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RECENT DEVELOPMENTS IN THE THEORY OF ALPHA-CALCULUS

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Abstract: We present new and more general results related to a fractional calculus introduced in four earlier papers. That alpha-calculus permits to construct a generalized physics, with an extension to alpha-spaces of the famous formula $E = mc^2$. The study is also made as a function of the parameter α.

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1. Introduction

A new fractional calculus is introduced in the papers [1, 2, 3, 4]. The α-derivative is defined by

$$b_\alpha(f(z);z) = \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k!} B_{k,\alpha} f^{(k)}(z), \tag{1}$$

where $B_{k,\alpha} := B_{k,\alpha}(0)$ and $B_{k,\alpha}(x)$ is the α-Bernoulli polynomial of degree k, with generating function

$$\frac{\exp((x-\frac{1}{2})z)}{g_\alpha(\frac{iz}{2})} = \sum_{k=0}^{\infty} \frac{B_{k,\alpha}(x)}{k!} z^k, \tag{2}$$

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where $g_\alpha(z) := 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}$. The α-derivative of order n is

$$b_\alpha^{(n)}(f(z);z) = \sum_{k=n}^{\infty} \frac{d_{k,\alpha}^{(n)}}{k!} f^{(k)}(z), \tag{3}$$

where $d_{k,\alpha}^{(1)} = \big((-1)^k - 1\big)B_{k,\alpha}$ and, for $n > 1$,

$$d_{k,\alpha}^{(n)} = \sum_{\ell=0}^{k} \binom{k}{\ell} \big((-1)^{k-\ell} - 1\big) B_{k-\ell,\alpha} d_{\ell,\alpha}^{(n-1)}. \tag{4}$$

We put $d_{0,\alpha}^{(0)} = 1$ and $d_{n,\alpha}^{(0)} = 0$, $n \geq 1$.

The fundamental polynomials $\phi_{n,\alpha}(z)$ can be introduced through the recurrence relation $\phi_{0,\alpha}(z) = 1$ ($\phi_{1,\alpha}(z) = z, \phi_{2,\alpha}(z) = z^2$) and, for $n \geq 1$,

$$\phi_{n,\alpha}(z) = z^n - \sum_{k=0}^{n-1} \frac{d_{n,\alpha}^{(k)}}{k!} \phi_{k,\alpha}(z). \tag{5}$$

The aim of this paper is to present new results related to the fractional calculus constructed from (1). The corresponding α-integral has several representations (see Section 3). A study is made as a function of the parameter α in Section 4. The α-physics proposed in Section 5 contains an α-extension of the formula $E = mc^2$; see also Proposition 5.1.1. Many special results are presented in Section 6.

2. General Fractional Calculus

In this section, we briefly compare our α-calculus with two of the main fractional calculus.

The Riemann–Liouville fractional integral is defined by

$$I_a^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)\,\mathrm{d}t, \tag{6}$$

where $\beta > 0$, $-\infty < a < \infty$ and $x > a$. The Riemann-Liouville fractional derivative is

$$D_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x), \tag{7}$$

where D is the ordinary derivative, $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. We have $D_a^1 f(x) = f'(x)$. An extensive study of (6) and (7) is made in [7].

The q-derivative (here we use the latin letter q and reserve greek letters for our alpha-calculus) of a function f is, for $|q| < 1$,

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{8}$$

with $D_1 f(x) = f'(x)$. The q-integral (also known as Jackson's integral) is defined by

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} q^n f(aq^n) \tag{9}$$

with

$$\int_0^a f(t) d_1 t = \int_0^a f(t)\, \mathrm{d}t.$$

The papers [5] and [6] contain some basic facts concerning (8) and (9). Other references are rapidly found with the internet.

It is interesting to examine in what extent the basic theorems of differential and integral calculus can be extended with generalized derivatives and integrals. An important result extended in both the Riemann–Liouville calculus and the q-calculus is Taylor's formula. See for examples [10] and [6]. The α-extension of Taylor's formula is an important result in [1]; see [2, p. 333] for the formula in two variables.

However, many classical theorems have not been generalized in the context of Riemann–Liouville or q-calculus. We are unable to find references for an extension of Fubini's Theorem. The curvilinear or surface integrals are not defined. Consequently, no version of Green–Riemann, Gauss and Stokes Theorems seems to be available. In complex variables, no version of Cauchy's Theorem is presented. All these theorems have a natural extension in the alpha-calculus.

3. The Alpha-Integral

Initially, the α-integral was defined by [1]

$${}_\alpha\!\int_a^b f(t)\, \mathrm{d}t = \sum_{n=1}^{\infty} \frac{b_\alpha^{(n-1)}(f(u); u=a)}{n!} \phi_{n,\alpha}(b-a). \tag{10}$$

We also have [2]

$${}_\alpha\!\int_a^b f(t)\, \mathrm{d}t = \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \int_a^b b_\alpha^{(p-1)}(f(t); t)\, \mathrm{d}t, \tag{11}$$

where

$$\psi_{p,\alpha} := \phi'_{p,\alpha}(0). \tag{12}$$

For an interval of the form $(a-\frac{1}{2}, a+\frac{1}{2})$ we have [3]

$$ {}_\alpha\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,\mathrm{d}t = \sum_{\ell=0}^{\infty} \frac{\Gamma(\alpha+1) f^{(2\ell)}(a)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)}. \tag{13}$$

In general, the additivity property

$$ {}_\alpha\!\int_a^b f(t)\,\mathrm{d}t = {}_\alpha\!\int_a^\infty f(t)\,\mathrm{d}t - {}_\alpha\!\int_b^\infty f(t)\,\mathrm{d}t$$

gives

$$ {}_\alpha\!\int_a^b f(t)\,\mathrm{d}t = {}_\alpha\!\int_0^\infty \big(f(t+a)-f(t+b)\big)\,\mathrm{d}t = \sum_{j=0}^{\infty} {}_\alpha\!\int_0^1 \big(f(t+j+a)-f(t+j+b)\big)\,\mathrm{d}t.$$

It follows from (13) that

Proposition 3.1. *We have the formula*

$$ {}_\alpha\!\int_a^b f(t)\,\mathrm{d}t = \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)} \big(f^{(2\ell)}(a+\tfrac{1}{2}+j) - f^{(2\ell)}(b+\tfrac{1}{2}+j)\big). \tag{14}$$

The representation [3]

$$ {}_\alpha\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,\mathrm{d}t = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} (1-4(t-a)^2)^{\alpha-\frac{1}{2}} f(t)\,\mathrm{d}t \tag{15}$$

is valid when $\mathrm{Re}(\alpha) > -\frac{1}{2}$. For example, we have

$$ {}_\alpha\!\int_0^1 t^\lambda\,\mathrm{d}t = \frac{2^{2\alpha}\Gamma(\alpha+1)\Gamma(\alpha+\lambda+\frac{1}{2})}{\sqrt{\pi}\Gamma(2\alpha+\lambda+1)}. \tag{16}$$

Formula (14), with the help of the Euler–MacLaurin formula

$$\sum_{n=1}^{\infty} \frac{B_n(u)}{n!} f^{(n-1)}(0) = \int_0^\infty f(t)\,\mathrm{d}t - \sum_{j=0}^{\infty} f(u+j), \tag{17}$$

gives the

Proposition 3.2. *We have the formula*

$$\begin{aligned} {}_\alpha\!\int_a^b f(t)\,\mathrm{d}t \\ = \int_a^b f(t)\,\mathrm{d}t + \sum_{\ell=1}^{\infty}\sum_{k=0}^{\ell} \frac{\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})(f^{(2\ell-1)}(b)-f^{(2\ell-1)}(a))}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!}. \end{aligned} \tag{18}$$

The preceding formula can also be obtained from (11) using the relation

$$\sum_{k=0}^{2\ell} \frac{\psi_{k+1,\alpha}}{(k+1)!} d_{2\ell,\alpha}^{(k)} = \sum_{k=0}^{\ell} \frac{(2\ell)!\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!}. \qquad (19)$$

This relation is interesting because it gives a link between typical quantities of alpha-calculus and the ordinary Bernoulli polynomials. It is a direct consequence of the

Lemma 3.2.1. *We have the relation*

$$\sum_{k=0}^{n} \frac{d_{n,\alpha}^{(k)}}{(k+1)!} \big(\phi_{k+1,\alpha}(b) - \phi_{k+1,\alpha}(a)\big)$$
$$= \sum_{\ell=0}^{[\frac{n}{2}]} \frac{n!\Gamma(\alpha+1)(B_{n-2\ell+1}(b+\frac{1}{2}) - B_{n-2\ell+1}(a+\frac{1}{2}))}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell+1)!}. \qquad (20)$$

The relation (20) is obtained by comparing the formula (105) of [1] and (18) of [4]. Applying the α-derivative on both sides of (20), with respect to b, and using [2, p. 341] $b_\alpha(B_n(z); z) = nB_{n-1,\alpha}(z)$, we obtain the recurrence relation [4, p. 5]

$$\sum_{\ell=0}^{[\frac{n}{2}]} \frac{n!\Gamma(\alpha+1)B_{n-2\ell,\alpha}(b)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell)!} = (b-\tfrac{1}{2})^n. \qquad (21)$$

The formula (102) of [2] is the case $n = 2m$, $a = 0$, $b = \frac{1}{2}$ of (20). A similar formula is obtained if we take $n = 2m+1$, $a = 0$, $b = \frac{1}{2}$:

$$\sum_{k=0}^{m} \frac{d_{2m+1,\alpha}^{(2k+1)}}{(2k+1)!} \phi_{2k+1,\alpha}\left(\frac{1}{2}\right) = \frac{(2m+1)!\Gamma(\alpha+1)}{2^{4m+1}m!\Gamma(\alpha+m+1)}. \qquad (22)$$

Remark 3.1. We have the asymptotic formula

$$\sum_{k=0}^{\ell} \frac{\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!} \sim \frac{2g_\alpha(\pi)}{(2\pi i)^{2\ell}} \qquad (23)$$

as $\ell \to \infty$. Also, it can be shown that

$$\sum_{k=0}^{\ell} \frac{(2\ell)!\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!} = {}_\alpha\!\int_0^1 B_{2\ell}(x)\,\mathrm{d}x. \qquad (24)$$

A representation analogous to (18) is given in [4, formula (16)]. We have

$${}_\alpha\!\int_a^b f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \Gamma(\alpha+1) \frac{(B_{n-2\ell+1}(b+\frac{1}{2}) - B_{n-2\ell+1}(a+\frac{1}{2}))}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell+1)!} f^{(n)}(0). \qquad (25)$$

In fact, a little work shows that (18) and (25) are equivalent by way of the addition formula $B_N(x+y) = \sum_{j=0}^{N} \binom{N}{j} B_{N-j}(x) y^j$.

