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MEAN SQUARE EXPONENTIAL STABILITY OF STOCHASTIC DELAY DIFFERENCE EQUATIONS

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Abstract: In this paper the authors investigate the mean square exponential stability of linear and nonlinear stochastic delay difference equations using difference inequalities. Examples are provided to illustrate the main results.

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1. Introduction

The stability behavior of solutions of stochastic difference equations with or without delay has been studied by many authors using Lyapunov functional technique, see, for example [1], [3], [4], [7], [8] and the references cited therein. In [5], [6], [9], [10] the authors studied the stability properties of stochastic difference equations using comparison method and difference inequalities. However it seems that there is no result available in the literature dealing with similar properties of solutions of stochastic delay difference equations using difference inequalities. This motivated our interest in studying the stability of stochastic delay difference equations of the form

$$x_{n+1} = \sum_{j=0}^{n+k} a_j x_{n-j} + \sum_{j=0}^{n+l} b_j x_{n-j}\xi_n, \quad n \in \mathbb{N}, \tag{1}$$

with initial condition

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$$x_n = \phi_n, \quad n \in Z_0,$$

and

$$x_{n+1} = \sum_{j=0}^{k} c_j x_{n-j} + \sum_{j=0}^{l} d_j x_{n-j}\xi_n + F(n, x_{n-m}, \dots, x_n), \; n \in \mathbb{N}\,, \quad (2)$$

with initial condition

$$x_n = \psi_n, \quad n \in Z_0,$$

using difference inequalities. Here $n \in Z_0 \cup \mathbb{N}$, $Z_0 = \{ -h, \dots, 0\}$, $\mathbb{N} = \{0, 1, \dots\}$, k, l, m are non-negative integral numbers, and $h = \max\{k, l\}$ (for equation (1)) or $h = \max\{k, l, m\}$ (for equation (2)) and the sequence of real numbers $\{x_n\}$ is a solution of equation (1) (resp. (2)). Let (Ω, P, σ) be a basic probability space, $f_n \subset \sigma, n \in \mathbb{N}_0$ be a sequence of σ- algebras, E be the mathematical expectation, $\xi_0, \xi_1, \dots$ be a sequence of mutually independent random variables, $\xi_n \in \mathbb{R}, \xi_n$ be f_{n+1} adapted and independent on f_n, $E\ \xi_n = 0$, $E\ \xi_n^2 = 1$, $n \in \mathbb{N}$.

Equations of the forms (1) and (2) are considered in [3], [7], [8] and their stability behavior is studied usimg the method of Lyapunov functionals construction. The purpose of this paper is to investigate the mean square (asymptotic) stability and mean square exponential stability of solutions of equations (1) and (2) using difference inequalities.

The plan of the paper is as follows. In Section 2, we present preliminary results, and in Section 3, we obtain conditions for the mean square (asymptotic) stability and mean square exponential stability of equations (1) and (2). Finally, in Section 4, we present some examples to illustrate the results.

2. Preliminary Results

Let $N[a, b] = \{a, a + 1, \dots, b\}$, where $a < b$ and a, b are integral numbers. Throughout this paper, we assume that equations (1) and (2) admit a zero soltuion. Next we will introduce some basic definitions.

Definition 1. The zero solution of equation (1) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $Ex_n^2 < \epsilon, n \in \mathbb{N}$, when the initial condition $\phi = (\phi_{-h}, \dots, \phi_0)^T$ satisfies $||\phi||^2 = \sup_{n \in \mathbb{N}_0} E\phi_n^2 < \delta$. If, besides, $\lim_{n\to\infty} Ex_n^2 = 0, n \in \mathbb{N}$, for all initial condition ϕ, then the zero solution of equation (1) is called asymptotically mean square stable.

Definition 2. The zero solution of equation (1) is called mean square exponential stable if there are positive constants $\beta < 1$ and M such that for any initial condition ϕ,

$$Ex_n^2 \le M||\phi||^2\beta^n, \quad n \in \mathbb{N}. \tag{3}$$

Remark 1. The stability definitions for equation (2) can be defined similarly.

