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EXISTENCE OF STRONG SOLUTIONS TO SOME QUASILINEAR ELLIPTIC EQUATIONS

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Abstract: Let $Lu = -\sum_{i,j=1}^{N} a_{ij}(x,u)D_{ij}u$. Consider the quasilinear elliptic equation $Lu + f(x,u,\nabla u) = 0$ on a bounded smooth domain Ω in $\mathbb{R}^N$. It is shown that if the oscillation of $a_{ij}(x,r)$ with respect to r is sufficiently small and $f(x,r,\xi)$ has a sub-linear growth in r and ξ, then there exists a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The existence of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solutions to the equation $Lu + c(x,u)u + f(x,u,\nabla u) = 0$, where $\beta \geq c(x,r) \geq \alpha > 0$, remains valid if f has a sub-quadratic growth in ξ.

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1. Introduction

Let Ω be a *bounded* $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 3$, and let L_v, L, D_v, D be *elliptic operators* defined by

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$$\begin{aligned} L_v u &= -\sum_{i,j=1}^{N} a_{ij}(x,v) D_{ij} u, \\ Lu &= L_u u, \\ D_v u &= -\sum_{i,j=1}^{N} D_i(a_{ij}(x,v) D_j u), \\ Du &= D_u u, \end{aligned}$$

where the coefficients a_{ij} and their derivatives $D_i a_{ij}$, $D_r a_{ij}$ are *bounded* Carathéodory functions, $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for some constant λ. We shall omit the summation notation $\sum_{i,j=1}^{N}$ and we shall use C for a *generic constant*.

Let $f(x,r,\xi)$ be a locally bounded Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. Consider the quasilinear elliptic equations

$$Lu + f(x,u,\nabla u) = 0 \tag{1}$$

and

$$Lu + c(x,u)u + f(x,u,\nabla u) = 0 \tag{1'}$$

in Ω, where $c(x,r) \geq \alpha > 0$ is a bounded Carathéodory function. The present paper aims to investigate the existence of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solutions to (1) and (1'). For simplicity, we denote in the sequel $s = (r,\xi)$, $|s| = |r| + |\xi|$, $f(x,r,\xi) = f(x,s)$, $W(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $B_t = \{v \in W(\Omega) | \|v\|_{2,p} \leq t\}$. For $u \in W^{2,p}(\Omega)$, $1 < p < \infty$, it is well known that in the linear case when $a_{ij} = a_{ij}(x)$, then one has

$$\|u\|_{2,p} \leq C(\|u\|_p + \|L_0 u\|_p),$$

where $L_0 u = -\sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u$. Let $F(v)$ denote $f(x,v,\nabla v)$. If $F(v) \in L^p(\Omega)$, $1 < p < \infty$, then there exists a unique solution $u \in W(\Omega)$ to the equation

$$L_v u + F(v) = 0 \tag{2}$$

and

$$\|u\|_{2,p} \leq C(\|u\|_p + \|F(v)\|_p). \tag{3}$$

Note that when $a_{ij} = a_{ij}(x,r)$, this estimate remains valid with C independent of v if the oscillations of $a_{ij}(x,r)$ with respect to r are sufficiently small [5]. We shall employ (3) together with the maximum principle of A.D. Aleksandrov [4] to show that if $f(x,s)$ has a sub-linear growth in $|s|$, then equation (1) has a solution $u \in W(\Omega)$.

In case that f has a sub-quadratic growth in $|\xi|$, one can reformulate (1') to the equation in divergence form

$$Du + c(x,u)u + \tilde{f}(x,u,\nabla u) = 0, \tag{4}$$

where $\tilde{f}$ has a quadratic growth in $|\xi|$. The result of [1] and [2] shows that (4) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, all such solutions lie in a bounded interval. We observe in Section 2 that in fact the $W(\Omega)$ solutions to the trucated approximating equations of (10) are L^∞-bounded as well as $W^{2,p}$-bounded. By passing to the limit, Theorem 2 concludes the existence of a solution $u \in W(\Omega)$ to equation (1').

