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THE GENERALIZED NONLINEAR HEAT EQUATION AND ITS SPECTRUM

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Abstract: In this paper, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t) + c^2(-\circledast)^k u(x,t) = f(x,t,u(x,t)),$$

where $\circledast^k$ is the operator iterated k-times, defined by

$$\circledast^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k,$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, k is a positive integer and c is a positive constant, f is the given function in nonlinear form depending on x,t and $u(x,t)$. On suitable conditions for f, p, q, k and the spectrum, we obtain the unique solution $u(x,t)$ of such equation. Moreover, if we put $q=0, k=1$, we obtain the solution of non-linear heat equation.

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1. Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x),$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y)dy \tag{1.2}$$

as the solution of (1.1).

Now, (1.2) can be written $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{1.3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [2], pp. 208-209.

In 1996, A. Kananthai [3] has introduced the diamond operator $\diamondsuit$ defined by

$$\diamondsuit = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \quad p+q = n,$$

or $\diamondsuit$ can be written as the product of the operators in the form $\diamondsuit = \triangle\Box = \Box\triangle$, where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourier transform of the diamond operator also has been studied and the elementary solution of such operator, see [4].

Next, K. Nonlaopon and A. Kananthai (see [5]) study the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t).$$

In this paper, we study the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) + c^2(-\circledast)^k u(x,t) = f(x,t,u(x,t)). \tag{1.4}$$

The operator $\circledast^k$ can be expressed in the form

$$
\begin{aligned}
\circledast^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\right. \\
&\quad \left.\cdot\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \triangle^k\left(\triangle^2 - \frac{3}{4}(\triangle + \Box)(\triangle - \Box)\right)^k = \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k,
\end{aligned}
$$

where

$$
\begin{aligned}
\triangle &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}, \\
\Box &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}, \\
\Diamond &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2,
\end{aligned}
$$

which is in the form of nonlinear heat equation. We consider the equation (1.4) with the following conditions on u and f as follows:

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with $6k$-derivatives.

(2) f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant with $0 < A < 1$.

(3) $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f and u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution of (1.4) where $E(x,t)$ is an elementary solution of (1.4).

2. Preliminaries

Definition 2.1. Let $f(x) \in L_1(\mathbb{R}^n)$ be the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$. Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f(\xi)} d\xi. \tag{2.2}$$

If f is a distribution with compact supports by [6], Theorem 7.4-3, p. 187 equation (2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{2.3}$$

Definition 2.2. The spectrum of the kernel $E(x,t)$ defined by (2.6) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 2.3. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and write

$$u = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by $\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and denote by $\overline{\Gamma}_+$ the closure of Γ_+. Let Ω be the spectrum of $E(x,t)$ for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[-c^2 t \left(\left(\displaystyle\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\displaystyle\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k \right], & \text{for } \xi \in \Gamma_+, \\ 0, & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.4}$$

Lemma 2.1. (The Fourier Transform of $(-\circledast)^k \delta$)

$$\mathcal{F}(-\circledast)^k \delta = \frac{(-1)^{4k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 \right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 \right)^3 \right]^k,$$

where $\mathcal{F}$ is the Fourier transform defined by equation (2.1) and if the norm of

ξ is given by $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$ then

$$\mathcal{F}(-\circledast)^k\delta \leq \frac{3}{(2\pi)^{n/2}}\|\xi\|^{6k},$$

that is $\mathcal{F}(-\circledast)^k$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by equation (2.2)

$$(-\circledast)^k\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3\right]^k.$$

Proof. By equation (2.3)

$$\begin{aligned}
\mathcal{F}(-\circledast)^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle(-\circledast)^k\delta, e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^k e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}(-\circledast)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{3}{4}\Diamond\Box - \frac{1}{4}\triangle^3\right)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{3}{4}\Diamond\Box\right)e^{-i(\xi.x)}\right\rangle \\
&+ \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{1}{4}\Box^3\right)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{3}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\right. \\
&\quad \left.\cdot\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)e^{-i(\xi.x)}\right\rangle \\
&+ \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{1}{4}(-1)^4\left(\sum_{i=1}^{n}\xi_i^2\right)^3 e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{3}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\right. \\
&\quad \left.\cdot\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right) + (\frac{1}{4}(-1)^4\left(\sum_{i=1}^{n}\xi_i^2\right)^3 e^{-i(\xi.x)}\right\rangle
\end{aligned}$$

$$= \frac{(-1)^4}{(2\pi)^{n/2}} \left\langle \delta, (-\circledast)^{k-1} \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) e^{-i(\xi.x)} \right\rangle.$$

