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SCHUR MULTIPLICATIVE AND HARMONIC CONVEXITIES OF GENERALIZED HERONIAN MEAN IN n VARIABLES AND THEIR APPLICATIONS

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Abstract: The Schur multiplicative and harmonic convexities of the generalized Heronian mean $H_w(x)$ in n variables are discussed. Some inequalities are established by the theory of majorization.

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1. Introduction

Throughout this paper, we use the sets of n-dimensional vectors over the reals $(n \geq 2)$, real number field by R^n, and $R^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$ and $R = R^1$.

For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in R^n_+$ and $\alpha > 0$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), \\ xy &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\ x^\alpha &= (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \end{aligned}$$

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$$\begin{aligned}\frac{1}{x} &= (\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}),\\ \log x &= (\log x_1, \log x_2, \cdots, \log x_n),\end{aligned}$$

and

$$e^x = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}).$$

For $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ and $w \geq 0$, the generalized Heronian mean $H_w(x)$ of x is defined by K.Z. Guan and H.T. Zhu [5] as follows:

$$H_w(x) = H_w(x_1, x_2, \cdots, x_n) = \begin{cases} \frac{nA_n(x)+wG_n(x)}{w+n}, & 0 \leq w < +\infty, \\ G_n(x), & w = +\infty, \end{cases} \tag{1.1}$$

where $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $G_n(x) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ denote the un-weighted arithmetic and geometric means of x, respectively.

In [5], K.Z. Guan and H.T. Zhu proved that $H_w(x)$ is Schur concave in R_+^n for any $w > 0$, and established several ratio inequalities and Ky Fan type inequalities involving the mean $H_w(x)$. The main purpose of this paper is to discuss the Schur multiplicative and harmonic convexities of $H_w(x)$, as applications, some new inequalities are established in the last section.

2. Preliminary Knowledge

For the sake of readability, in this section we introduce some definitions and well-known results as follows.

Definition 2.1. Let $E \subseteq R^n$ be a set, a real-valued function F on E is called a Schur convex function if

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the i-th largest component of x. A function F is called Schur concave if $-F$ is Schur convex.

Definition 2.2. Let $E \subseteq R_+^n$ be a set, $F : E \to R_+$ is called Schur multiplicatively convex on E if $F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n)$ for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\log x \prec \log y$. F is called Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

Definition 2.3. Let $E \subseteq R_+^n$ be a set, $F : E \to R_+$ is called Schur harmonic convex (or Schur harmonic concave, respectively) on E if

$$F(x_1, x_2, \cdots, x_n) \leq (\text{or} \geq \text{, respectively}) F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$.

Definitions 2.1, 2.2, and 2.3 have the following consequences.

Remark 2.1. Let $E \subseteq R_+^n$ be a set, and $H = \log E = \{\log x : x \in E\}$. Then $f : E \to R_+$ is Schur multiplicatively convex (or Schur multiplicatively concave, respectively) on E if and only if $\log f(e^x)$ is Schur concave (or Schur convex, respectively) on H.

Remark 2.2. Let $E \subseteq R_+^n$ be a set, and $H = \frac{1}{E} = \{\frac{1}{x} : x \in E\}$. Then $f : E \to R_+$ is Schur harmonic convex (or Schur harmonic concave, respectively) on E if and only if $\frac{1}{f(\frac{1}{x})}$ is Schur concave (or Schur convex, respectively) on H.

Schur convexity was introduced by I. Schur in 1923 [9], it has many applications in inequality theory [6], [12], [1]. Recently, the Schur multiplicative convexity was investigated in [2], [4], [7], but no one has ever researched the Schur harmonic convexity.

The following well-known result was proved by A.W. Marshall and I. Olkin [8].

Theorem A. *Let $E \subseteq R^n$ be a symmetric convex set with nonempty interior* $\operatorname{int} E$ *and $f : E \to R$ be a continuous symmetric function. If f is differentiable on* $\operatorname{int} E$, *then f is Schur convex on E if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, x_2, \cdots, x_n) \in \operatorname{int} E$. Here E is a symmetric set means that $x \in E$ implies $Px \in E$ for any $n \times n$ permutation matrix P.

Remark 2.3. Since f is symmetric, the Schur's condition in Theorem A, i.e. (2.1) can be reduced as

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0.$$

The following Theorems B and C can be derived from Remarks 2.1-2.3 and Theorem A.