Simple expressions for the α-integral are available for certain values of the parameter α. For instances, we have

$$ {}_{\infty}\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,\mathrm{d}t = f(a) \tag{26}$$

and

$$ {}_{-\frac{1}{2}}\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,\mathrm{d}t = \frac{1}{2}\big(f(a+\tfrac{1}{2}) + f(a-\tfrac{1}{2})\big). \tag{27}$$

A more general expression for the ∞-integral is [4, formula (19)]

$$ {}_{\infty}\!\int_{a}^{b} f(t)\,\mathrm{d}t = \sum_{j=0}^{\infty}\big(f(a+\tfrac{1}{2}+j) - f\big(b+\tfrac{1}{2}+j\big)\big). \tag{28}$$

The representation (27) is the particular case $p=0$ of [3, formula (50)]

$$ {}_{-p-\frac{1}{2}}\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,\mathrm{d}t = \frac{1}{2}\sum_{j=0}^{p}\binom{p}{j}\frac{(2p-j)!}{(2p)!}\big(f^{(j)}(a-\tfrac{1}{2}) + (-1)^j f^{(j)}(a+\tfrac{1}{2})\big). \tag{29}$$

It is also a consequence of [4, formula (20)]

$$ {}_{-\frac{1}{2}}\!\int_{a}^{b} f(t)\,\mathrm{d}t = \tfrac{1}{2}\big(f(a) - f(b)\big) + \sum_{j=1}^{\infty}\big(f(a+j) - f(b+j)\big). \tag{30}$$

We present a result that contains both (29) and (30).

Proposition 3.3. *For each integer $p \geq 0$, we have*

$$ {}_{-p-\frac{1}{2}}\!\int_{a}^{b} f(t)\,\mathrm{d}t = \sum_{j=0}^{[\frac{p}{2}]}\sum_{\nu=1}^{\infty}\binom{p}{2j}\frac{(2p-2j)!}{(2p)!}\big(f^{(2j)}(a+\nu) - f^{(2j)}(b+\nu)\big) + \frac{1}{2}\sum_{k=0}^{p}\binom{p}{k}\frac{(2p-k)!}{(2p)!}\big(f^{(k)}(a) - f^{(k)}(b)\big). \tag{31}$$

Proof. The additivity property gives

$$ {}_{-p-\frac{1}{2}}\!\int_{a}^{b} f(t)\,\mathrm{d}t = \sum_{\nu=0}^{\infty} {}_{-p-\frac{1}{2}}\!\int_{0}^{1}\big(f(t+a+\nu) - f(t+b+\nu)\big)\,\mathrm{d}t. \tag{32}$$

The relation (29) is equivalent to

$$ {}_{-p-\frac{1}{2}}\int_0^1 F(t)\,\mathrm{d}t = \frac{1}{2}\sum_{k=0}^{p}\binom{p}{k}\frac{(2p-k)!}{(2p)!}\big(F^{(k)}(0)+(-1)^k F^{(k)}(1)\big). \tag{33} $$

We apply (33) to the function $F(t) = f(t+a+\nu) - f(t+b+\nu)$ and substitute in (32). We get

$$ {}_{-p-\frac{1}{2}}\int_a^b f(t)\,\mathrm{d}t = \frac{1}{2}\sum_{\nu=0}^{\infty}\sum_{k=0}^{p}\binom{p}{k}\frac{(2p-k)!}{(2p)!}\big(\big(f^{(k)}(a+\nu) - f^{(k)}(b+\nu)\big) + (-1)^k\big(f^{(k)}(a+\nu+1) - f^{(k)}(b+\nu+1)\big)\big). \tag{34} $$

The result follows after elementary transformations of the summations. ☐

A consequence of (31) and (28) (with Stirling's formula and a Taylor's expansion) is

$$ {}_{-\infty}\int_a^b f(t)\,\mathrm{d}t = {}_{\infty}\int_a^b f(t)\,\mathrm{d}t. \tag{35} $$

The formula (19) can also be written differently in the spherical case $\alpha = p+\frac{1}{2}$ or $\alpha = -p-\frac{1}{2}$, $p = 0,1,2,\ldots$. In order to do that, we use two relations which are independent properties of Bernoulli polynomials.

Proposition 3.4. *If α is a number of the form $\alpha = -p-\frac{1}{2}$, $p = 0,1,2,\ldots$, then we have*

$$ \sum_{k=0}^{\ell}\frac{(2\ell)!\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!} = \sum_{\nu=0}^{p}\frac{p!(2p-\nu)!}{(2p)!(p-\nu)!}\binom{2\ell}{\nu}B_{2\ell-\nu}. \tag{36} $$

The foregoing relation can be proved by mathematical induction. Note that $p = 0$ is a consequence of the addition formula for Bernoulli polynomials.

Proposition 3.5. *If α is a number of the form $\alpha = p+\frac{1}{2}$, $p = 0,1,2,\ldots$, then we have*

$$ \sum_{k=0}^{\ell}\frac{\Gamma(\alpha+1)B_{2\ell-2k}(\frac{1}{2})}{2^{4k}k!\Gamma(\alpha+k+1)(2\ell-2k)!} = 2(-1)^{p+1}\frac{(2p+1)!}{p!}\sum_{j=0}^{p}\frac{(2j)!\binom{p+j}{2j}}{j!(2\ell+p+j+1)!}B_{2\ell+p+j+1}. \tag{37} $$

The relation (37) is the special case $x = -\frac{1}{2}$ of the following identity.

Proposition 3.6. *For $\ell = 0,1,2,\ldots$ and $p = 0,1,2,\ldots$, we have*

$$\sum_{k=0}^{\ell} \frac{2^p(k+p)!B_{2\ell-2k}(\frac{1}{2})}{k!(2k+2p+1)!(2\ell-2k)!}x^{2k+2p+1}$$
$$= \sum_{j=0}^{p} \frac{(-1)^j(2j)!\binom{p+j}{2j}x^{p-j}}{2^j j!(2\ell+p+j+1)!}B_{2\ell+p+j+1}(x+\tfrac{1}{2}). \quad (38)$$

Proof. The case $p = 0$ follows from the addition formula for Bernoulli polynomials. Assume that (38) is valid for a given $p > 0$. We multiply both sides by x and integrate between 0 and x. We then integrate by parts $(p-j+1)$ times the integral in the right-hand member. We obtain

$$\sum_{k=0}^{\ell} \frac{2^{p+1}(k+p+1)!B_{2\ell-2k}(\frac{1}{2})}{k!(2k+2p+3)!(2\ell-2k)!}x^{2k+2p+3}$$
$$= \sum_{j=0}^{p} \sum_{r=0}^{p-j+1} \frac{(-1)^j(2j)!\binom{p+j}{2j}(-1)^r(p+j+1)!B_{2\ell+p+j+r+2}(x+\frac{1}{2})}{2^j j!(p-j+1-r)!(2\ell+p+j+r+2)!}$$
$$\times x^{p-j+1-r}. \quad (39)$$

Now we replace r by $(r-j)$ in (39) and permute the order of summation. The result follows by using the combinatorial identity (which is very easy to prove by mathematical induction)

$$\sum_{r=0}^{j} \frac{(p-r+1)(p+r)!}{2^r r!} = \frac{(p+j+1)!}{2^j j!}, \quad \text{for } 0 \leq j \leq p+1. \quad (40)$$

□

If we divide both members of (38) by $x^{2\ell+2p+1}$, and let $x \to \infty$, then we obtain

$$\sum_{j=0}^{p} \frac{(-1)^j(2j)!\binom{p+j}{2j}}{2^j j!(2\ell+p+j+1)!} = \frac{2^p(\ell+p)!}{\ell!(2\ell+2p+1)!}. \quad (41)$$

4. The Parameter Alpha

In the papers [1, 3, 4] several results are given in terms of the parameter α. Here we present new and more general results of the same kind.

4.1.

In this subsection, we prove some formulas for expressions of the form $Q_{\alpha,m}$, where

$$Q_{\alpha,m} := \frac{\partial^{m-1} Q_\alpha}{\partial \alpha^{m-1}}. \tag{42}$$

Proposition 4.1.1. *For $m = 2, 3, 4, \ldots$, we have*

$$\begin{aligned}
&\lim_{\alpha\to\infty} \alpha^m \,{}_{\alpha,m}\!\int_a^b f(t)\,\mathrm{d}t \\
&\quad = \frac{(-1)^{m-1}}{16}(m-1)! \sum_{\ell=0}^{\infty} \frac{B_{2\ell}(\frac{1}{2})}{(2\ell)!}\big(f^{(2\ell+1)}(b) - f^{(2\ell+1)}(a)\big) \\
&\quad = \frac{(-1)^{m-1}}{16}(m-1)! \sum_{j=0}^{\infty} \big(f''(a+j+\tfrac{1}{2}) - f''(b+j+\tfrac{1}{2})\big).
\end{aligned} \tag{43}$$

Proof. The partial fraction expansion

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{(j-1)!(k-j)!(\alpha+j)}$$

and (18) give, for $m > 1$,

$$\begin{aligned}
&{}_{\alpha,m}\!\int_a^b f(t)\,\mathrm{d}t \\
&= \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} \sum_{j=1}^{k} \frac{(-1)^{j-1} B_{2\ell-2k}(\frac{1}{2})(-1)^{m-1}(m-1)!(f^{(2\ell-1)}(b) - f^{(2\ell-1)}(a))}{(j-1)!(k-j)!2^{4k}k!(2\ell-2k)!(\alpha+j)^m}.
\end{aligned} \tag{44}$$

The relation $\sum_{j=1}^{k}(-1)^{j-1}\binom{k-1}{j-1} = 0$, $k > 1$, readily gives the first equality in (43). The second equality follows with the help of (17). □

If $b = a+1$ then the second series in (43) is telescoping and we obtain

$$\lim_{\alpha\to\infty} \alpha^m \,{}_{\alpha,m}\!\int_a^{a+1} f(t)\,\mathrm{d}t = \frac{(-1)^{m-1}}{16}(m-1)! f''(a+\tfrac{1}{2}). \tag{45}$$

Now we prove an explicit formula in the case $f(t) = t^n$.

