-3 \geq g$. For each integer $a \geq 2$ let d_a (resp. D_a) be the minimal (resp. maximal) integer a such that $L_{d,a}$ is balanced. We have $d_a = (g-1)(2a-1)$ and $D_a = (g-1)(2a+1)$. For eall integers a, d such that $a \geq 2$ and $d_a \leq d \leq D_a$ the line bundle $L_{d,a}$ is balanced.

Example 2. Take the set-up of Example 1 and let $v : U \to X$ be the partial normalization of X in which we normalize only the point P. Thus $U \cong C \sqcup D$. Hence $h^0(X, v_*(\mathcal{O}_U)) = h^0(U, \mathcal{O}_U) = 2$. The depth 1 sheaf $v_*(\mathcal{O}_U)$ is a depth 1 sheaf on X with pure rank 1. Riemann-Roch applied to U and to X gives $\deg(v_*(\mathcal{O}_U)) = 1$. Obviously any line bundle with a subsheaf isomorphic to $v_*(\mathcal{O}_U)$ has at least 2 linearly independent section. We may take as such line bundle a line bundle ,$\mathcal{O}_X(Z)(W)$, where Z is an effective Cartier divisor such thatt $\text{length}(Z) = 2$ and $Z_{red} = \{P\}$ and W is any effective (or empty) Cartier divisor. Since $\deg(\mathcal{O}_X(Z)|C) = \deg(\mathcal{O}_X(Z)|D) = 1$, among these line bundles we find all numerical types of line bundles $L_{d,a}$ with $d > a > 0$. Taking $a = 1$ we get an $(g-1)$-dimensional irreducible subset of $W^1_{g-1}(X)$: Z depends from 1-parameter and we may take as W a general subset of C with $\sharp(W) = g-2$. Notice that many of these line bundles have $h^0 \geq 3$, but that decreasing their degree we lose their balancedeness.

Example 3. Take the set-up of Example 1. For any $L \in \text{Pic}(X)$ we have an exact sequence

$$0 \to L \to L|C \otimes L|D \to L\{P\} \to 0\,. \tag{3}$$

From (3) we get the inequalities

$$h^i(C, L|C) + h^i(D, L|D) - 1 \leq h^i(X, L) \leq h^i(C, L|C) + h^i(D, L|D) \tag{4}$$

for $i = 0, 1$ and the first inequality is an equality for $i = 0$ if and only if it is an equality for $i = 0$ if and only if at least one of the line bundles $L|C$ and $L|D$ have not P in their base locus. Thus if $L|D \cong \mathcal{O}_D$ we have $h^0(X, L) = h^0(C, L|C)$. To get a balanced line bundle $L_{d,0}$ we need $-g+1 \leq d \leq g-1$. Thus we get in this way elements of $W^1_d(X)$ with d lower than the ones coming from Example 3. For each integer $a \geq 2$ let δ_a (resp. Δ_a) be the minimal (resp. maximal) integer a such that $M_{d,a}$ is balanced.

Lemma 1. *Fix integers $a \geq 1$, $d \geq 2g + a - 1$. If $a = 1$ assume $d \geq 2g + 1$. Then $h^1(X, L_{d,a}) = 0$.*

Proof. Set $L := L_{d,a}$. Since $\deg(|C) = d - a \geq 2(g-1) - 1$, we have $h^1(C, L|C) = 0$. Since $\deg(L|D) = a > 0$ and $p_a(D) = 1$, $h^1(D, L|D) = 0$. Our assumptions implies that at least one of the restriction maps $H^0(C, L|C) \to H^0(\{P\}, L|\{P\})$, $H^0(D, L|D) \to H^0(\{P\}, L|\{P\})$. Apply (3). □

Lemma 2. *Take the set-up of Example 1. A balanced $L_{d,a}$ is very ample if and only if $a \geq 3$. If $a \geq 3$ and $d \geq d_a$, then any $L_{d,a}$ is very ample.*