2. Existence of Strong Solutions

Let $F(v) = f(x,v,\nabla v)$. Consider the equation

$$L_v u + F(v) = 0$$

in Ω; if $p > N$, then there exists a unique solution $u \in W(\Omega)$. We start with a $W^{2,p}$-estimate for the solutions in $W(\Omega)$.

Let $x \in \Omega$ be fixed. Denote $\operatorname{osc} a_{ij}(x,r)$ the *oscillations* of $a_{ij}(x,r)$ with respect to r for $r \in \mathbb{R}$, i.e.

$$\operatorname{osc} a_{ij}(x,r) = \sup\{|a_{ij}(x,r_1) - a_{ij}(x,r_2)| : r_1, r_2 \in \mathbb{R}\},$$

and $\operatorname{osc} a(x,r) = \max_{1 \le i,j \le N} \operatorname{osc} a_{ij}(x,r)$. For operators L_v, we quote the following result from [5].

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ which is $C^{1,1}$ diffeomorphic to a ball in $\mathbb{R}^N$, and the cofficients $a_{ij} \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, $|a_{ij}| \le \Lambda$, where Λ is a positive constant, i, j= 1, $\ldots, N$. Assume that $\operatorname{osc} a(x,r)$ is sufficiently small with respect to r and uniformly for $x \in \bar{\Omega}$. Then if $u \in W(\Omega)$ and $L_v u \in L^p(\Omega)$, $1 < p < \infty$, one has the estimate*

$$\|u\|_{2,p} \le C(\|L_v u\|_p + \|u\|_p), \tag{5}$$

where C is a constant (independent of v) dependent on $N, p, \lambda, \Lambda, \partial\Omega, \Omega$, the diffeomorphism and the moduli of continuity of $a_{ij}(x,r)$ with respect to x in $\bar{\Omega}$.

Remark 1. The magnitude of $\operatorname{osc} a(x,r)$ that fulfills the purpose of Lemma 1 can be found in [5, p. 191].

Remark 2. If v is confined in a bounded subset S in $W^{1,p}(\Omega)$, then (5) holds without the constraint on $\operatorname{osc} a(x,r)$. This can be derived from the compact imbedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, the equicontinuity of v in S by Azela-Ascoli Theorem.

In view of Lemma 1, if $u \in W(\Omega)$ is the solution to (2), then (3) holds. Furthermore, an application of the weak maximum principle of A.D. Aleksandrov [4, p. 220] implies that

$$\|u\|_\infty \leq C\|\frac{F(v)}{\mathcal{D}^*}\|_N,$$

where $\mathcal{D}^*$ is the geometric mean of the eigenvalues of the matrix $[a_{ij}]$, and C depends on N and the diameter of Ω. By ellipticity, $\mathcal{D}^* \geq \lambda > 0$, so

$$\|u\|_p \leq C\|F(v)\|_p. \tag{6}$$

Combining (3) and (6), one gets

$$\|u\|_{2,p} \leq C\|F(v)\|_p. \tag{7}$$

Lemma 2. *For $p > N$, the map $\tilde{F}$ which assigns to $v \in B_t$ the solution $u \in W(\Omega)$ to (2) is continuous in $W^{1,p}(\Omega)$.*