Proposition 4.1.2. *If $m > 1$ and $n > 1$ are integers then we have*

$${}_{\alpha,m}\!\int_0^1 t^n\,\mathrm{d}t = \sum_{\ell=1}^{[\frac{n}{2}]} \frac{(-1)^{m-\ell}(m-1)!(2n-2\ell)!}{(\ell-1)!2^{2n}(n-2\ell)!(n-\ell)!(\alpha+\ell)^m}. \tag{46}$$

Proof. The evaluation [4, p. 6]

$$ {}_{\alpha}\!\int_0^1 t^n \,\mathrm{d}t = \sum_{\ell=0}^{[\frac{n}{2}]} \frac{n!\Gamma(\alpha+1)(B_{n-2\ell+1}(\frac{3}{2}) - B_{n-2\ell+1}(\frac{1}{2}))}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell+1)!} \tag{47} $$

and the formula $B_m(z+1) - B_m(z) = mz^{m-1}$ give, for $m > 1$,

$$ {}_{\alpha,m}\!\int_0^1 t^n \,\mathrm{d}t = \sum_{\ell=1}^{[\frac{n}{2}]} \sum_{j=1}^{\ell} \frac{(-1)^{m-1}(m-1)!n!(-1)^{j-1}}{2^{n+2\ell}\ell!(n-2\ell)!(j-1)!(\ell-j)!(\alpha+j)^m}. \tag{48} $$

Permuting the order of summation, we obtain

$$ {}_{\alpha,m}\!\int_0^1 t^n \,\mathrm{d}t = \sum_{\ell=1}^{[\frac{n}{2}]} \sum_{j=\ell}^{[\frac{n}{2}]} \frac{(-1)^{m-1}(m-1)!n!(-1)^{\ell-1}}{2^{n+2j}j!(n-2j)!(\ell-1)!(j-\ell)!(\alpha+\ell)^m}. \tag{49} $$

From (49), we see that the result follows in view of the combinatorial identity

$$ \sum_{j=\ell}^{[\frac{n}{2}]} \frac{1}{2^{n+2j}j!(n-2j)!(j-\ell)!} = \frac{(2n-2\ell)!}{2^{2n}n!(n-2\ell)!(n-\ell)!}. \tag{50} $$

Here is a direct proof of (50). We start with the identity

$$ \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k}\binom{n-k}{k} a^k = \frac{1}{n!}\frac{d^n}{dz^n}\big(\sqrt{a} + z + \sqrt{a}z^2\big)^n\Big|_{z=0}, $$

which gives (note that $g^{(n)}(0) = \frac{1}{(\sqrt{a})^n}\frac{d^n}{du^n}g(\sqrt{a}u)\big|_{u=0}$):

$$ \begin{aligned} \sum_{k=\ell}^{[\frac{n}{2}]} \frac{\binom{n-\ell}{k}\binom{n-k-\ell}{k-\ell}n!}{2^{2k-2\ell}(n-\ell)!} &= \frac{1}{n!}\frac{d^\ell}{da^\ell}\frac{d^n}{dz^n}\big(\sqrt{a} + z + \sqrt{a}z^2\big)^n\Big|_{\substack{z=0\\ a=\frac{1}{4}}} \\ &= \frac{1}{n!}\frac{d^\ell}{da^\ell}\frac{d^n}{du^n}(1+u+au^2)^n\Big|_{\substack{u=0\\ a=\frac{1}{4}}} \\ &= \frac{1}{(n-\ell)!}\frac{d^n}{du^n}u^{2\ell}\big(1+\tfrac{u}{2}\big)^{2n-2\ell}\Big|_{u=0} \\ &= \frac{1}{(n-\ell)!}\binom{n}{2\ell}\frac{(2\ell)!(2n-2\ell)!}{2^{n-2\ell}n!}. \end{aligned} $$

We thus have

$$ \sum_{k=\ell}^{[\frac{n}{2}]} \frac{\binom{n-\ell}{k}\binom{n-k-\ell}{k-\ell}}{2^{n+2k}} = \frac{1}{2^{2n}}\binom{2n-2\ell}{n}, \tag{51} $$

which is equivalent to (50). □

Remark 4.1.1. The same idea used to prove (51) gives, more generally,

$$\sum_{\substack{s=0\\ s\geq n-m}}^{[\frac{n}{2}]} \frac{\binom{n-s}{s}\binom{m}{n-s}}{2^{n+2s}} = \frac{1}{2^{2n}}\binom{2m}{n}, \quad \text{for } m \leq n. \tag{52}$$

A partial result is obtained in [4, p. 30] for the quantities

$$\lim_{\alpha\to\infty} \alpha^m \phi_{n,\alpha,m}(z),$$

where $\phi_{n,\alpha}(z)$ are the fundamental polynomials defined by (5). We have

$$\lim_{\alpha\to\infty} \alpha^2 \phi_{n,\alpha,2}(z) = -\frac{nz}{2}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}\frac{(\frac{\zeta+\sqrt{\zeta^2+4}}{2})^{2z}}{\sqrt{\zeta^2+4}}\left(\ln\left(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\right)\right)^2\Bigg|_{\zeta=0}. \tag{53}$$

Here is a more precise result.

Proposition 4.1.3. *For $m = 2, 3, 4, \dots,$ we have*

$$\lim_{\alpha\to\infty} \alpha^m \phi_{n,\alpha,m}(z) = \frac{(-1)^m}{8}(m-1)!n!\sum_{\ell=1}^{[\frac{n-1}{2}]}\frac{(-1)^\ell((\ell-1)!)^2 z\Gamma(z+\frac{n}{2}-\ell)}{(2\ell)!(n-1-2\ell)!\Gamma(z+1-\frac{n}{2}+\ell)}. \tag{54}$$

Proof. We derive $(m-1)$ times, with respect to α, both sides of the α-expansion [1, p. 292]

$$f(z) = \sum_{n=0}^{\infty}\frac{b_\alpha^{(n)}(f(u); u=0)}{n!}\phi_{n,\alpha}(z). \tag{55}$$

Using Leibniz' rule and multiplying each term by α^m, we get

$$\begin{aligned}0 = \sum_{n=1}^{\infty}\frac{1}{n!}b_\alpha^{(n)}(f(u); u=0)\alpha^m\phi_{n,\alpha,m}(z)\\ + \sum_{n=1}^{\infty}\sum_{j=1}^{m-2}\frac{1}{n!}\binom{m-1}{j}\alpha^j b_{\alpha,j+1}^{(n)}(f(u); u=0)\alpha^{m-j}\phi_{n,\alpha,m-j}(z)\\ + \sum_{n=1}^{\infty}\frac{1}{n!}\alpha^m b_{\alpha,m}^{(n)}(f(u); u=0)\phi_{n,\alpha}(z).\end{aligned} \tag{56}$$

Note that the function f can be any polynomial, so that no problem of convergence will appear in the reasoning. From (56), the relations (6) and (16) of [1], and the relation (113) of [4], we deduce that

$$0=\sum_{n=3}^{\infty}\frac{1}{n!}\Delta^n f\left(-\tfrac{n}{2}\right)\lim_{\alpha\to\infty}\alpha^m\phi_{n,\alpha,m}(z)$$
$$+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{n}{16}(-1)^m(m-1)!\Delta^n f''\left(-\tfrac{n}{2}\right)\frac{z\Gamma(z+\frac{n}{2})}{\Gamma(z+1-\frac{n}{2})}. \quad (57)$$

A formula similar to (53) can be obtained with (57) but it is simpler to observe from (57) that

$$\lim_{\alpha\to\infty}\frac{(-1)^m}{(m-1)!}\alpha^m\phi_{n,\alpha,m}(z)$$

does not depend on m. Thus, we have

$$\lim_{\alpha\to\infty}\frac{(-1)^m}{(m-1)!}\alpha^m\phi_{n,\alpha,m}(z)$$
$$=-\frac{nz}{2}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}\frac{(\frac{\zeta+\sqrt{\zeta^2+4}}{2})^{2z}}{\sqrt{\zeta^2+4}}\left(\ln\left(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\right)\right)^2\Bigg|_{\zeta=0}. \quad (58)$$

The result follows by evaluating the derivatives of order $(n-1)$, at $\zeta=0$, in (58). In order to do that, we can use the expansions

$$\frac{(\frac{\zeta+\sqrt{\zeta^2+4}}{2})^{2z}}{\frac{\sqrt{\zeta^2+4}}{2}}=\sum_{j=0}^{\infty}\frac{\Gamma(z+\frac{(j+1)}{2})}{j!\Gamma(z-\frac{(j-1)}{2})}\zeta^j \quad (59)$$

and

$$\left(\ln\left(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\right)\right)^2=\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}((\ell-1)!)^2}{2(2\ell)!}\zeta^{2\ell}. \quad (60)$$

The expansion (59) is more easily obtained if we see it as the derivative of a Stirling's expansion,

$$F(z)=\sum_{j=0}^{\infty}\frac{\Delta^j F(-\frac{j}{2})}{j!}\frac{z\Gamma(z+\frac{j}{2})}{\Gamma(z+1-\frac{j}{2})},$$

where $F(z)=\left(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\right)^{2z}$; we have $\Delta^j F\left(-\frac{j}{2}\right)=\zeta^j$. The expansion (60) may be obtained by noting that the function $G(u):=\left(\ln(u+\sqrt{u^2+1})\right)^2$ satisfies the differential equation $(1+u^2)G''(u)+uG'(u)=2$, which gives $G^{(2\ell)}(0)=(-1)^{\ell-1}\left((\ell-1)!\right)^2 2^{2\ell-1}$ for $\ell=1,2,3,\ldots$. □

Remark 4.1.2. If we substitute the right-hand member of (54) in (57) then we obtain the identity

$$\sum_{n=3}^{\infty}\sum_{\ell=1}^{[\frac{n-1}{2}]}\frac{(-1)^{\ell-1}((\ell-1)!)^2 z\Gamma(z+\frac{n}{2}-\ell)}{(2\ell)!(n-1-2\ell)!\Gamma(z+1-\frac{n}{2}+\ell)}\Delta^n f\left(-\tfrac{n}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{z\Gamma(z+\frac{n}{2})}{2(n-1)!\Gamma(z+1-\frac{n}{2})} \Delta^n f''\left(-\tfrac{n}{2}\right). \quad (61)$$

This formula holds of course for any polynomial. It can be shown that it also holds for all entire functions of exponential type $\tau < 2$.

4.2.

Some asymptotic results for quantities of the form $Q_{\frac{\alpha}{m},m}$, $m \to \infty$, are presented in [4]. Here $Q_{\frac{\alpha}{m},m}$ means that we first calculate $Q_{\alpha,m} = \frac{\partial^{m-1}}{\partial\alpha^{m-1}} Q_\alpha$ and then replace α by $\frac{\alpha}{m}$. We prove a result similar to (43).