Theorem B. (see [2]) *Let $E \subseteq R_+^n$ be a symmetric multiplicatively convex set with nonempty interior* $\operatorname{int} E$ *and $f : E \to R_+$ be a continuous symmetric function. If f is differentiable on* $\operatorname{int} E$*, then f is Schur multiplicatively convex on E if and only if*

$$(\log x_1 - \log x_2)(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}) \geq 0$$

for all $(x_1, x_2, \cdots, x_n) \in \operatorname{int} E$. Here $E \subseteq R_+^n$ is a multiplicatively convex set means that $x^{\frac{1}{2}} y^{\frac{1}{2}} \in E$ whenever $x, y \in E$.

Theorem C. *Let $E \subseteq R_+^n$ be a symmetric harmonic convex set with nonempty interior* $\operatorname{int} E$ *and $f : E \to R_+$ be a continuous symmetric function. If f is differentiable on* $\operatorname{int} E$*, then f is Schur harmonic convex on E if and only if*

$$(x_1 - x_2)(x_1^2 \frac{\partial f}{\partial x_1} - x_2^2 \frac{\partial f}{\partial x_2}) \geq 0$$

for all $(x_1, x_2, \cdots, x_n) \in \operatorname{int} E$. Here $E \subseteq R_+^n$ is a harmonic convex set means that $\frac{2xy}{x+y} \in E$ whenever $x, y \in E$.

3. Main Results

Theorem 3.1. *If $w \geq 0$, then $H_w(x)$ is:*

(i) Schur multiplicatively convex in R_+^n;

(ii) Schur harmonic convex in R_+^n.

Proof. It is easy to see that $H_w(x)$ is symmetric and has continuous partial derivatives in R_+^n for $w \geq 0$.

If $0 \leq w < +\infty$, then (1.1) leads to that

$$\frac{\partial H_w(x)}{\partial x_i} = \frac{1}{w+n} \left(1 + \frac{w G_n(x)}{n x_i} \right), \quad i = 1, 2, \cdots, n, \tag{3.1}$$

$$(\log\, x_1 - \log\, x_2) \left(x_1 \frac{\partial H_w(x)}{\partial x_1} - x_2 \frac{\partial H_w(x)}{\partial x_2} \right) = \frac{1}{w+n} (\log\, x_1 - \log\, x_2)(x_1 - x_2) \geq 0 \tag{3.2}$$

and

$$(x_1 - x_2)\left(x_1^2\frac{\partial H_w(x)}{\partial x_1} - x_2^2\frac{\partial H_w(x)}{\partial x_2}\right)$$
$$= \frac{1}{(w+n)}(x_1-x_2)^2\left(x_1 + x_2 + \frac{w}{n}G_n(x)\right) \geq 0. \quad (3.3)$$

If $w = +\infty$, then (1.1) yields that

$$\frac{\partial H_w(x)}{\partial x_i} = \frac{w}{nx_i}G_n(x), \quad i = 1, 2, \cdots, n, \quad (3.4)$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_w(x)}{\partial x_1} - x_2\frac{\partial H_w(x)}{\partial x_2}\right) = 0 \quad (3.5)$$

and

$$(x_1 - x_2)\left(x_1^2\frac{\partial H_w(x)}{\partial x_1} - x_2^2\frac{\partial H_w(x)}{\partial x_2}\right) = \frac{w}{n}(x_1-x_2)^2 G_n(x) \geq 0. \quad (3.6)$$

Therefore, Theorem 3.1 (i) follows from (3.2), (3.5) and Theorem B, and Theorem 3.1 (ii) follows from (3.3), (3.6), and Theorem C. □

Theorem 3.2. *The function $\phi_w(x) = \frac{H_w(x)}{H_{w-1}(x)}$ is Schur multiplicatively concave in R_+^n for $w \geq 1$.*

Proof. We clearly see that $\phi_w(x)$ is symmetric and has continuous partial derivatives in R_+^n. If $w = +\infty$, then Theorem 3.2 is trivial. If $1 \leq w < +\infty$, then (3.1) implies that

$$\frac{\partial \phi_w(x)}{\partial x_i} = \frac{G_n(x)}{(w+n)(w-1+n)H_{w-1}^2(x)}\left(\frac{A_n(x)}{x_i} - 1\right), \quad i = 1, 2, \cdots, n, \quad (3.7)$$

and

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial \phi_w(x)}{\partial x_1} - x_2\frac{\partial \phi_w(x)}{\partial x_2}\right)$$
$$= -\frac{(x_1-x_2)(\log x_1 - \log x_2)}{(w+n)(w-1+n)H_{w-1}^2(x)}G_n(x) \leq 0. \quad (3.8)$$