To prove the main results, we need the following assumptions:

(H_1) $\{a_n\}$, $\{b_n\}$ are sequences of real numbers;

(H_2) for any $0 < \lambda < 1$, $\lim_{n\to\infty} \sum_{j=0}^{n+h} (a|a_j| + b|b_j|)\lambda^{-(j+1)} = \mu_1 < 1$, where $a = \sum_{j=0}^{\infty} |a_j|$ and $b = \sum_{j=0}^{\infty} |b_j|$;

(H_3) for any $n \in \mathbb{N}$, there exists positive constant $e_j(n)$ such that

$$|F_n(x_{n-m}, \ldots, x_n)| < \sum_{j=0}^{m} e_j(n)|x_{n-j}|;$$

(H_4) $\sup\{(c(n) + e(n))^2 + d^2(n)\} = \mu_2 < 1$, where

$$c(n) = \sum_{j=0}^{h} |c_j(n)|, \quad d(n) = \sum_{j=0}^{h} |d_j(n)| \quad \text{and} \quad e(n) = \sum_{j=0}^{h} |e_j(n)|;$$

(H_5) there exists a constant $\alpha > 1$ such that

$$1 < \lambda < \left(\frac{1}{\mu_2}\right)^{\frac{1}{h}+1}.$$

3. Stability Results

It is well known that difference inequalities play very important role in studying the qualitative behavior of solutions of deterministic difference equations. Therefore, we establish some difference inequalities to study the stability behavior of the solutions of the stochastic difference equations (1) and (2).

Lemma 1. *Suppose $\{f_n\}$ is a sequence of nonnegative real numbers and there exists a constant $0 < \beta \le 1$ such that $\lim_{n\to\infty} \sum_{j=0}^{n+h} f_j\beta^{j+1} = f < 1$ satisfying the following inequality*

$$u_{n+1} \le \sum_{j=0}^{n+h} f_j u_{n-j}, \quad n \ge n_1 \in \mathbb{N}. \tag{4}$$

Then

$$u_n \leq \eta\ \beta^n, \quad n \geq n_1 \in \mathbb{N},$$

provided

$$u_n \leq \eta\beta^n,\ \textit{for}\ n \in N[-h, n_1], \tag{5}$$

where $n_1 \in \mathbb{N},\ d \in R^+$.

Proof. Let $y_n = \frac{u_n}{\beta^n}$. Then from (5), we have $y_n \leq \eta, n \in N[-h, n_1]$.

We next show that for any $n \geq n_1$,

$$y_n \leq \eta. \tag{6}$$

If this is not true, then there must be a positive integral number $n_2 \geq n_1$ such that

$$y_{n_2+1} > \eta \ \text{ and } y_n \leq \eta, \quad n \in N[-h, n_2]. \tag{7}$$

By (4) and (7), we have

$$y_{n_2+1} = \frac{u_{n_2+1}}{\beta^{n_2+1}} \leq \frac{1}{\beta^{n_2+1}} \sum_{j=0}^{n_2+h} f_j\ u_{n_2-j} \leq \sum_{j=0}^{n_2+h} f_j\ y_{n_2-j}\ \frac{\beta^{n_2-j}}{\beta^{n_2+1}}$$

$$\leq \eta \sum_{j=0}^{n+h} f_j\ \beta^{-(j+1)} \leq \eta,$$

which contradicts the first inequality of (7). Thus (6) holds for any $n \geq n_2$. Therefore, we have $u_n \leq \eta\beta^n, n \geq n_1$. The proof is now complete. □

Lemma 2. *Suppose* $p_j(n) \in \mathbb{R}^+,\ n \in \mathbb{N},\ j \in N[0, h]$ *and* $\sup_{n\in\mathbb{N}} \left\{ \sum_{j=0}^{h} p_j(n) \right\} = p < 1$. *Let* $\{u_n\}$ *be a sequence of real numbers satisfying the following inequality*

$$u_{n+1} \leq \sum_{j=0}^{h}\ p_j(n)\ u_{n-j}, \quad n \geq n_1 \in \mathbb{N}. \tag{8}$$

Then

$$u_n \leq \eta\alpha^{-n}, \quad n \geq n_1 \in \mathbb{N}, \tag{9}$$

provided

$$u_n \leq \eta\alpha^{-n}, \quad n \in N[n_1 - h, n_1], \tag{10}$$

where $n \in \mathbb{N}, d \in R^+$ *and* α *satisfies*

$$1 < \alpha < \left(\frac{1}{p}\right)^{\frac{1}{h}+1}. \tag{11}$$

Proof. Since $p < 1$, there exists a constant α satisfying the inequality (11). Then $\alpha^{h+1}p \le 1$. Let $y_n = u_n\alpha^n$. Then from (10), we have

$$y_n \le \eta, \quad n \in N[n_1 - h, n_1].$$

We next show that for any $n \ge n_1$,$y_n \le \eta$. If this is not true, then there exists positive integer $n_2 \ge n_1$ such that

