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A COEFFICIENT INEQUALITY FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS

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Abstract: Sharp bound for $|a_1 - \mu a_0^2|$ is derived for a certain class of meromorphic functions. Also, an application of the main result for a class of functions defined through Ruscheweyh derivatives is obtained.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are *analytic* and *univalent* in the punctured open unit disk

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$$\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \Delta - \{0\},$$

where Δ is the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \Sigma$ is said to be *meromorphic univalent starlike of order* α if

$$-\Re \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \Delta;\ 0 \le \alpha < 1)$$

and the class of all such meromorphic univalent starlike functions in Δ^* is denoted by $\Sigma^*(\alpha)$. Many researchers including [4, 10, 12, 13] have studied about several subclasses of $\Sigma^*(\alpha)$.

Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let $\Sigma^*(\phi)$ be the class of functions $f \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \Delta),$$

where $\prec$ denotes subordination between analytic functions. Very recently, this class was studied by Silverman et al [15]. They have obtained the Fekete-Szegö like inequality for functions in the class $\Sigma^*(\phi)$.

Motivated by the aforementioned work, in this paper, we obtain Fekete-Szegö like inequality for a new class of meromorphic functions, which we define below. Also we give applications of our results to certain functions defined through Ruscheweyh derivatives.

Definition 1.1. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let $MR^*_\alpha(\phi)$ be the class of functions $f \in \Sigma$ for which

$$\frac{-(1-2\alpha)zf'(z) + \alpha z^2 f''(z)}{f(z)} \prec \phi(z)$$

$$(z \in \Delta;\ \alpha \in \mathbb{C} - (0,1];\ \Re(\alpha) \ge 0),$$

where $\prec$ denotes subordination between analytic functions.

Some of the interesting subclasses of $MR^*_\alpha(\phi)$ are:

1. $MR^*_0(\phi) = \Sigma^*(\phi)$,
2. $MR^*_0\left(\frac{1+(1-2\alpha)z}{1-z}\right) = \Sigma^*(\alpha) \quad (0 \le \alpha < 1)$.
3. $MR^*_0\left(\frac{1+\beta(1-2\alpha\gamma)z}{1+\beta(1-2\gamma)z}\right) = \Sigma(\alpha, \beta, \gamma)$,

$(0 \le \alpha < 1, 0 < \beta \le 1, 1/2 \le \gamma \le 1)$ studied by Kulkarni and Joshi [9].

4. $MR_0^*\left(\frac{1+Aw(z)}{1+Bw(z)}\right) = K_1(A,B)$ $(0 \leq B < 1; -B < A < B)$ studied by Karunakaran [7].

To prove our result, we need the following lemma.

Lemma 1.2. (see [8]) *If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is a function with positive real part in Δ, then for any complex number μ,*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |1 - 2\mu|\}.$$

2. Coefficient Inequality

By making use of Lemma 1.2, we prove the following bounds for the class $MR_\alpha^*(\phi)$.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1) belongs to $MR_\alpha^*(\phi)$, then for any complex number μ,*

$$(i) \quad |a_1 - \mu a_0^2| \leq \left|\frac{B_1}{2(1-\alpha)}\right| \max\left\{1, \left|\frac{B_2}{B_1} - (1 - 2(1-\alpha)\mu)\, B_1\right|\right\}, \quad B_1 \neq 0, \tag{2}$$

$$(ii) \quad |a_1 - \mu a_0^2| \leq \left|\frac{1}{(1-\alpha)}\right|, \quad B_1 = 0. \tag{3}$$

The bounds obtained are sharp.

Proof. If $f(z) \in MR_\alpha^*(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\frac{-(1-2\alpha)zf'(z) + \alpha z^2 f''(z)}{f(z)} = \phi(w(z)) \quad (\alpha \in \mathbb{C} - (0,1],\ \Re(\alpha) \geq 0). \tag{4}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p(z)) > 0$ and $p(0) = 1$. Therefore

$$\begin{aligned}
\phi(w(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\
&= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right) z^3 + \cdots\right]\right) \\
&= 1 + \frac{1}{2} B_1 c_1 z + \left(\frac{1}{2} B_1 \left(c_2 - \frac{1}{2} c_1^2\right) + \frac{1}{4} B_2 c_1^2\right) z^2 + \cdots.
\end{aligned} \tag{5}$$

Now using (5) and (1) in (4), and comparing the coefficients, we have

$$a_0 + \frac{1}{2}B_1c_1 = 0,$$

$$-a_1(1-2\alpha) = a_1 + \frac{1}{2}a_0B_1c_1 + \frac{1}{2}B_1c_2 - \frac{1}{4}(B_1-B_2)c_1^2;$$

or equivalently,

$$a_0 = -\frac{1}{2}B_1c_1,$$

$$a_1 = -\frac{1}{2(1-\alpha)}\left(\frac{1}{2}B_1c_2 + \frac{1}{4}(B_2-B_1-B_1^2)c_1^2\right).$$

Therefore,

$$a_1 - \mu a_0^2 = -\frac{B_1}{4(1-\alpha)}\left\{c_2 - vc_1^2\right\},$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + (1-2\mu(1-\alpha))\,B_1\right].$$

Now, the result (2) follows by an application of Lemma 1.2. Also, if $B_1 = 0$, then $a_0 = 0$ and $a_1 = \left(-\frac{1}{8(1-\alpha)}\right)B_2c_1^2$. Since $p(z)$ has positive real part, $|c_1| \leq 2$, so that $|a_1 - \mu a_0^2| \leq |\frac{B_2}{2(1-\alpha)}|$. Since $\phi(z)$ also has positive real part, $|B_2| \leq 2$. Thus $|a_1 - \mu a_0^2| \leq \left|\frac{1}{(1-\alpha)}\right|$, proving (3).