Therefore, Theorem 3.2 follows from (3.8) and Theorem B together with Definition 2.2. □

Theorem 3.3. *If $n = 2$, then the function $\phi_w(x) = \frac{H_w(x)}{H_{w-1}(x)}$ is Schur harmonic concave in R_+^2 for $w \geq 1$.*

Proof. If $w = +\infty$, then Theorem 3.3 is trivial. If $1 \leq w < +\infty$, then (3.7) leads to that

$$(x_1 - x_2)\left(x_1^2\frac{\partial \phi_w(x)}{\partial x_1} - x_2^2\frac{\partial \phi_w(x)}{\partial x_2}\right) = -\frac{(x_1+x_2)(x_1-x_2)^2}{n(w+n)(w+n-1)H_{w-1}^2(x)}G_n(x)$$
$$\leq 0. \quad (3.9)$$

Therefore, Theorem 3.3 follows from (3.9) and Theorem C together with Definition 2.3. □

Remark 3.1. It is not difficult to see that $\frac{H_w(x)}{H_{w-1}(x)}$ is neither Schur harmonic convex nor Schur harmonic concave in R_+^n for any $w \geq 1$ and $n \geq 3$.

4. Inequalities Involving $H_w(x)$

Theorem 4.1. *Suppose that $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, and $\sum_{i=1}^{n} x_i = s$. If $c \geq s$ and $w \geq 0$, then*

$$H_w(\frac{1}{x}) \geq \left(\frac{nc}{s} - 1\right) H_w(\frac{1}{c-x}).$$

Proof. According to [3, Lemma 2.3] we have

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, \frac{c-x_2}{\frac{nc}{s}-1}, \cdots, \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.1}$$

Therefore, Theorem 4.1 follows from (4.1) and Theorem 3.1 (ii). □

Theorem 4.2. *Suppose that $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, and $\sum_{i=1}^{n} x_i = s$. If $c \geq 0$ and $w \geq 0$, then*

$$H_w(\frac{1}{x}) \geq \left(\frac{nc}{s} + 1\right) H_w(\frac{1}{c+x}).$$

Proof. According to [3, Lemma 2.4] we have

$$\frac{c+x}{\frac{nc}{s}+1} = \left(\frac{c+x_1}{\frac{nc}{s}+1}, \frac{c+x_2}{\frac{nc}{s}+1}, \cdots, \frac{c+x_n}{\frac{nc}{s}+1}\right) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.2}$$

Therefore, Theorem 4.2 follows from Lemma (4.2) and Theorem 3.1(ii). □

Theorem 4.3. *Suppose that $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, and $\sum_{i=1}^{n} x_i = s$. If $0 \leq \lambda \leq 1$ and $w \geq 0$, then:*

(i) $H_w(\frac{1}{x}) \geq (n-\lambda) H_w(\frac{1}{s-\lambda x})$;

(ii) $H_w(\frac{1}{x}) \geq (n+\lambda) H_w(\frac{1}{s+\lambda x})$.

Proof. A result due to S.H. Wu [11, Lemma 2] gives

$$\frac{s-\lambda x}{n-\lambda} = \left(\frac{s-\lambda x_1}{n-\lambda}, \frac{s-\lambda x_2}{n-\lambda}, \cdots, \frac{s-\lambda x_n}{n-\lambda}\right) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.3}$$

It is not difficult to verify that

$$\frac{s+\lambda x}{n+\lambda}=\left(\frac{s+\lambda x_1}{n+\lambda},\frac{s+\lambda x_2}{n+\lambda},\cdots,\frac{s+\lambda x_n}{n+\lambda}\right)\prec(x_1,x_2,\cdots,x_n)=x. \quad (4.4)$$

Therefore, Theorem 4.3(i) follows from (4.3) and Theorem 3.1 (ii), and Theorem 4.3 (ii) follows from (4.4) and Theorem 3.1 (ii). □

Theorem 4.4. *Suppose that $A=A_1A_2\cdots A_{n+1}$ be an n-dimensional simplex in R^n $(n\geq 3)$. Let P be an arbitrary point in the interior of A, and B_i stand for the intersection point of straight line A_iP and the hyperplane $\Sigma_i=A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_nA_{n+1}$, $i=1,2,\cdots,n+1$. If $w\geq 0$, then:*

(i) $H_w\left(\frac{A_1B_1}{PB_1},\ \frac{A_2B_2}{PB_2},\ \cdots,\ \frac{A_{n+1}B_{n+1}}{PB_{n+1}}\right)\geq n+1$;

(ii) $H_w\left(\frac{A_1B_1}{A_1P},\ \frac{A_2B_2}{A_2P},\ \cdots,\ \frac{A_{n+1}B_{n+1}}{A_{n+1}P}\right)\geq \frac{n+1}{n}$.