Proof. Let $v_n \to v$ in $W^{1,p}(\Omega)$, $u_n = \tilde{F}(v_n)$ and $u = \tilde{F}(v)$. By passing to subsequences, we may assume $v_n \to v$, $\nabla v_n \to \nabla v$ a.e.. Since f is locally bounded, the estimate (7) implies that (u_n) is $W^{2,p}$-bounded. By the compact imbedding of $W^{2,p}(\Omega)$ in $W^{1,p}(\Omega)$, one can extract a subsequence, still denoted by (u_n), such that $u_n \to w$ in $W^{1,p}(\Omega)$, $u_n \to w$ and $\nabla u_n \to \nabla w$ a.e.. We claim that w is a weak solution to equation (2). By reformulating (2) as an equation in divergence form, it suffices to show that for every $\phi \in C_0^\infty(\Omega)$

$$\int a_{ij}(v)D_j w D_i\phi + \int [D_i a_{ij}(x,v) + D_r a_{ij}(x,v)\cdot D_i v]D_j w\cdot\phi + \int F(v)\phi = 0.$$

Since $u_n = \tilde{F}(v_n)$,

$$\int a_{ij}(v_n)D_j u_n D_i\phi + \int [D_i a_{ij}(x,v_n) + D_r a_{ij}(x,v_n)\cdot D_i v_n]D_j u_n\cdot\phi + \int F(v_n)\phi = 0.$$

But a_{ij}, $D_i a_{ij}$, $D_r a_{ij}$ are bounded Carathéodory functions and $u_n \to w$, $\nabla u_n \to \nabla w$ a.e., by applying Lebesque Dominated Convergence Theorem, one gets

$$\int a_{ij}(v)D_j w D_i\phi + \int [D_i a_{ij}(x,v) + D_r a_{ij}(x,v)\cdot D_i v]D_j w\cdot\phi + \int F(v)\phi = 0$$

for all $\phi \in C_0^\infty(\Omega)$. Hence $w \in W(\Omega)$ is a solution to (2). It follows from the uniqueness of solutions that $u = w$ and $u_n \to u$ in $W^{1,p}(\Omega)$. Therefore, $\tilde{F}$ is continuous in $W^{1,p}(\Omega)$. □

We proceed to the following theorem.

Theorem 1. *Let $f(x,s)$ be a locally bounded Carathéodory function satisfying $f(x,s) = o(|s|)$. Then there exists a solution $u \in W(\Omega)$ to equation (1) provided $\operatorname{osc} a(x,r)$ is sufficiently small uniformly for $x \in \bar{\Omega}$.*

Proof. Let $p > N$. f is locally bounded, so $\|F(v)\|_p$ is bounded for $v \in B_t$. By the assumption that $f(x,s) = o(|s|)$,

$$\|F(v)\|_p \leq C_\varepsilon + \varepsilon \|v\|_{1,p}, \qquad 0 < \varepsilon < 1.$$

Thus, it follows from (7) that

$$\|\tilde{F}(v)\|_{2,p} \leq C\|F(v)\|_p < t$$

for large t, i.e. $\tilde{F} : B_t \to B_t$. Since B_t is compact in $W^{1,p}(\Omega)$, one concludes from the Schauder Fixed Point Theorem that there exists a $u \in B_t$ such that $\tilde{F}(u) = u$. Therefore, $u \in W(\Omega)$ is a solution to (1). □

Remark 3. As an example, if $f(x,s) = C + \frac{|s|}{\ln(|s|+2)}$, then $f(x,s) = o(|s|)$, (1) has a solution in $W(\Omega)$.

In the sequel, let the operators $\tilde{L}_v$, $\tilde{D}_v$, $\tilde{L}$ and $\tilde{D}$ be defined by

$$\begin{aligned} \tilde{L}_v u &= L_v u + c(x,v)u, \\ \tilde{D}_v u &= D_v u + c(x,v)u, \\ \tilde{L} &= \tilde{L}_u, \\ \tilde{D} &= \tilde{D}_u. \end{aligned}$$

Consider now the equation

$$\tilde{L}_u + f(x,u,\nabla u) = 0 \tag{8}$$

in Ω. If $f(x,r,\xi) = O(|\xi|^2)$, (8) can be reformulated as an equation in divergence form

$$\tilde{D}_u + [D_i a_{ij}(x,u) + D_r a_{ij}(x,u) D_i u] D_j u + f(x,u,\nabla u) = 0 \tag{9}$$

which has a quadratic growth in the gradient term.