Proof. Since $L_{d,a}|D$ is a degree a line bundle on the integral genus 1 curve D, the "only if " part is obvious. Fix integers $a \geq 3$ and d such that $d \geq d_a = (2a-1)(g-1)$. Fix any $L \cong L_{d,a}$ and any zero-dimensional scheme $Z \subset X$ such that length$(Z) = 2$. First assume $Z \subset D \backslash \{P\}$. Since $a \geq 3$ and $p_a(D) = 1$, we have $h^1(D, \mathcal{I}_Z \otimes (R|D)) = 0$. Hence the restriction map $H^0(D, R|D) \to R|Z$ is surjective. Then we use the surjectivity of the restriction map $H^0(X, R) \to H^0(D, R|D)$, which follows from the exact sequence (3) for $L := R$ and the surjectivity of the restriction map $\rho : H^0(C, R|C) \to H^0(\{P\}, R|\{P\})$; ρ is surjective, because C is integral, $p_a(C) = g-1$, $P \in C_{reg}$ and $\deg(\mathcal{I}_{\{P\}} \otimes (R|C)) = d-a-1 \geq (2a-1)(g-1)-a-1 \geq 3g-4 \geq 2g-1$. In a similar way we check the case $Z \subset C \backslash \{P\}$. Even the case Z reduced and $P \in Z_{red}$ is similar, taking as first curve the curve containing Z. Now assume $Z_{red} = \{P\}$. If either $Z \subset C$ or $Z \subset D$, then we do the same proof. If none of these containement is true, then Z is a Cartier divisor. We have $\mathcal{I}_Z \otimes R \cong L_{d-1,a-1}$. Hence it is sufficient to use Lemma 1. □

We may also describe the very ampleness and Property N_p for balanced line bundles associated to ω_X-semistable sheaves (see [1]) which are not locally free at P. If Y and D are smooth, then these sheaves are all the non-locally free depth 1 sheaves with pure rank 1 on X. We first describe them.

Example 4. Let X be the stable curve with 2 irreducible components, C, D such that $p_A(D) = 1$, $p_a(C) = g-1 \geq 3$ and $\sharp(D \cap C) = 1$. Hence $p_a(X) = g$ and $P := C \cap D$ is a disconnecting node of X. Let Y be the only quasistable curve with a unique exceptional component E, X as its stable model and such that the contracting map $u : Y \to X$ satisfies $u(E) = \{P\}$. Call A, B the irreducible components of Y such that $u(A) = C$ and $u(B) = D$. Fix integers d, a. Let $M_{d,a}$ be any line bundle on Y such that $\deg(M_{d,a}|E) = 1$,$\deg(M_{d,a}) = d$ and $\deg(M_{d,a}|B) = a$. Thus $\deg(M_{d,a}|A) = d-a-1$, $w_A = w_{A \cup E} = 2g-3$, $w_B = w_{B \cup E} = 1$, and $k_A = k_B = k_{A \cup E} = k_{B \cup E} = 1$. The line bundle $M_{d,a}$ is balanced if and and only if it is semibalanced if and only if the inequalities (2) and

$$|(2g-2)(a+1) - d| \leq g-1 \tag{5}$$

hold. If $a < 0$ and $M_{d,a}$ is balanced, then $d-1 < 0$ (at least if $g \geq 4$). Thus $h^0(Y, M_{d,a}) = 0$ if $a < 0$ and $M_{d,a}$ is balanced. The line bundle $M_{d,0}$ is balanced if and only if $d = g-1$, while if $M_{d,a}$ is balanced and $a \geq 1$, then $d \geq 3g-3 \geq g$.

Lemma 3. *Take the set-up of Example 4. A balanced $M_{d,a}$ is very ample* *if and only if* $a \geq 3$.