Proposition 4.2.1. *We have the asymptotic relations*

$$\begin{aligned} &\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!} e^{\alpha} \ {}_{\frac{\alpha}{m},m}\!\int_a^b f(t)\,\mathrm{d}t \\ &= \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} \frac{B_{2\ell-2k}(\frac{1}{2})}{(k-1)!2^{4k}k!(2\ell-2k)!} \left(f^{(2\ell-1)}(b) - f^{(2\ell-1)}(a)\right) \quad (62) \\ &= \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(\ell-1)!2^{4\ell}\ell!} \left(f^{(2\ell)}(a+j+\tfrac{1}{2}) - f^{(2\ell)}(b+j+\tfrac{1}{2})\right). \end{aligned}$$

Proof. The first equality is a direct consequence of (44). The second equality follows from (17) after a permutation of the summations in the first equality. □

The second double series in (62) is telescoping when $b-a$ is an integer. We can obtain, for $n = 1, 2, 3, \ldots,$

$$\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!} e^{\alpha} \ {}_{\frac{\alpha}{m},m}\!\int_0^1 t^n \,\mathrm{d}t = \frac{(n-1)}{2^{2n}} \binom{2n-2}{n-1}. \quad (63)$$

This relation is a more direct consequence of (46). We also have, for $k = 0, 1, 2, \ldots,$

$$\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!} e^{\alpha} \ {}_{\frac{\alpha}{m},m}\!\int_{-\frac{1}{2}}^{\frac{1}{2}} t^{2k} \,\mathrm{d}t = \frac{k}{2^{4k}} \binom{2k}{k}. \quad (64)$$

The exact behavior of $e^{\alpha} \lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!m^s} \phi_{n,\frac{\alpha}{m},m}(z) =: A_n(z)$, where $s := [\frac{n-3}{2}]$, is more difficult to obtain. In [4, Remark 4.2] we give empirical calculations, such as $A_3(z) = \frac{3}{8}z$, $A_4(z) = \frac{3}{2}z^2$, $A_5(z) = \frac{15}{16}z$, $A_6(z) = \frac{225}{32}z^2, \ldots.$ We conjecture that

$$A_{2m+1}(z) = \frac{(2m)!(2m+1)!z}{(m-1)!m!(m+1)!2^{4m}}, \quad \text{for } m = 1, 2, 3, \dots \text{ and} \tag{65}$$

$$A_{2m}(z) = \frac{((2m)!)^2 z^2}{(m-2)!m!(m+1)!2^{4m-3}}, \quad \text{for } m = 2, 3, 4, \dots. \tag{66}$$

Remark 4.2.1. It is interesting to observe that two sequences very similar to (65) and (66) appear when we study the quantities

$$\lim_{\alpha \to -1} (\alpha+1)^{[\frac{n-1}{2}]} \phi_{n,\alpha}(z) =: a_n(z).$$

We apparently have $a_{2m+1}(z) = (m-1)!A_{2m+1}(z)$ and $a_{2m}(z) = (m-2)!A_{2m}(z)$.

4.3.

The representation (11) of the α-integral can be used to obtain a special identity. We derive both sides with respect to α, take $\alpha = \frac{1}{2}$, and compare with the formula (103) of [4]. We obtain the identity

$$\sum_{p=3}^{\infty} \frac{\psi_{p,\frac{1}{2},2}}{p!} \int_a^b f^{(p-1)}(t)\,\mathrm{d}t$$
$$= 2\int_a^b f(t)\,\mathrm{d}t + \int_0^\infty \ln\big(\{t\}(1-\{t\})\big)\big(f(t+a) - f(t+b)\big)\,\mathrm{d}t, \tag{67}$$

where $\{t\} = t - [t]$ is the fractional part of t. Both members of (67) are equal to

$$_{\frac{1}{2},2}\!\int_a^b f(t)\,\mathrm{d}t.$$

In particular,

$$_{\frac{1}{2},2}\!\int_0^1 f(t)\,\mathrm{d}t = 2\int_0^1 f(t)\,\mathrm{d}t + \int_0^1 \ln\big(t(1-t)\big) f(t)\,\mathrm{d}t. \tag{68}$$

The coefficient $\psi_{p,\frac{1}{2},2} = \lim_{\alpha \to \frac{1}{2}} \frac{\partial}{\partial\alpha}\psi_{p,\alpha}$, in (67), can be evaluated in terms of Bernoulli numbers.

Proposition 4.3.1. *We have, for* $p = 2, 3, 4, \dots,$

$$\psi_{p,\frac{1}{2},2} = \big((-1)^p - 1\big) \sum_{k=1}^{p} \frac{1}{k}\binom{p}{k} B_{p-k}. \tag{69}$$

Proof. We observe that

$$\int_0^1 \ln\bigl(t(1-t)\bigr) f(t)\,\mathrm{d}t = -\sum_{n=1}^{\infty} \frac{f^{(n-1)}(0)}{n!}\left(\frac{1}{n} + \sum_{j=1}^{n}\frac{1}{j}\right), \tag{70}$$

where $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$. This formula follows from the elementary evaluation

$$\int_0^1 t^n \ln\bigl(t(1-t)\bigr)\,\mathrm{d}t = -\frac{1}{(n+1)^2} - \frac{1}{(n+1)}\sum_{j=1}^{n+1}\frac{1}{j}.$$

The choice $f(t) = B_{p-1}(t)$, in (70), gives

$$\int_0^1 \ln\bigl(t(1-t)\bigr) B_{p-1}(t)\,\mathrm{d}t = -\frac{1}{p}\sum_{n=1}^{p}\binom{p}{n} B_{p-n}\left(\frac{1}{n} + \sum_{j=1}^{n}\frac{1}{j}\right), \tag{71}$$

which can also be written as

$$p\bigl((-1)^p - 1\bigr)\int_0^1 \ln(t) B_{p-1}(t)\,\mathrm{d}t = \sum_{n=1}^{p}\binom{p}{n} B_{p-n}\left(\frac{1}{n} + \sum_{j=1}^{n}\frac{1}{j}\right). \tag{72}$$

On the other hand, if we derive both members of (67) with respect to b, and take $f(t) = \exp(-ct)$, $c > 0$, then we obtain

$$\begin{aligned}\sum_{p=3}^{\infty} \frac{(-1)^{p-1}}{p!}\psi_{p,\frac{1}{2},2} c^{p-1} &= 2 + c\int_0^{\infty} \ln\bigl(\{t\}(1-\{t\})\bigr)\exp(-ct)\,\mathrm{d}t \\ &= 2 + c\sum_{k=0}^{\infty}\int_k^{k+1} \ln\bigl(\{t\}(1-\{t\})\bigr)\exp(-ct)\,\mathrm{d}t,\end{aligned}$$

whence

$$\sum_{p=3}^{\infty} \frac{(-1)^{p-1}}{p!}\psi_{p,\frac{1}{2},2} c^{p-1} = 2 + \int_0^1 \ln\bigl(t(1-t)\bigr)\frac{c\exp(-ct)}{1-\exp(-c)}\,\mathrm{d}t. \tag{73}$$

The integrand in (73) contains the generating function of the Bernoulli polynomials. Thus, we have

$$\sum_{p=3}^{\infty} \frac{(-1)^{p-1}}{p!}\psi_{p,\frac{1}{2},2} c^{p-1} = \sum_{p=2}^{\infty}\frac{(-1)^{p-1}}{(p-1)!}\int_0^1 \ln\bigl(t(1-t)\bigr) B_{p-1}(t)\,\mathrm{d}t\, c^{p-1},$$

whence

$$\int_0^1 \ln\bigl(t(1-t)\bigr) B_{p-1}(t)\,\mathrm{d}t = \frac{1}{p}\psi_{p,\frac{1}{2},2},$$

i.e.,

$$p(1-(-1)^p)\int_0^1 \ln(t) B_{p-1}(t)\,\mathrm{d}t = \psi_{p,\frac{1}{2},2}, \quad \text{for } p = 2,3,4,\ldots. \tag{74}$$

The result follows from (72) and (74) if we note that

$$\sum_{k=1}^{p}\frac{1}{k}\binom{p}{k}B_{p-k}$$
$$=\int_0^1\frac{(B_p(t)-B_p)}{t}\,\mathrm{d}t=\int_0^1\int_0^t\frac{B_p'(u)}{t}\,\mathrm{d}u\,\mathrm{d}t=-p\int_0^1\ln(t)B_{p-1}(t)\,\mathrm{d}t.\quad\square$$

We conclude this section with some independent observations concerning the definitions (8) and (9). To the best of our knowledge, no systematic study of the q-calculus has been made as a function of the parameter q. Let us assume that f is analytic in a neighborhood of $x=0$, including $x=a$. The definition (8) and (9) can be written in the form

$$D_q f(x)=\sum_{k=1}^{\infty}\frac{(q^k-1)}{(q-1)}f^{(k)}(0)\frac{x^{k-1}}{k!}\tag{75}$$

and

$$\int_0^a f(x)d_qx=\sum_{k=0}^{\infty}\frac{(1-q)}{(1-q^{k+1})}f^{(k)}(0)\frac{a^{k+1}}{k!}.\tag{76}$$

Using the same notation as in (42), we deduce from (75) that

$$\lim_{q\to1}D_{q,m}f(x)=\frac{1}{m}x^{m-1}f^{(m)}(x),\quad\text{for } m=1,2,3,\ldots.\tag{77}$$

We also have, from (76),

$$\int_0^a f(x)d_{q,2}x=-\frac{1}{2}\int_0^a xf'(x)\,\mathrm{d}x\tag{78}$$

and

$$\int_0^a f(x)d_{q,3}x=\frac{a^2}{4}f'(a)-\frac{1}{12}\int_0^a x^2f''(x)\,\mathrm{d}x.\tag{79}$$

5. The Alpha-Physics

The Riemann–Liouville integral (6) has its origin in physics (Abel's mechanical problem, oscillating pendulum, ...). The q-derivative (8) appears in quantum analysis (see for example [9]). We use the α-calculus to construct a generalized physics, which we call the alpha-physics.

In the α-physics, we consider the universe as being constituated of α-spaces. An α-space is itself constituated of particles whose mass depend on α. Our usual Euclidian space corresponds to $\alpha=\frac{1}{2}$.

5.1.