If $p > \frac{N}{2}$ and $u \in W(\Omega)$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. According to Theorem 3.3 of [2], one has

Lemma 3. *If $p > \frac{N}{2}$, then every solution $u \in W(\Omega)$ to (9) lies in a bounded interval I a.e..*

Let f_n be the truncation of f by $\pm n$. For $v \in W^{1,p}(\Omega)$, the Dirichlet problem

$$\tilde{L}_v u + f_n(x,v,\nabla v) = 0$$

has a unique solution $u_{n,v} \in W(\Omega)$. We note here that the results of Lemma 1 and Lemma 2 remain valid if one replaces the operator L_v there by $\tilde{L}_v$. So if $\operatorname{osc} a_{ij}(x,r)$ is sufficiently small, the estimates (3) and (7) hold. Lemma 2 and the Schauder Fixed Point Theorem imply that there exists a solution $u_n \in W(\Omega)$ to the truncated equation $\tilde{L}u + f_n(x,u,\nabla u) = 0$.

Based on Lemma 3, we proceed to the following $W^{2,p}$-estimate of (u_n).

Lemma 4. *If $f(x,r,\xi) = o(|\xi|^2)$, then the approximating solutions (u_n) to (8) are $W^{2,p}$-bounded.*

Proof. Since $f(x,r,\xi) = o(|\xi|^2)$ and (u_n) is L^∞-bounded by Lemma 3,

$$|F(u_n)| \quad \leq \quad C_\varepsilon + \varepsilon|\nabla u_n|^2.$$

Also, $u_n \in L^\infty \cap W^{2,p}(\Omega)$, from the Interpolation Theorem of Gagliardo-Nirenberg [3], p. 194, we obtain

$$\|\nabla u_n\|_{2p}^2 \leq C\|u\|_\infty \|u\|_{2,p}.$$

So

$$\|F(u_n)\|_p \leq C + \varepsilon_1 \|u_n\|_{2,p}. \tag{10}$$

Combining (7) and (12), one deduces that

$$\begin{aligned} \|u_n\|_{2,p} \quad &\leq \quad C(\|u_n\|_p + \|F(u_n)\|_p) \\ &\leq \quad C + \varepsilon_2 \|u\|_{2,p}. \end{aligned}$$

Therefore, (u_n) is $W^{2,p}$-bounded. □

The existence of solutions in $W(\Omega)$ can be deduced from the above lemmas.

Theorem 2. *Let Ω be $C^{1,1}$-smooth in $\mathbb{R}^N$, $N \geq 3$, $a_{ij} \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, $D_i a_{ij}(x,r)$, $D_r a_{ij}(x,r)$ be bounded Carathéodory functions. Assume that $f(x,r,\xi) = o(|\xi|^2)$. Then there exists a solution $u \in W(\Omega)$ to equation (1) provided $\operatorname{osc} a(x,r)$ is sufficiently small uniformly for $x \in \bar{\Omega}$.*

Proof. By Lemma 4, we get $W^{2,p}$-bounded approximating solutions to (8). It then follows from the compact imbedding $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ that there exists a subsequence, still denoted by (u_n), such that $u_n \to u$, $\nabla u_n \to \nabla u$ a.e. and $u_n \to u$ in $W_0^{1,p}(\Omega)$. Now since $\|u_n\|_{2,p} \leq t$ for some $t > 0$ and the set B_t is closed in $W^{1,p}(\Omega)$, the limit u of (u_n) belongs to $W^{2,p}(\Omega)$. By passing to the limit and using Vitali Convergence Theorem, one deduces that $\tilde{L}u_n \to \tilde{L}u$ in $\mathfrak{D}'(\Omega)$ and $f_n(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ in $L^1(\Omega)$ [1] which proves that $u \in W(\Omega)$ is a solution to (8). □

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