$$y_{n_2+1} > \eta \ \text{ and } \ y_n \le \eta, \quad n \in N[n_1 - h, n_2].$$

Now

$$y_{n_2+1} = u_{n_2+1}\alpha^{n_2+1} \le \alpha^{n_2+1}\sum_{j=0}^{h} p_j(n_2)u_{n_2-j}$$

$$\le \alpha^{n_2+1}\sum_{j=0}^{h} p_j(n_2)y_{n_2-j}\alpha^{-(n_2-j)} \le \alpha^{h+1}\eta\sum_{j=0}^{h} p_j(n_2) \le \alpha^{h+1}p\eta \le \eta,$$

which is a contradiction. Thus $y_n \ge \eta$ for all $n \ge n_1$. Therefore, we have $u_n \le \eta\alpha^{-n}, n \ge n_1$. This completes the proof. □

Theorem 3. *Assume that conditions* (H_1) *and* (H_2) *hold. Then the zero solution of equation (1) is mean square exponential stable.*

Proof. From (1) and the Hölder inequality [2], we have

$$Ex_{n+1}^2 \le E\left(\sum_{j=0}^{n+h}|a_j||x_{n-j}|\right)^2 + E\left(\sum_{j=0}^{n+h}|b_j||x_{n-j}|\right)^2 \le \sum_{j=0}^{n+h}|a_j|\sum_{j=0}^{n+h}|a_j|E|x_{n-j}|^2$$

$$+\sum_{j=0}^{n+h}|b_j|\sum_{j=0}^{n+h}|b_j|E|x_{n-j}|^2 = \sum_{j=0}^{n+h}[a|a_j| + b|b_j|]Ex_{n-j}^2. \quad (12)$$

For the initial condition $x_n = \phi_n, n \in Z_0 = N[-h, 0]$, we have

$$Ex_n^2 \le ||\phi_n||^2\beta^n, \quad n \in N[-h, 0], \quad (13)$$

where $||\phi_n||^2 = \sup_{n\in Z_0} E\phi_n^2 < \delta$. From the condition ($H_2$) and (12), we have

$$\lim_{n\to\infty}\sum_{j=0}^{n+h}[a|a_j| + b|b_j|]\beta^{-(j+1)} = \mu_1 < 1.$$

Then all the conditions of Lemma 1 are satisfied and hence we have

$$Ex_n^2 \le ||\phi||^2\beta^n < \delta\beta^n, \quad n \in \mathbb{N}.$$

The proof is now complete. □

In [4], the author consider equation (1) and obtain that the condition

$$a^2 + b^2 < 1 \quad (14)$$

guarantee the zero solution is stable in probability using the Lyapunov functional method. On the other hand, by Theorem 3, we can obtain the following theorem.

Theorem 4. *If condition (14) holds, then the zero solution of equation (1) is mean square stable.*

Proof. If we take $\beta = 1$, then from the proof of Theorem 3, we have

$$Ex_n^2 \leq ||\phi||^2, \quad n \in \mathbb{N}$$

whenever $Ex_n^2 \leq ||\phi||^2$, $n \in N[-h, 0]$. Let $\delta = \epsilon$. Then, we have $Ex_n^2 \leq ||\phi||^2 < \delta$ for $n \in N[-h, 0]$, implies $Ex_n^2 < \epsilon$ for all $n \in \mathbb{N}$ and the proof is complete. □

Theorem 5. *Assume that conditions (H$_3$) - (H$_5$) hold, then the zero solution of equation (2) is mean square exponential stable.*

Proof. From equation (2), (H$_3$) and the Hölder inequality [2], we have

$$\begin{aligned}
Ex_{n+1}^2 \leq\ & E\left(\sum_{j=0}^{k}|a_j(n)||x_{n-j}|\right)^2 + E\left(\sum_{j=0}^{l}|b_j||x_{n-j}|\right)^2 \\
& + E\left(\sum_{j=0}^{m}|c_j(n)||x_{n-j}|\right)^2 + 2E\left(\sum_{j=0}^{k}|a_j(n)||x_{n-j}|\sum_{j=0}^{m}|c_j(n)||x_{n-j}|\right) \\
\leq\ & E\left(\sum_{j=0}^{h}(|a_j(n)| + |c_j(n)|)|x_{n-j}|\right)^2 + E\left(\sum_{j=0}^{h}|b_j||x_{n-j}|\right)^2 \\
\leq\ & \sum_{j=0}^{n+h}(|a_j| + |c_j(n)|)\sum_{j=0}^{h}(|a_j| + |c_j(n)|)E|x_{n-j}|^2 \\
& + \sum_{j=0}^{h}|b_j(n)|\sum_{j=0}^{h}|b_j(n)|E|x_{n-j}|^2 \\
=\ & \sum_{j=0}^{h}\{[|a(n)| + |c(n)|][|a_j(n)| + |c_j(n)|] + |b(n)||b_j(n)|\}Ex_{n-j}^2.
\end{aligned}$$