Proof. Adapt the proof of Lemma 1. Fix any $M \in \mathrm{Pic}(Y)$. Here we need to use two Mayer-Vietoris exact sequence, for instance the following ones

$$0 \to M|(B \cup E) \to M|B \oplus M|E \to M|E \cap B \to 0\,, \tag{6}$$

$$0 \to M \to M|(B \cup E) \oplus M|A \to M|E \cap A \tag{7}$$

in which the scheme-theoretic intersections $E \cap B$ and $E \cap A$ are just one point with the reduced structure. One could also interchange the role of B and A, i.e. first use the pair (A, E) and then the pair $(A \cup E, B)$. □

Lemma 4. *Take* $X = C \cup D$ *as in the statement of Theorem 1. Fix an embedding* $X \hookrightarrow \mathbb{P}^n$ *such that* $\langle C \rangle \cap \langle D \rangle = \{P\}$. *Fix a spanned vector bundle* G *on* $\mathbb{P}^n$ *such that the restriction maps* $H^0(\mathbb{P}^n, G) \to H^0(C, G|C)$ *and* $H^0(\mathbb{P}^n, G) \to H^0(D, G|D)$ *are surjective. Then the restriction map* $H^0(\mathbb{P}^n, G) \to H^0(X, G|X)$ *is surjective.*

Proof. Use the Mayer-Vietoris exact sequence (3) with $L := G|X$. □

Proof of Theorem 1. Since $\deg(L|Y) \geq 2p_a(Y) - 1$, $h^1(Y, L|Y) = 0$. Since $a = \deg(L|D) \geq 1$ and $p_a(Y) = 1$, $h^1(D, L|D) = 0$ and the restriction map $H^0(D, L|D) \to H^0(\{P\}, L|\{P\})$ is surjective. Hence (3) gives $h^1(X, L) = 0$. The very ampleness of L may be proved as in Lemma 2. See X as embedded into $\mathbb{P}^n$, $n := h^0(X, L) - 1 = d - g$, by the complete linear system $|L|$. By assumption X is linearly normal in $\mathbb{P}^n$ and $h^1(X, L) = 0$. To prove Property N_p for $p \leq a - 2$ (with N_0 meaning that X is arithmetically Cohen-Macaulay) apply Lemma 4 with respect to the spanned vector bundle $G := \Omega^p_{\mathbb{P}^n}(p + 2)$. We use that C (resp. D) has property N_p with $p := d - a - 1 - (g - 1) \geq a - 2$ (resp. $p = a - 2$) in its linear span, because the proof given in [4] for the case of a smooth curve works for an arbitrary integral curve. Now we check that X has not Property N_{a-1}. Let $\eta : H^0(X, \Omega^{a-1}_{\mathbb{P}^n}(a+1)|X) \to H^0(D, \Omega^{a-1}_{\mathbb{P}^n}(a+1)|D)$ denote the restriction map. Since $\Omega^{a-1}_{\mathbb{P}^n}(a+1))$ is spanned, ther Mayer-Vietoris exact sequence (3) with $L := \Omega^{a-1}_{\mathbb{P}^n}(a + 1))|X$ gives the surjectivity of η. The curve D has not Property N_{a-1} in its linear span $\langle D \rangle \cong \mathbb{P}^{a-1}$. Hence it is sufficient to use the surjectivity of the map η. □

Proof of Theorem 2. Use Example 2, Lemma 2 and the statement of Theorem 1. □

Theorem 3. *Take* Y *as in Example 4 and let* $M \in \mathrm{Pic}(Y)$ *such that* $M \cong M_{d,a}$. *If* M *is balanced, then it is very ample if and only if* $a \geq 3$. *If* $a \geq 3$ *and* $d \geq (2a-1)(g-1)$, *then* M *is very ample. If* $a \geq 3$ *and* $d \geq (2a-1)(g-1)$, *then* L *is arithmetically Cohen-Macaulay; it has Property* N_p *if and only if*

$p \leq a - 2$.

Proof. Use Lemma 3 and its proof to adapt the proof of Theorem 1. □

Remark 1. A statement similar to Theorem 3 is true if we take the partial normalization $C \sqcup D$ of X in which we normalize only the point P instead of the quasistable curve Y.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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