Proof. One can easily see that $\sum\limits_{i=1}^{n+1}\frac{PB_i}{A_iB_i}=1$ and $\sum\limits_{i=1}^{n+1}\frac{A_iP}{A_iB_i}=n$. Therefore, Theorem 4.4 follows from Theorem 3.1 (ii) and (1.1) together with the fact that

$$(\frac{1}{n+1},\frac{1}{n+1},\cdots,\frac{1}{n+1})\prec\left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\cdots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right)$$

and

$$(\frac{n}{n+1},\frac{n}{n+1},\cdots,\frac{n}{n+1})\prec\left(\frac{A_1P}{A_1B_1},\frac{A_2P}{A_2B_2},\cdots,\frac{A_{n+1}P}{A_{n+1}B_{n+1}}\right).$$

□

Theorem 4.5. *Suppose that $A\in M_n(C)$ $(n\geq 2)$ is a complex matrix, $\lambda_1,\lambda_2,\cdots,\lambda_n$ and $\sigma_1,\sigma_2,\cdots,\sigma_n$ are the eigenvalues and singular values of A, respectively. If A is a positive definite Hermitian matrix and $w\geq 0$, then:*

(i) $H_w(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\cdots,\frac{1}{\lambda_n})\geq\frac{n}{\operatorname{tr}A}$;

(ii) $H_w(\lambda_1,\lambda_2,\cdots,\lambda_n)\geq\sqrt[n]{\det A}$;

(iii) $H_w(1+\lambda_1,1+\lambda_2,\cdots,1+\lambda_n)\geq\sqrt[n]{\det(I+A)}$;

(iv) $H_w(\sigma_1,\sigma_2,\cdots,\sigma_n)\geq H_w(\lambda_1,\lambda_2,\cdots,\lambda_n)$;

(v) $\frac{H_{w+1}(\sigma_1,\sigma_2,\cdots,\sigma_n)}{H_w(\sigma_1,\sigma_2,\cdots,\sigma_n)}\leq\frac{H_{w+1}(\lambda_1,\lambda_2,\cdots,\lambda_n)}{H_w(\lambda_1,\lambda_2,\cdots,\lambda_n)}$.

Proof. We clearly see that $\lambda_i>0$, $\sigma_i>0$ $(i=1,2,\cdots,n)$, $\sum\limits_{i=1}^{n}\lambda_i=\operatorname{tr}A$, $\prod\limits_{i=1}^{n}\lambda_i=\det A$ and $\prod\limits_{i=1}^{n}(1+\lambda_i)=\det(I+A)$. These lead to that

$$\left(\frac{\operatorname{tr}A}{n},\frac{\operatorname{tr}A}{n},\cdots,\frac{\operatorname{tr}A}{n}\right)\prec(\lambda_1,\lambda_2,\cdots,\lambda_n), \quad (4.5)$$

$$\log\left(\sqrt[n]{\det A}, \sqrt[n]{\det A}, \cdots, \sqrt[n]{\det A}\right) \prec \log(\lambda_1, \lambda_2, \cdots, \lambda_n) \tag{4.6}$$

and

$$\begin{aligned}\log(\sqrt[n]{\det (I+A)}, \sqrt[n]{\det (I+A)}, \cdots, \sqrt[n]{\det (I+A)}) \\ \prec \log(1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n).\end{aligned} \tag{4.7}$$

A result due to H. Weyl [10] gives

$$\log(\lambda_1, \lambda_2, \cdots, \lambda_n) \prec \log(\sigma_1, \sigma_2, \cdots, \sigma_n). \tag{4.8}$$

Therefore, Theorem 4.5 (i) follows from (4.5), Theorem 3.1 (ii) and (1.1); Theorem 4.5 (ii) follows from (4.6), Theorem 3.1 (i) and (1.1); Theorem 4.5 (iii) follows from (4.7), Theorem 3.1 (i) and (1.1); Theorem 4.5 (iv) follows from (4.8) and Theorem 3.1 (i); and Theorem 4.5 (v) follows from (4.8) and Theorem 3.2. □

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