From condition (H$_4$), we obtain

$$\sup_{n\in\mathbb{N}}\left\{\sum_{j=0}^{h}[|a(n)| + |c(n)| + |c_j(n)| + |b(n)| + |b_j(n)|]\right\}$$
$$= \sup_{n\in\mathbb{N}}[(|a(n)| + |c(n)|)^2 + b^2(n)] = \mu_2 < 1.$$

From the initial condition $x_n = \phi_n, n \in Z_0 = N[-h, 0]$, we have

$$Ex_n^2 \leq ||\phi_n||^2 \alpha^{-n}, \quad n \in N[-h, 0],$$

where $||\phi_n||^2 = \sup_{n \in Z_0} E\phi_n^2 < \delta$. Then all conditions of Lemma 2 are satisfied and we obtain that

$$Ex_n^2 \leq ||\phi||^2 \alpha^{-n}, \quad n \in \mathbb{N}.$$

This implies that the zero solution of equation (2) is mean square exponential stable. □

Remark 2. *If $F_n \equiv 0$ and $k = l = h$ in equation (2), then the result obtained in Theorem 5 is reduced to that in [10].*

4. Examples

In this section we provide some examples to illustrate the results obtained in Section 3.

Example 1. Consider the following stochastic difference equation

$$x_{n+1} = \sum_{j=0}^{n+1} \frac{1}{4^{j+2}} x_{n-j} + \sum_{j=0}^{n+1} \frac{1}{5^{j+2}} x_{n-j} \xi_n, \quad n \in \mathbb{N}_0. \tag{15}$$

Here $h = 1, a_j = \frac{1}{4^{j+2}}, b_j = \frac{1}{5^{j+2}}, a = \frac{1}{12}, b = \frac{1}{20}$. Let $\beta = \frac{1}{2}$. Then

$$\lim_{n \to \infty} \sum_{j=0}^{n+1} \left[\frac{1}{12} \left(\frac{1}{4^{j+2}} \right) + \frac{1}{20} \left(\frac{1}{5^{j+2}} \right) \right] 2^{j+1} = \frac{33}{1200} < 1.$$

Thus all conditions of Theorem 3 are satisfied and so the zero solution of equation (15) is mean square exponential stable.

Example 2. Consider the following stochastic difference equation

$$x_{n+1} = \sum_{j=0}^{n+2} \left(\frac{1}{2} \right)^{j+2} x_{n-j} + \sum_{j=0}^{n+3} \left(\frac{1}{3} \right)^{j+2} x_{n-j} \xi_n, n \in \mathbb{N}_0. \tag{16}$$

Here $k = 2, l = 3, a_j = \left(\frac{1}{2}\right)^{j+2}, b_j = \left(\frac{1}{3}\right)^{j+2}$. Thus, we have $a^2 + b^2 = \frac{5}{18} < 1$. Therefore by Theorem 4, the zero solution of equation (16) is mean square stable.

Example 3. Consider the following stochastic difference equation

$$x_{n+1} = \frac{1}{4} x_n - \frac{1}{5} x_{n-1} + \frac{1}{2} x_n \xi_n + \frac{1}{5} \frac{x_n}{(1 + x_n^2)}, n \in \mathbb{N} \tag{17}$$

Here $h = 1, a_0(n) = \frac{1}{4}, a_1(n) = \frac{1}{3}, b_0(n) = \frac{1}{2}, g_n(x_n) = \frac{1}{5} \frac{x_n}{(1+x_n^2)}$ and hence

$|g_n(x_n)| \leq \frac{1}{5}|x_n|; n \in \mathbb{N}$. So $\mu = \frac{3109}{3600} < 1$.

Let $\lambda = 1.15 \leq \left(\frac{1}{\mu}\right)^{\frac{1}{h+1}} = \left(\frac{3600}{3109}\right)^{\frac{1}{2}}$. Then condition ($H_5$) is satisfied. Hence by Theorem 5, we can get that the zero solution of equation (17) is mean square exponential stable.

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