An interesting link between time and mass is already implicit in [3, Section 7]. The α-speed of a particle is defined by

$$b_\alpha(\vec{r}(t);t) = \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k!} B_{k,\alpha} \frac{d^k \vec{r}(t)}{dt^k}, \tag{80}$$

where $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is the vector position. The unit associated to the α-speed is $\frac{m}{s_\alpha}$, where the α-second s_α is given by

$$s_\alpha = \frac{g_\alpha(\frac{i}{2s})}{\exp(\frac{1}{2s}) - \exp(-\frac{1}{2s})}. \tag{81}$$

The α-mass of a two-dimensional object R is

$$M_\alpha := {}_\alpha\!\!\int_\alpha\!\!\int_R \rho(x,y)\,\mathrm{d}A \tag{82}$$

where $\rho(x,y)$ is the density of the object. The unit associated to the α-mass is

$$(kg)_\alpha = kg\left(\frac{g_\alpha(\frac{i}{2m})}{m(\exp(\frac{1}{2m}) - \exp(-\frac{1}{2m}))}\right)^2. \tag{83}$$

For an object whose euclidian dimension is n, the unit of α-mass is

$$(kg)_\alpha = kg\left(\frac{g_\alpha(\frac{i}{2m})}{m(\exp(\frac{1}{2m}) - \exp(-\frac{1}{2m}))}\right)^n. \tag{84}$$

It is clear that we have the following relation.

Proposition 5.1.1. *Let* $f_1(s) := \frac{s_\alpha}{s}$ *and* $\big(f_2(m)\big)^n := \frac{(kg)_\alpha}{kg}$, *where* s_α *and* $(kg)_\alpha$ *are defined by* (81) *and* (84). *We have*

$$f_1 = f_2. \tag{85}$$

Precisely,

$$f_1(z) = f_2(z) = \frac{\frac{1}{z} g_\alpha(\frac{i}{2z})}{\exp(\frac{1}{2z}) - \exp(-\frac{1}{2z})}. \tag{86}$$

In our usual space ($\alpha = \frac{1}{2}$) the relation (85) is trivial since, in that case, $f_1(z) = f_2(z) \equiv 1$. It is necessary to introduce the concept of α-space in order to see the link (85) between time and mass.

5.2.

In the α-physics, the notion of absolute vacuum is theoretical. What is called vacuum in the ordinary physical language is only an appearance of emptiness. There are always particles that cannot be detected in a given α-space. The appearance of vacuum should be termed "α-vacuum" in general.

The speed of light in the theoretical vacuum is infinite. It is a constant c in our $\frac{1}{2}$-space. In order to obtain a concrete expression for the speed of light, in the α-vacuum, we will generalize the formula $E = mc^2$ from relativity. We use the notation of [8, Part V]. In that context, the equation of motion is written in the from $\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{v}_0)$, where $\vec{v}_0 = u\vec{i} + v\vec{j} + w\vec{k}$, $u = \frac{\mathrm{d}x}{\mathrm{d}t}$, $v = \frac{\mathrm{d}y}{\mathrm{d}t}$, $w = \frac{\mathrm{d}z}{\mathrm{d}t}$ and $m = \frac{m_0}{\sqrt{1-\beta^2}}$. Here, m_0 is the so-called "rest mass" and $\beta^2 = \frac{u^2+v^2+w^2}{c^2}$.

The work done by a particle that moves along a trajectory C is given, in a general α-space, by

$$W_\alpha(C) = {}_\alpha\!\int_C \vec{F} \cdot d\vec{r}, \tag{87}$$

where C is the curve with vectorial parametrization $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $A \le t \le B$. Thus we have (see [2, Subsection 4.1])

$$\begin{aligned} W_\alpha(C) = m_0 \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \int_C \Bigg(& \frac{\partial_\alpha^{p-1}}{\partial_\alpha x^{p-1}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{u}{\sqrt{1-\beta^2}} \right) \right) \mathrm{d}x \\ & + \frac{\partial_\alpha^{p-1}}{\partial_\alpha y^{p-1}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{v}{\sqrt{1-\beta^2}} \right) \right) \mathrm{d}y \\ & + \frac{\partial_\alpha^{p-1}}{\partial_\alpha z^{p-1}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{w}{\sqrt{1-\beta^2}} \right) \right) \mathrm{d}z \Bigg). \end{aligned} \tag{88}$$

We have $\psi_{p,\alpha} = 0$, $p > 1$, when $\alpha = \frac{1}{2}$; in that case, the identity

$$\begin{aligned} u\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\sqrt{1-\beta^2}}\right) + v\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{\sqrt{1-\beta^2}}\right) + w\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w}{\sqrt{1-\beta^2}}\right) \\ = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c^2}{\sqrt{1-\beta^2}}\right) \end{aligned} \tag{89}$$

can be used to obtain (writing W for $W_{\frac{1}{2}}(C)$):

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \Bigg|_{t=A}^{t=B}, \tag{90}$$

which gives the famous relation $E = m_0 c^2$, for the "rest energy", by taking

$\beta = \|\vec{v}_0\| = 0$.

Let us denote by $E_\alpha(C)$ the rest energy, in a general α-space, with respect to the trajectory C. The α-extension of $E = m_0c^2$ is a formula of the form

$$E_\alpha(C) = m_0 c_\alpha^2, \tag{91}$$

where c_α is the speed of light in the α-space, along the trajectory C. The quantity c_α is independent of the curve C when $\alpha = \frac{1}{2}$, but not for general α. Here are two examples where the calculations can be done explicitly.

Example 5.2.1. Let C be the curve parametrized by $x(t) = c^2\sqrt{1 + \frac{(t+k)^2}{c^2}}$, $y(t) = 0$, $z(t) = 0$, $A \le t \le B$, where k is a constant. We have $\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{u}{\sqrt{1-\beta^2}}\big) \equiv 1$ whence, from (88),

$$W_\alpha(C) = m_0 \int_A^B \mathrm{d}x(t) = m_0 x(t)\Big|_A^B.$$

The energy at rest is obtained when $u = 0$, i.e. $t = -k$, which gives

$$E_\alpha(C) = m_0 x(-k) = m_0 c^2.$$

Example 5.2.2. Let C_a be the curve parametrized by $x(t) = \frac{c}{a}\ln\big(\cosh(at)\big)$, $y(t) = 0$, $z(t) = 0$, $A \le t \le B$, where a is a small positive constant. We have $\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{u}{\sqrt{1-\beta^2}})\big) = ac\exp(\frac{a}{c}x)$ whence, from (88) and the formula (57) of [1],

$$\begin{aligned} W_\alpha(C_a) &= m_0 ac \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \int_{C_a} \frac{\partial_\alpha^{p-1}}{\partial_\alpha x^{p-1}} \exp\left(\frac{a}{c}x\right) \mathrm{d}x \\ &= m_0 c^2 \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \left(\frac{\exp(\frac{a}{2c}) - \exp(-\frac{a}{2c})}{g_\alpha(\frac{ia}{2c})}\right)^{p-1} \exp\left(\frac{a}{c}x(t)\right)\Big|_A^B. \end{aligned}$$

The relation (61) of [2] then gives

$$W_\alpha(C_a) = m_0 c^2 \frac{\frac{a}{c} g_\alpha(\frac{ia}{2c})}{\exp(\frac{a}{2c}) - \exp(-\frac{a}{2c})} \exp\left(\frac{a}{c}x(t)\right)\Big|_A^B. \tag{92}$$

The energy at rest is obtained when $u = 0$, i.e. $t = 0$, which gives

$$E_\alpha(C_a) = m_0 c^2 \frac{\frac{a}{c} g_\alpha(\frac{ia}{2c})}{\exp(\frac{a}{2c}) - \exp(-\frac{a}{2c})}. \tag{93}$$

Thus, the speed of light c_α (C_a) in a general α-space, along the trajectory C_a, is

$$c_\alpha^2(C_a) = c^2 \frac{\frac{a}{c} g_\alpha(\frac{ia}{2c})}{\exp(\frac{a}{2c}) - \exp(-\frac{a}{2c})}. \tag{94}$$

As $\alpha \to \infty$ we obtain

$$c_{\infty}^2(C_a) = c^2 \frac{\frac{a}{c}}{\exp(\frac{a}{2c}) - \exp(-\frac{a}{2c})}, \tag{95}$$

with $\frac{\frac{a}{c}}{\exp(\frac{a}{2c})-\exp(-\frac{a}{2c})} < 1$ for $a > 0$. We also have

$$c_{-\frac{1}{2}}(C_a) = c^2 \frac{\frac{a}{2c}}{\tanh(\frac{a}{2c})}, \tag{96}$$

with $\frac{\frac{a}{2c}}{\tanh(\frac{a}{2c})} > 1$ for $a > 0$. We see from (95) and (96) that the speed of light can take any positive value in some α-spaces.

In general, it is difficult to make explicit calculations with (88), even with $y(t) = 0$ and $z(t) = 0$, because the quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\sqrt{1-\beta^2}}\right) = \frac{1}{(1-\beta^2)^{\frac{3}{2}}} \frac{\mathrm{d}u}{\mathrm{d}t} \tag{97}$$

must be a function of x if we want to calculate

$$\frac{\partial_\alpha^{p-1}}{\partial_\alpha x^{p-1}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\sqrt{1-\beta^2}}\right)\right).$$

Let us assume that $u = cf(x)$ for a suitable function f. Then we have $\frac{\mathrm{d}u}{\mathrm{d}t} = cf'(x)u = c^2 f'(x) f(x)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{\sqrt{1-\beta^2}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c^2}{\sqrt{1-f^2(x)}}\right). \tag{98}$$

In that case, it follows from (88) that

$$\begin{aligned} W_\alpha(C) &= m_0 c^2 \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \int_C \frac{\partial_\alpha^{p-1}}{\partial_\alpha x^{p-1}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{1-f^2(x)}}\right)\mathrm{d}x \\ &= m_0 c^2 \sum_{p=1}^{\infty} \sum_{k=p-1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \frac{d_{k,\alpha}^{(p-1)}}{k!} \int_C \frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{\sqrt{1-f^2(x)}}\right)\mathrm{d}x \\ &= m_0 c^2 \sum_{p=1}^{\infty} \sum_{k=p-1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \frac{d_{k,\alpha}^{(p-1)}}{k!} \frac{\mathrm{d}^k}{dx^k}\left(\frac{1}{\sqrt{1-f^2(x)}}\right)\Bigg|_{x(A)}^{x(B)}. \end{aligned}$$

Since the rest energy corresponds to $u = 0$, we obtain

$$E_\alpha(C) = m_0 c^2 \sum_{p=1}^{\infty} \sum_{k=p-1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \frac{d_{k,\alpha}^{(p-1)}}{k!} \frac{\mathrm{d}^k}{dx^k}\left(\frac{1}{\sqrt{1-f^2(x)}}\right)\Bigg|_{f(x)=0}. \tag{99}$$

The associated speed of light (in the α-space, along the trajectory C with

$u = cf(x)$) is

$$c_\alpha^2(C) = c^2 \sum_{p=1}^{\infty} \sum_{k=p-1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \frac{d_{k,\alpha}^{(p-1)}}{k!} \frac{\mathrm{d}^k}{dx^k} \left(\frac{1}{\sqrt{1-f^2(x)}} \right) \Bigg|_{f(x)=0}. \tag{100}$$