By keeping on operator $(-\circledast)$ with $k-1$ times, we obtain

$$\mathcal{F}(-\circledast)^k \delta = \frac{(-1)^{4k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right]^k.$$

Now,

$$\begin{aligned} |\mathcal{F}(-\circledast)^k \delta| &= \frac{1}{(2\pi)^{n/2}} \left| \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right|^k \\ &\leq \frac{1}{(2\pi)^{n/2}} \left| \xi_1^2 + \ldots + \xi_n^2 \right|^k \left| 3 \left(\xi_1^2 + \ldots + \xi_n^2\right)^2 \right|^k \leq \frac{3}{(2\pi)^{n/2}} \|\xi\|^{6k}, \end{aligned}$$

where $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$, $\xi_i (i = 1, 2, \ldots, n) \in \mathbb{R}$. Hence we obtain $\mathcal{F}(-\circledast)^k \delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.2)

$$\circledast \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right].$$

That completes the proof. □

Lemma 2.2. *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} + c^2 (-\circledast)^k, \tag{2.5}$$

where $\circledast^k$ is the operator iterated k-times defined by

$$\circledast^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k,$$

$p+q = n$ is the dimension of $\mathbb{R}^n$ $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, k is a positive integer and c is the positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \times \int_\Omega \exp \left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k + i(\xi, x) \right] d\xi \tag{2.6}$$

as the elementary solution of (1.4) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $LE(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of the operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) + c^2(-\circledast)^k E(x,t) = \delta(x)\delta(t)$$

takes the Fourier transform defined by (2.1) to both sides of the equation

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} + c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right],$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right],$$

so we have

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi.$$

By (2.3),

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi,$$

where Ω is the spectrum of $E(x,t)$. Thus

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k + i(\xi,x)\right]d\xi.$$

for $t > 0$. □

Definition 2.4. We can extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n}\displaystyle\int_{\Omega}\exp\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right], & \text{for } t > 0, \\ 0, & \text{for } t \le 0. \end{cases}$$

Lemma 2.3. (The Properties of $E(x,t)$) *The kernel $E(x,t)$ defined by (2.6) have the following properties:*

(1) $E(x,t) \in C^{\infty}$ is the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely ditterentiable.

(2) $\left(\frac{\partial}{\partial t} + c^2(-\circledast)^k\right) E(x,t) = 0$ for $t > 0$.

(3) $|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(p/2)\Gamma(q/2)}$ for $t > 0$ where $M(t)$ is a function of t in the spectrum and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(4) $\lim\limits_{t \to 0} E(x,t) = \delta$.

Proof. (1) From (2.6)

$$\frac{\partial^n}{\partial x^n} E(x,t)$$
$$= \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k + i(\xi, x)\right] d\xi.$$

Thus $E(x,t) \in C^{\infty}$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} + c^2(-\circledast)^k\right) E(x,t) = 0.$$

(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k + i(\xi, x)\right] d\xi.$$

Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[-c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k\right] d\xi.$$

By changing to bipolar coordinates $\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$ and $\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_{p+q} = sw_{p+q}$, where

$$\sum_{i=1}^{p} w_i^2 = 1 \quad \text{and} \quad \sum_{j=p+1}^{p+q} w_j^2 = 1.$$

Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[-c^2 t \left(r^6 + s^6\right)^k\right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and suppose $0 \le r \le R$ and $0 \le s \le L$, where R and L are constants. Thus we obtain

$$
\begin{aligned}
|E(x,t)| &\le \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp\left[-c^2 t\left(r^6+s^6\right)^k\right] r^{p-1}s^{q-1}drds \\
&= \frac{\Omega_p \Omega_q}{(2\pi)^n} M(t) \qquad \text{for any fixed } t>0 \text{ in the spectrum } \Omega \\
&= \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(p/2)\Gamma(q/2)},
\end{aligned} \tag{2.7}
$$

where $M(t) = \int_0^R \int_0^L \exp\left[-c^2 t\left(r^6+s^6\right)^k\right] r^{p-1}s^{q-1}drds$ is a function for $t>0$, $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus for any fixed $t>0$, $E(x,t)$ is bounded.