The bounds are sharp for the functions $F_1(z)$ and $F_2(z)$ defined by

$$\frac{-(1-2\alpha)zF_1'(z) + \alpha z^2F_1''(z)}{F_1(z)} = \phi(z^2),\ \text{where } F_1(z) = \frac{1+z^2}{z(1-z^2)},$$

and

$$\frac{-(1-2\alpha)zF_2'(z) + \alpha z^2F_2''(z)}{F_2(z)} = \phi(z),\ \text{where } F_2(z) = \frac{1+z}{z(1-z)}.$$

Clearly $F_1(z), F_2(z) \in \Sigma$. □

For $\alpha = 0$, in Theorem 2.1, we get the following result obtained by Silverman et al [15].

Corollary 2.2. *Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$. If $f(z)$ given by (1) belongs to $\Sigma^*(\phi)$, then for any complex number μ,*

$$(i)\quad |a_1 - \mu a_0^2| \leq \frac{|B_1|}{2}\max\left\{1, \left|\frac{B_2}{B_1} - (1-2\mu)B_1\right|\right\},\quad B_1 \neq 0, \tag{6}$$

and

$$(ii)\qquad |a_1 - \mu a_0^2| \leq 1,\quad B_1 = 0. \tag{7}$$

The result is sharp.

3. Applications to Functions Defined by Ruscheweyh Derivatives

In this section, we introduce the class $MR^*_{\alpha,\lambda}(\phi)$ of meromorphic functions defined by Ruscheweyh derivatives and obtain coefficient bounds for functions in this class.

Let $f \in \Sigma$ be given by (1) and $g \in \Sigma$ be given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k,$$

then the Hadamard product of f and g is defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).$$

In terms of the Hadamard product of two functions, the analogue of the familiar Ruscheweyh derivative [14] is defined as

$$D^\lambda f(z) := \frac{1}{z(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1; f \in \Sigma), \tag{8}$$

so that

$$D^\lambda f(z) = \frac{1}{z}\left(\frac{z^{\lambda+1} f(z)}{\lambda!}\right)^{(\lambda)} \quad (\lambda > -1; f \in \Sigma). \tag{9}$$

Here and in what follows λ is an integer (> -1), that is $\lambda \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

It follows from (8) and (9) that

$$D^\lambda f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \delta(\lambda, k), a_k z^k \quad (f \in \Sigma), \tag{10}$$

where $f \in \Sigma$ is given by (1) and

$$\delta(\lambda, k) := \binom{\lambda + k + 1}{k + 1}.$$

The above defined operator D^λ for $\lambda \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ was also studied by Cho [3] and Padmanabhan [11]. For various developments involving the operator D^λ for functions belonging to Σ, the reader may be referred to the recent works of Uralegaddi et al [5, 16, 17] and others [1, 2, 6].

Using (10), under the same conditions as Definition 1.1, we define the class $MR^*_{\alpha,\lambda}(\phi)$ as follows.

Definition 3.1. A function $f \in \Sigma$ is in the class $MR^*_{\alpha,\lambda}(\phi)$ if

$$\frac{-(1-2\alpha)z[D^\lambda f(z)]' + \alpha z^2[D^\lambda f(z)]''}{[D^\lambda f(z)]} \prec \phi(z),$$

$$(z \in \Delta;\ \alpha \in \mathbb{C} - (0,1];\ \Re(\alpha) \geq 0).$$

For the class $MR^*_{\alpha,\lambda}(\phi)$, using methods similar to those in the proof of Theorem 2.1, we obtain the following results.

Theorem 3.2. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1) belongs to $MR^*_{\alpha,\lambda}(\phi)$, then for any complex number μ,*

$$(i)\quad |a_1 - \mu a_0^2| \leq \left|\frac{B_1}{(1-\alpha)(\lambda+1)(\lambda+2)}\right| \max\left\{1, \left|\frac{B_2}{B_1} - \left(1 - \frac{(1-\alpha)(\lambda+2)}{\lambda+1}\mu\right) B_1\right|\right\}, \quad B_1 \neq 0, \tag{11}$$

$$(ii)\quad |a_1 - \mu a_0^2| \leq \left|\frac{2}{(1-\alpha)(\lambda+1)(\lambda+2)}\right|, \quad B_1 = 0. \tag{12}$$

The bounds are sharp.

Remark 3.3. For $\lambda = 0$ in (11), (12), we get the results (2) and (3) respectively. Further, when $\lambda = \alpha = 0$, in (11), (12), we get the results (6), (7) of Silverman et al [15].

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