Finally, the relation (19) gives

$$E_\alpha(C) = m_0 c^2 \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \frac{\Gamma(\alpha+1) B_{2\ell-2k}(\frac{1}{2})}{2^{4k} k! \Gamma(\alpha+k+1)(2\ell-2k)!} \frac{d^{2\ell}}{dx^{2\ell}} \left(\frac{1}{\sqrt{1-f^2(x)}} \right) \Bigg|_{f(x)=0} \tag{101}$$

with

$$c_\alpha^2(C) = c^2 \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \frac{\Gamma(\alpha+1) B_{2\ell-2k}(\frac{1}{2})}{2^{4k} k! \Gamma(\alpha+k+1)(2\ell-2k)!} \frac{d^{2\ell}}{dx^{2\ell}} \left(\frac{1}{\sqrt{1-f^2(x)}} \right) \Bigg|_{f(x)=0}. \tag{102}$$

The expressions for $E_\alpha(C)$ and $c_\alpha(C)$, in Examples 5.2.1 and 5.2.2, correspond respectively to

$$f(x) = \frac{c}{x} \sqrt{\frac{x^2}{c^2} - c^2}, \quad \text{and} \quad f(x) = \sqrt{1 - \exp\left(\frac{-2ax}{c} \right)}.$$

In the second case, the formulas (94) and (102) are equivalent to the expansion

$$\frac{z g_\alpha(\frac{iz}{2})}{\exp(\frac{z}{2}) - \exp(-\frac{z}{2})} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \frac{\Gamma(\alpha+1) B_{2\ell-2k}(\frac{1}{2}) z^{2\ell}}{2^{4k} k! \Gamma(\alpha+k+1)(2\ell-2k)!}, \quad \text{for } |z| < 2\pi. \tag{103}$$

Other extensions of classical physics formulas could be given in our context. Sometime we need only to replace the (differential) operator appearing in the physics formula by its counterpart in the α-calculus. This procedure leads us to study differential equations of infinite order (see Subsection 6.6).

6. Miscellaneous Results

6.1. Generalized Legendre Polynomials

In Section 6 of [3], we introduced some generalized orthogonal polynomials with a Gram–Schmidt procedure applied to the set of functions $\{x^n : n = 0, 1, 2, \dots\}$, with the scalar product

$$(f,g)_\alpha = {}_\alpha\!\int_a^b \rho(t)f(t)g(t)\,\mathrm{d}t, \tag{104}$$

where a, b and the weight function $\rho(t)$ are suitably chosen. For instances, the first three normalized α-Legendre polynomials $P_n^*(x,\alpha)$ are $P_0^*(x,\alpha) = \frac{1}{\sqrt{2}}$, $P_1^*(x,\alpha) = \frac{2x}{\sqrt{\frac{2\alpha+3}{\alpha+1}}}$ and

$$P_2^*(x,\alpha) = \frac{(8x^2-3)+\alpha(8x^2-2)}{\sqrt{\frac{2(8\alpha^2+26\alpha+17)}{\alpha+2}}}.$$

We observed that, apparently,

$$\lim_{\alpha\to\infty} \frac{P_n^*(x,\alpha)}{(\sqrt{\alpha})^{[\frac{n}{2}]}} = c_n x^n \left(4 - \frac{1}{x^2}\right)^{[\frac{n}{2}]}. \tag{105}$$

The calculations of the constants c_n can be very long, even with the help of a mathematical software. Here are the results of elaborate computations. We obtain $c_0 = \frac{1}{\sqrt{2}}$, $c_1 = \sqrt{2}$, $c_2 = \frac{1}{2}$, $c_3 = 1$, $c_4 = \frac{1}{4}$, $c_5 = \frac{1}{2}$, $c_6 = \frac{1}{4\sqrt{6}}$ and $c_7 = \frac{1}{2\sqrt{6}}$. We thus have, for $m = 0,1,2,3$, the values $c_{2m} = \frac{1}{2^{\frac{(m+1)}{2}}\sqrt{m!}}$ and (note the misprint at the line following (156) in [3]) $c_{2m+1} = \frac{1}{2^{\frac{(m-1)}{2}}\sqrt{m!}}$.

6.2. Generalized Bernoulli Polynomials

We saw in this paper that the Bernoulli polynomials play a major role in the α-calculus. Here, we present an extension of the formula

$$\sum_{k=0}^{n} \binom{n}{k} B_k(z)B_{n-k}(w) = n(w+z-1)B_{n-1}(w+z) - (n-1)B_n(w+z). \tag{106}$$

Proposition 6.2.1. *For $n = 0,1,2,\ldots,$ we have the identity*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} B_{k,\alpha}(z)B_{n-k}(w) &= n(w+z-1)B_{n-1,\alpha}(w+z) - nB_{n,\alpha}(w+z) \\ &\quad - 2\sum_{\ell=0}^{[\frac{n}{2}]} \binom{n}{2\ell} B_{2\ell+1,\alpha}B_{n-2\ell}(w+z).\end{aligned} \tag{107}$$

Proof. We apply the α-derivative, with respect to z, on both sides of (106) and use the relation $b_\alpha(B_m(u);u) = mB_{m-1,\alpha}(u)$. We obtain

$$\sum_{k=1}^{n}\binom{n}{k}kB_{k-1,\alpha}(z)B_{n-k}(w) = nb_\alpha\big((w+z-1)B_{n-1}(w+z);z\big) - n(n-1)B_{n-1}(w+z). \quad (108)$$

Now we use the formula (91) of [1], namely

$$b_\alpha^{(N)}(zg(z);z) = zb_\alpha^{(N)}(g(z);z) + \sum_{m=N}^{\infty} c_{1,m}(N,\alpha)b_\alpha^{(m-1)}(g(z);z), \quad (109)$$

where

$$c_{1,m}(N,\alpha) := \frac{1}{(m-1)!}b_\alpha^{(N)}(u\phi_{m-1,\alpha}(u);u=0).$$

Using (109) with $N=1$ and $g(z) = B_{n-1}(w+z)$, we get from (108),

$$\begin{aligned} n\sum_{k=1}^{n}\binom{n-1}{k-1}B_{k-1,\alpha}(z)B_{n-k}(w) &= n(n-1)(w+z-1)B_{n-2,\alpha}(w+z) - n(n-1)B_{n-1,\alpha}(w+z) \\ &\quad + n\sum_{m=1}^{n}\sum_{j=0}^{n-1} c_{1,m}(1,\alpha)\frac{d_{j,\alpha}^{(m-1)}}{j!}\frac{(n-1)!}{(n-1-j)!}B_{n-1-j}(w+z), \end{aligned} \quad (110)$$

whence

$$\begin{aligned} \sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,\alpha}(z)B_{n-1-k}(w) &= (n-1)(w+z-1)B_{n-2,\alpha}(w+z) - (n-1)B_{n-1,\alpha}(w+z) \\ &\quad + \sum_{j=0}^{n-1}\sum_{m=1}^{n} c_{1,m}(1,\alpha)d_{j,\alpha}^{(m-1)}\binom{n-1}{j}B_{n-1-j}(w+z). \end{aligned} \quad (111)$$

The relation (109), applied to $g(z) = z^j$, $z=0$, gives

$$d_{j+1,\alpha}^{(N)} = \sum_{m=N}^{j+1} c_{1,m}(N,\alpha)d_{j,\alpha}^{(m-1)}. \quad (112)$$

Using (112) with $N=1$, we deduce from (111) that

$$\sum_{k=0}^{n-1}\binom{n-1}{k}B_{k,\alpha}(z)B_{n-1-k}(w) = (n-1)(w+z-1)B_{n-2,\alpha}(w+z) - (n-1)B_{n-1,\alpha}(w+z)$$

$$+\sum_{j=0}^{n-1} d_{j+1,\alpha}^{(1)} \binom{n-1}{j} B_{n-1-j}(w+z). \quad (113)$$

The identity (107) follows from (113) $\big($recall that $d_{j+1,\alpha}^{(1)} = \big((-1)^{j+1}-1\big)B_{j+1,\alpha}\big)$ if we replace n by $(n+1)$. □

The identity (106) is obtained from (107) with $\alpha = \frac{1}{2}$. The limiting case $\alpha \to \infty$ of (107) gives, since $\lim_{\alpha\to\infty} B_{m,\alpha}(u) = (u-\frac{1}{2})^m$, the formula

$$B_n(w-\tfrac{1}{2}) = \sum_{\ell=0}^{[\frac{n}{2}]} \frac{\binom{n}{2\ell}}{2^{2\ell}} B_{n-2\ell}(w) - \frac{n}{2}(w-\tfrac{1}{2})^{n-1}. \quad (114)$$

In view of the addition formula

$$B_N(x+y) = \sum_{k=0}^{N} \binom{N}{k} B_{N-k}(x) y^k, \quad (115)$$

we see that (114) is equivalent to

$$\sum_{\ell=0}^{[\frac{n-1}{2}]} \frac{\binom{n}{2\ell+1}}{2^{2\ell+1}} B_{n-2\ell-1}(w) = \frac{n}{2}(w-\tfrac{1}{2})^{n-1}. \quad (116)$$

Formula (116) can be proved directly by mathematical induction, as follows. We assume that (116) is true for a given $n > 1$ and integrate both sides between $\frac{1}{2}$ and w. We readily see that (116) will be true for the integer $(n+1)$ if

$$\sum_{\ell=0}^{[\frac{n}{2}]} \frac{\binom{n+1}{2\ell+1}}{2^{2\ell+1}} B_{n-2\ell}(\tfrac{1}{2}) = 0, \quad (117)$$

which is evident if n is odd. If $n = 2m$ is even then (117) follows from (115) where $N = 2m+1$ and $x = y = \frac{1}{2}$.

6.3. The Discrete Cauchy–Riemann Equations

The α-calculus has been used to put in evidence a discrete version of the Cauchy–Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad (118)$$

where $u(x,y)$ and $v(x,y)$ are the real and imaginary parts of an analytic function $f(z) = u + iv$. The other equation, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, is obtained by applying (118) to the function $if(z) = -v + iu$.

The α-extension of (118) is [2, p. 362]

$$ib_\alpha(u(x,y);x) = b_\alpha(v(x,it);t=-iy). \tag{119}$$

The discrete version is obtained by letting $\alpha \to \infty$ in (119):

$$i\big(u(x+\tfrac{1}{2},iy) - u(x-\tfrac{1}{2},iy)\big) = v\big((x,i(y+\tfrac{1}{2})\big) - v\big(x,i(y-\tfrac{1}{2})\big). \tag{120}$$

We observe that a more general result than (120), which also implies (118), can be proved easily.