(4) From (2.5),

$$
\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{i(\xi,x)}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}d\xi = \delta(x),
$$

for $x \in \mathbb{R}^n$, see [1], p. 396, equation (10.2.19b). □

3. Main Results

Theorem 3.1. *Given the nonlinear equation*

$$
\frac{\partial}{\partial t}u(x,t) + c^2(-\circledast)^k u(x,t) = f(x,t,u(x,t)) \tag{3.1}
$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, k is a positive number and with the following conditions on u and f as follows:

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t>0$, where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with $6k$-derivative.

(2) f satisfies the Lipchitz condition,

$$
|f(x,t,u) - f(x,t,w)| \le A|u-w|,
$$

where A is constant with $0<A<1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x=(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, $0<t<\infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then obtain the convolution

$$
u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.2}
$$

as a unique solution of (3.1) for $x \in \Omega$, where Ω is a compact subset of $\mathbb{R}^n$ and $0 \le t \le T$ with T is constant and $E(x,t)$ is an elementary solution defined by (2.6) and also $u(x,t)$ is bounded for any fixed $t > 0$. In particular, if we put $k = 1$ and $q = 0$ in (3.1), then (3.1) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t) - c^2\triangle^3 u(x,t) = f(x,t,u(x,t))$$

which is relate to the heat equation.

Proof. Convolving both sides of (3.1) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial}{\partial t}u(x,t) + c^2(-\circledast)^k u(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial}{\partial t}E(x,t) + c^2(-\circledast)^k E(x,t)\right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$\begin{aligned} u(x,t) &= E(x,t) * f(x,t,u(x,t)) \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s)f(x-r,t-s,u(x-r,t-s))drds, \end{aligned}$$

where $E(r,s)$ is given by definition (2.5). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds \\ &\le \frac{2^{2-n}NM(t)}{\pi^{n/2}\Gamma(p/2)\Gamma(q/2)} \qquad \text{by condition (3) and (2.6),} \end{aligned}$$

where $N = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. To show that $u(x,t)$ is unique. Next we show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (3.1), then

$$w(x,t) = E(x,t) * f(x,t,w(x,t))$$

for $(x,t) \in \Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $E(x,t)$ is defined by (2.6).

Now, define $\|u(x,t)\| = \sup\limits_{\substack{x\in\Omega_0 \\ 0<t\le T}} |u(x,t)|$.

Now,

$$\begin{aligned}
|u(x,t)-w(x,t)| &= |E(x,t)*f(x,t,u(x,t))-E(x,t)*f(x,t,w(x,t))| \\
&\leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)|\cdot|f(x-r,t-s,u(x-r,t-s)) \\
&\qquad -f(x-r,t-s,w(x-r,t-s))|drds \\
&\leq A|E(r,s)|\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}|u(x-r,t-s)-w(x-r,t-s)|drds
\end{aligned}$$

by (2.6) and the condition (2) of the theorem. Now, for $(x,t)\in\Omega_0\times(0,T]$ we have

$$\begin{aligned}
|u-w| &\leq A|E(r,s)|\|u-w\|\int_0^T ds\int_{\Omega_0} dr \\
&= A|E(r,s)|TV(\Omega_0)\|u-w\|, \qquad (3.3)
\end{aligned}$$

where $V(\Omega_0)$ is the volume of the surface on Ω_0.

Choose $A|E(r,s)|TV(\Omega_0)\leq 1$ or $A\leq\frac{1}{|E(r,s)|TV(\Omega_0)}$. Thus from (3.3),

$$\|u-w\|\leq\alpha\|u-w\|, \quad \text{where} \quad \alpha=A|E(r,s)|TV(\Omega_0)\leq 1.$$

It follows that $\|u-w\|=0$, thus $u=w$. That is the solution u of (3.1) is unique. In particular, if we put $k=1$ and $q=0$ in (3.1), then (3.1) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t)-c^2\triangle^3 u(x,t)=f(x,t,u(x,t))$$

which has solution

$$u(x,t)=E(x,t)*f(x,t,u(x,t)),$$

where $E(x,t)$ is defined by (2.6) with $k=1$ and $q=0$. □

Acknowledgments

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