Proposition 6.3.1. *If $u(x,y)$ and $v(x,y)$ are the real and imaginary parts of an analytic function then, for all reals a, b, x, y, we have*

$$i\big(u(x+a,i(y+b)\big)-u\big(x-a,i(y-b)\big) = v(x+b,i(y+a)\big)-v\big(x-b,i(y-a)\big). \tag{121}$$

Proof. Let $f(z) = u+iv = \sum_{n=0}^{\infty} a_n z^n$, so that

$$u(x,y) = \sum_{n=0}^{\infty} \mathrm{Re}(a_n z^n) = \frac{1}{2}\sum_{n=0}^{\infty}(a_n(x+iy)^n + \bar{a}_n(x-iy)^n) \tag{122}$$

and

$$v(x,y) = \sum_{n=0}^{\infty} \mathrm{Im}(a_n z^n) = \frac{1}{2i}\sum_{n=0}^{\infty}(a_n(x+iy)^n - \bar{a}_n(x-iy)^n). \tag{123}$$

The equation (121) is a simple algebraic verification using (122) and (123). □

For $b=0$, the equation (121) becomes

$$i\big(u(x+a,iy) - u(x-a),iy\big) = v\big(x,i(y+a)\big) - v\big(x,i(y-a)\big). \tag{124}$$

The relation (120) is the particular case $a = \frac{1}{2}$ of (124). If we divide both members of (124) by a, and let $a \to 0$, then we are led to the traditional equation (118).

6.4. The Fundamental Theorem of Line Alpha-Integrals

The generalized version of Gauss and Stokes Theorems are presented in [2, Theorem 4.2] and [3, Theorem 2.1]. The results take the form

$$\begin{aligned}{}_\alpha\!\!\int_{S}{}_\alpha\!\!\int & P(x,y,z)\,\mathrm{d}x\,\mathrm{d}y + N(x,y,z)\,\mathrm{d}x\,\mathrm{d}z + M(x,y,z)\,\mathrm{d}y\,\mathrm{d}z \\ &= {}_\alpha\!\!\int_{R}{}_\alpha\!\!\int{}_\alpha\!\!\int \left(\frac{\partial_\alpha M}{\partial_\alpha x} + \frac{\partial_\alpha N}{\partial_\alpha y} + \frac{\partial_\alpha P}{\partial_\alpha z}\right)\mathrm{d}V \end{aligned} \tag{125}$$

and

$$
{}_{\alpha}\!\iint_{S}{}_{\alpha}\left(\frac{\partial_\alpha N}{\partial_\alpha x}-\frac{\partial_\alpha M}{\partial_\alpha y}\right)\mathrm{d}x\,\mathrm{d}y+\left(\frac{\partial_\alpha M}{\partial_\alpha z}-\frac{\partial_\alpha P}{\partial_\alpha x}\right)\mathrm{d}x\,\mathrm{d}z
+\left(\frac{\partial_\alpha P}{\partial_\alpha y}-\frac{\partial_\alpha N}{\partial_\alpha z}\right)\mathrm{d}y\,\mathrm{d}z={}_{\alpha}\!\oint_{C} M\,\mathrm{d}x+N\,\mathrm{d}y+P\,\mathrm{d}z. \quad (126)
$$

The α-divergence and α-curl of the vector field $\vec{F}(x,y,z)=M\vec{i}+N\vec{j}+P\vec{k}$ are defined by

$$
\nabla_\alpha\cdot\vec{F}=\frac{\partial_\alpha M}{\partial_\alpha x}+\frac{\partial_\alpha N}{\partial_\alpha y}+\frac{\partial_\alpha P}{\partial_\alpha z} \quad (127)
$$

and

$$
\nabla_\alpha\wedge\vec{F}=\left(\frac{\partial_\alpha P}{\partial_\alpha y}-\frac{\partial_\alpha N}{\partial_\alpha z}\right)\vec{i}+\left(\frac{\partial_\alpha M}{\partial_\alpha z}-\frac{\partial_\alpha P}{\partial_\alpha x}\right)\vec{j}+\left(\frac{\partial_\alpha N}{\partial_\alpha x}-\frac{\partial_\alpha M}{\partial_\alpha y}\right)\vec{k}. \quad (128)
$$

The α-gradient of a scalar field $f(x,y,z)$ is defined by

$$
\nabla_\alpha f=\frac{\partial_\alpha f}{\partial_\alpha x}\vec{i}+\frac{\partial_\alpha f}{\partial_\alpha y}\vec{j}+\frac{\partial_\alpha f}{\partial_\alpha z}\vec{k}. \quad (129)
$$

The vector field $\vec{F}=M\vec{i}+N\vec{j}+P\vec{k}$ is said to be α-conservative if there exists a scalar field f such that

$$
\vec{F}(x,y,z)=\nabla_\alpha f(x,y,z). \quad (130)
$$

Theorem 6.4.1. *Assume that the vector field $\vec{F}$ is α-conservative and that* $\mathrm{dom}(\vec{F})$ *is connected. We then have*

$$
{}_{\alpha}\!\int_{C} M\,\mathrm{d}x+N\,\mathrm{d}y+P\,\mathrm{d}z=f(P_2)-f(P_1), \quad (131)
$$

where f is the α-potential and P_1, P_2 are the extremities of the curve C.

Proof. By hypothesis we have

$$
M=\frac{\partial_\alpha f}{\partial_\alpha x},\quad N=\frac{\partial_\alpha f}{\partial_\alpha y},\quad \text{and}\quad P=\frac{\partial_\alpha f}{\partial_\alpha z}.
$$

Thus,

$$
\begin{aligned}
{}_{\alpha}\!\int_{C} M\,\mathrm{d}x+N\,\mathrm{d}y+P\,\mathrm{d}z&=\sum_{p=1}^{\infty}\frac{\psi_{p,\alpha}}{p!}\int_C \frac{\partial_\alpha^{p-1}M}{\partial_\alpha x^{p-1}}\,\mathrm{d}x+\frac{\partial_\alpha^{p-1}N}{\partial_\alpha y^{p-1}}\,\mathrm{d}y+\frac{\partial_\alpha^{p-1}P}{\partial_\alpha z^{p-1}}\,\mathrm{d}z\\
&=\sum_{p=1}^{\infty}\frac{\psi_{p,\alpha}}{p!}\int_C \frac{\partial_\alpha^{p}f}{\partial_\alpha x^{p}}\,\mathrm{d}x+\frac{\partial_\alpha^{p}f}{\partial_\alpha y^{p}}\,\mathrm{d}y+\frac{\partial_\alpha^{p}f}{\partial_\alpha z^{p}}\,\mathrm{d}z.
\end{aligned}
$$

We now use the formula (61) of [2], namely

$$g'(z) = \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} b_{\alpha}^{(p)}(g(z); z), \tag{132}$$

to obtain

$$ {}_{\alpha}\int_{C} M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \int_{C} \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z. \tag{133}$$

The result follows from (133) and the application of the usual fundamental theorem of line integrals:

$$ {}_{\alpha}\int_{C} M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \int_{C} \nabla f \cdot d\vec{r} = f(P_2) - f(P_1). \qquad \square$$

For a vector field $\vec{F}$ whose domain is simply connected, it follows from the α-extension (126) of Stokes Theorem that $\vec{F}$ is *conservative if*

$$\nabla_{\alpha} \wedge \vec{F} = \vec{0}. \tag{134}$$

Under suitable conditions on f and $\vec{F}$, we have the identities

$$\nabla_{\alpha} \wedge (\nabla_{\alpha} f) = \vec{0} \tag{135}$$

and

$$\nabla_{\alpha} \cdot (\nabla_{\alpha} \wedge \vec{F}) = 0. \tag{136}$$

6.5. Generalized Mean Value Theorem and Integral Inequalities

The representation (see (15))

$$ {}_{\alpha}\int_{0}^{1} f(t)\,\mathrm{d}t = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{0}^{1} \big(t(1-t)\big)^{\alpha-\frac{1}{2}} f(t)\,\mathrm{d}t, \tag{137}$$

where $\alpha > -\frac{1}{2}$, can be used to obtain interesting properties of the α-integral. An α-extension of the mean value theorem for integrals readily follows from (137).

Proposition 6.5.1. *Let* $\alpha > -\frac{1}{2}$. *If* $g(t)$ *has a constant sign in* $[0,1]$ *and if* f *is continuous in* $[0,1]$ *then there exists a number* $c \in (0,1)$ *such that*

$$ {}_{\alpha}\int_{0}^{1} f(t)g(t)\,\mathrm{d}t = f(c)\, {}_{\alpha}\int_{0}^{1} g(t)\,\mathrm{d}t. \tag{138}$$

Using the additivity property of the α-integral, and the intermediate value theorem, we can extend (138) to

$$ {}_{\alpha}\!\int_a^{\infty} f(t)g(t)\,\mathrm{d}t = f(c)\,{}_{\alpha}\!\int_0^{\infty} g(t)\,\mathrm{d}t, \quad \text{for some } c \in (a,\infty). \tag{139} $$

For the remainder of this subsection, we suppose that f and g are positive functions in the interval under consideration.

Let $1 \le p < \infty$. Applying Minkowski inequality, namely

$$ \left(\int_a^b \big(f(t)+g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \le \left(\int_a^b \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_a^b \big(g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}, \tag{140} $$

we deduce from (137) that

$$ \begin{aligned} &{}_{\alpha}\!\int_0^1 \big(f(t)+g(t)\big)^p\,\mathrm{d}t \\ &\quad = \int_0^1 \left(\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\right)^{\frac{1}{p}} \big(t(1-t)\big)^{\frac{1}{p}(\alpha-\frac{1}{2})}\big(f(t)+g(t)\big)\right)^p \mathrm{d}t \\ &\quad \le \left(\left(\int_0^1 \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(t(1-t)\big)^{\alpha-\frac{1}{2}}\big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right. \\ &\qquad\qquad \left. + \left(\int_0^1 \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(t(1-t)\big)^{\alpha-\frac{1}{2}}\big(g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right)^p \\ &\quad = \left(\left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_0^1 \big(g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right)^p. \end{aligned} $$

We thus have the following α-extension of Minkowski inequality for integrals.

Proposition 6.5.2. *For $\alpha > -\frac{1}{2}$ and $1 \le p < \infty$, we have*

$$ \left({}_{\alpha}\!\int_0^1 \big(f(t)+g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \le \left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_0^1 \big(g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{141} $$

The Minkowski inequality for finite sums,

$$ \left(\sum_{k=1}^n (x_k+y_k)^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n y_k^p\right)^{\frac{1}{p}}, \tag{142} $$

where $x_k \ge 0$, $y_k \ge 0$ for $1 \le k \le n$, leads us to the more general inequality

$$ \left({}_{\alpha}\!\int_M^N \big(f(t)+g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \le \left({}_{\alpha}\!\int_M^N \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_M^N \big(g(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}, \tag{143} $$

where M and N ($M \le N$) are integers. We give the reasoning for $M = 0$,

$N = 2$. We write

$$ {}_{\alpha}\!\int_0^2 \big(f(t)+g(t)\big)^p \,\mathrm{d}t = {}_{\alpha}\!\int_0^1 \big(f(t)+g(t)\big)^p \,\mathrm{d}t + {}_{\alpha}\!\int_0^1 \big(f(t+1)+g(t+1)\big)^p \,\mathrm{d}t $$

and apply (141) two times. We obtain

$$ {}_{\alpha}\!\int_0^2 \big(f(t)+g(t)\big)^p \,\mathrm{d}t \le \left(\left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_0^1 \big(g(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \right)^p + \left(\left({}_{\alpha}\!\int_1^2 \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_1^2 \big(g(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \right)^p. \tag{144} $$

The inequality (142) with $n = 2$ is

$$ \big((x_1+y_1)^p + (x_2+y_2)^p\big)^{\frac{1}{p}} \le (x_1^p + x_2^p)^{\frac{1}{p}} + (y_1^p + y_2^p)^{\frac{1}{p}}. \tag{145} $$

Applying the inequality (145) in (144), we get

$$ \begin{aligned} \left({}_{\alpha}\!\int_0^2 \big(f(t)+g(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} &\le \left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p \,\mathrm{d}t + {}_{\alpha}\!\int_1^2 \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \\ &\quad + \left({}_{\alpha}\!\int_0^1 \big(g(t)\big)^p \,\mathrm{d}t + {}_{\alpha}\!\int_1^2 \big(g(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \\ &= \left({}_{\alpha}\!\int_0^2 \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} + \left({}_{\alpha}\!\int_0^2 \big(g(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}}, \end{aligned} $$

which is the inequality (143) where $M = 0$, $N = 2$.

We observe that the inequatily (142) is the limiting case $\alpha \to \infty$ of (143) (see (26)). We can write the inequality (141) in the form

$$ \|f+g\|_{\alpha,p} \le \|f\|_{\alpha,p} + \|g\|_{\alpha,p}, \tag{146} $$

where

$$ \|f\|_{\alpha,p} := \left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}}. \tag{147} $$

The real-valued function $\|\ \|_{\alpha,p}$ is in fact a norm defined on an appropriate linear space.

The Hölder inequalities for integrals and finite sums are

$$ \int_a^b f(t)g(t)\,\mathrm{d}t \le \left(\int_a^b \big(f(t)\big)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \left(\int_a^b \big(g(t)\big)^q \,\mathrm{d}t \right)^{\frac{1}{q}} \tag{148} $$

and

$$ \sum_{k=1}^n x_k y_k \le \left(\sum_{k=1}^n x_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n y_k^q \right)^{\frac{1}{q}}, \tag{149} $$

where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following result is a consequence of (137) and (148).

Proposition 6.5.3. *For $\alpha > -\frac{1}{2}$ and $1 < p < \infty$, we have*

$$ {}_{\alpha}\!\int_0^1 f(t)g(t)\,\mathrm{d}t \le \left({}_{\alpha}\!\int_0^1 \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \left({}_{\alpha}\!\int_0^1 \big(g(t)\big)^q\,\mathrm{d}t\right)^{\frac{1}{q}}, \tag{150} $$

where q is defined by the relation $\frac{1}{p} + \frac{1}{q} = 1$.

As for the Minkowski inequality, we can use the finite version (149) to obtain a more general result:

$$ {}_{\alpha}\!\int_M^N f(t)g(t)\,\mathrm{d}t \le \left({}_{\alpha}\!\int_M^N \big(f(t)\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}} \left({}_{\alpha}\!\int_M^N \big(g(t)\big)^q\,\mathrm{d}t\right)^{\frac{1}{q}}, \tag{151} $$

where M and N ($M \le N$) are integers. After the event, the inequality (149) becomes the limiting case $\alpha \to \infty$ of (151).

6.6. Alpha-Differential Equations

In principle, the ordinary differential equations can be α-extended by replacing the ordinary derivative by the α-derivative. For the common partial differential equations, we may replace the differential operators by their counterparts of α-calculus: the partial derivatives are replaced by the partial α-derivatives, the gradient by the α-gradient (129), the divergence by the α-divergence (127), the curl by the α-curl (128) and so on. For example, the Maxwell's equations of electromagnetism are α-extended in a natural way using this procedure. The familiar diffusion and wave equations can be generalized as

$$ \nabla_\alpha^2 u = \frac{\partial_\alpha u}{\partial_\alpha t} \tag{152} $$

and

$$ \nabla_\alpha^2 u = \frac{\partial_\alpha^2 u}{\partial_\alpha t^2}, \tag{153} $$

where $u = u(x, y, z, t)$ and

$$ \begin{aligned} \nabla_\alpha^2 u &:= \nabla_\alpha \cdot (\nabla_\alpha u) \\ &= \frac{\partial_\alpha^2 u}{\partial_\alpha x^2} + \frac{\partial_\alpha^2 u}{\partial_\alpha y^2} + \frac{\partial_\alpha^2 u}{\partial_\alpha z^2} \end{aligned} $$

is the vector α-laplacian (see [3, p. 204]). If we want to find solutions of these partial α-differential equations then we are led to study, in Cartesian coordinates, ordinary α-differential equations of the form

$$a\,b_\alpha^{(2)}(y(x);x) + b\,b_\alpha^{(1)}(y(x);x) + c\,y(x) = 0, \tag{154}$$

where a, b, c are real numbers. We can find explicit solutions of (154) in the cases $\alpha = \frac{1}{2}$, $\alpha = -\frac{1}{2}$ and $\alpha \to \infty$. We examine some examples.

Let us consider the α-differential equation

$$b_\alpha(y(x);x) - a\,y(x) = 0. \tag{155}$$

It is a differential equation of infinite order since

$$b_\alpha(y(x);x) = \sum_{k=1}^{\infty} \frac{d_{k,\alpha}^{(1)}}{k!} y^{(k)}(x).$$

A solution of the form

$$y(x) = \exp(\lambda x) \tag{156}$$

exists whenever

$$\frac{\exp(\frac{\lambda}{2}) - \exp(-\frac{\lambda}{2})}{g_\alpha(\frac{i\lambda}{2})} = a. \tag{157}$$

The equation (157) reduces to $\lambda = a$ for $\alpha = \frac{1}{2}$. If we let $\alpha \to \infty$ then (155) becomes the functional equation

$$y(x + \tfrac{1}{2}) - y(x - \tfrac{1}{2}) - a\,y(x) = 0 \tag{158}$$

and (157) becomes $\exp(\frac{\lambda}{2}) - \exp(-\frac{\lambda}{2}) = a$, whose solutions are

$$\lambda_k = 2\ln\left(\frac{a \pm \sqrt{a^2+4}}{2}\right) + 4k\pi i,$$

where k is an integer. The corresponding solutions (156), of (158), are

$$y_k(x) = \left(\frac{a \pm \sqrt{a^2+4}}{2}\right)^{2x} \exp(4k\pi i x).$$

Any linear combination is also a solution:

$$y(x) = \sum_{k=-M}^{N} c_k \exp(4k\pi i x)\left(\frac{a \pm \sqrt{a^2+4}}{2}\right)^{2x}. \tag{159}$$

If we let $M \to \infty$, $N \to \infty$, in (159), then we recognize $\sum_{k=-\infty}^{\infty} c_k \exp(4k\pi i x)$ as being a typical representation of a periodic function, with period $\frac{1}{2}$. We conclude that any function of the form

$$y(x) = \phi(x)\left(\frac{a \pm \sqrt{a^2+4}}{2}\right)^{2x}, \tag{160}$$

where ϕ is a periodic function with period $\frac{1}{2}$, is a solution of the ∞-differential equation (158). Of course we can verify this fact directly by substituting (160) in (158).

For $\alpha = -\frac{1}{2}$, the equation (155) can be written in the form (see [1, p. 302])

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{m-1}} \Delta^m y(x) - a\, y(x) = 0, \tag{161}$$

where $\Delta^1 y(x) = y(x+1) - y(x)$ and $\Delta^m y(x) = \Delta^1\big(\Delta^{m-1} y(x)\big)$ for $m > 1$. The equation (157) becomes $\frac{2(\exp(\frac{\lambda}{2})-\exp(-\frac{\lambda}{2}))}{\exp(\frac{\lambda}{2})+\exp(-\frac{\lambda}{2})} = a$, whose solutions are $\lambda_k = \ln(\frac{2+a}{2-a}) + 2k\pi i$, where k is an integer (no solution exists for $a = \pm 2$). Here, we conclude that any function of the form

$$y(x) = \phi(x)\left(\frac{2+a}{2-a}\right)^x, \tag{162}$$

where ϕ is a periodic function with period 1, is a solution of the $(-\frac{1}{2})$-differential equation (156).

More generally, the α-differential equation (154) has a solution of the form (156) whenever

$$a\left(\frac{\exp(\frac{\lambda}{2}) - \exp(-\frac{\lambda}{2})}{g_\alpha(\frac{i\lambda}{2})}\right)^2 + b\left(\frac{\exp(\frac{\lambda}{2}) - \exp(-\frac{\lambda}{2})}{g_\alpha(\frac{i\lambda}{2})}\right) + c = 0. \tag{163}$$

This equation can be solved explicitly for $\alpha = \frac{1}{2}$, $\alpha = -\frac{1}{2}$, or $\alpha \to \infty$. For example, any function of the fom

$$y(x) = \phi(x)\left(\frac{\sqrt{5} \pm 1}{2}\right)^{2x}, \tag{164}$$

where ϕ is a periodic function with period 1, is a solution of the ∞-differential equation $(a = 1, b = 0, c = -1)$

$$y(x+1) - 3y(x) + y(x-1) = 0. \tag{165}$$

7. Conclusion

In this paper, and the earlier work [1, 2, 3, 4] we have presented a generalized calculus. We showed that the α-calculus contains all the basic results of what should be considered as an extension of the classical differential and integral calculus. We also studied some α-quantities as a function of the parameter α.

We used the α-calculus, in Section 5, to construct an α-physics, with results such as (85) and (101). Many open problems remain to be solved. In particular, the development of an α-differential equations theory seems to be of interest.

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