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ON REPRESENTATION OF CERTAIN REAL NUMBERS USING COMBINATORIAL IDENTITIES

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Abstract: In this paper we present identities used to represent real numbers of the form $a\,b^n + c$ for appropriately chosen real numbers a, b, c, and non-negative integers n. Multiple forms of these identities using single and triple sums will be presented.

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1. Introduction

There are many examples of combinatorial identities that represent real numbers of certain type (see [8], [7], [6]). These identities were derived from studying the asymptotic behavior of the roots of a sequence of the generalized Fibonacci

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polynomials given by

$$F_j(x) = x^{j-1} - \cdots - x - 1.$$

It is known that the positive roots of $F_j(x)$ are of the form $2 - O(2^{-j})$ (see [2], [4]). Grossman and Narayan [5] have proven that the single negative zero of $F_j(x)$ has the form $-1 + O(j^{-1})$ for j even and tends to -1 monotonically as $j \to \infty$. More general results on all of the zeros of derivatives and integrals of F_j are given in [13]. Using the fact that

$$F_j(x) = (x - 2 + \varepsilon_j)(x + 1 - \delta_j)(x^{j-2} + a_{j-3}\, x^{j-3} + \cdots + a_1\, x + a_0),$$

where δ_j and ε_j are positive, decreasing sequences for $j = 4, 6, \ldots$. Grossman and Zeleke [8] have found an explicit form for the coefficients a_i, where $i = 0, 1, \ldots, j-3$, in terms of δ and ε. The solution was found by solving a non-homogeneous linear recurrence relation of the form

$$-a_n + b\, a_{n+1} + c\, a_{n+2} = 1, \text{ where } b = \varepsilon - 1 - \delta,\ c = (1-\delta)(2-\varepsilon).$$

As byproduct of this solution, several combinatorial identities were formulated by considering even and odd indexed terms of a_n.

Our goal in this paper is to apply similar techniques to the infinite sequence of $F_j'(x)$. It is known that the positive and negative roots of $F_j'(x)$ are of the form $2 - O(j^{-1}), 1 + O(j^{-1})$ (see [5]). Thus, one can write

$$F_j'(x) = (x - 2 + \epsilon_j)(x + 1 - \delta_j) \sum_{k=0}^{j-3} a_k x^k. \tag{1}$$

By multiplying out and equating like powers of x in (1) (suppressing subscripts) we get a system of equations

$$\begin{aligned}
&a_0 = \frac{1}{(1-\delta)(2-\epsilon)}\ , \\
&(1-\delta)(2-\epsilon)a_1 - (\epsilon - 1 - \delta)a_0 = 2\ , \\
&(1-\delta)(2-\epsilon)a_2 - (\epsilon - 1 - \delta)a_1 - a_0 = 3\ , \\
&\vdots \\
&(1-\delta)(2-\epsilon)a_j - (\epsilon - 1 - \delta)a_{j-1} - a_{j-2} = j + 1.
\end{aligned}$$

If we let $b = \epsilon - 1 - \delta$ and $c = (1-\delta)(2-\epsilon)$, and consider the even and odd indexed recurrences separately, then the middle $j - 1$ equations can be written as

$$-a_{2n} - b a_{2n+1} + c a_{2n+2} = 2n + 3,$$

and

$$-a_{2n+1} - ba_{2n+2} + ca_{2n+3} = 2n + 4.$$

By solving these recurrence relations using standard techniques and considering special cases we derive our main results.

2. Main Results

The following identities hold for any $n \geq 0$.

Theorem 1.

$$\frac{1}{4^{2(n+1)}}\sum_{k=0}^{n} 9^k \binom{2n+1}{2k} - \frac{1}{4^{2(n+1)}}\sum_{k=0}^{n} 9^k \binom{2n+1}{2k+1}$$
$$= \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k} \binom{i+1+k}{j} \frac{(-1)^{j+1}}{2^{i+2+k}} - \sum_{i=1}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k} \binom{i+1+k}{j} \frac{(-1)^{j}}{2^{i+2+k}}$$
$$= \frac{1}{12}\left(1 - \frac{1}{4^n}\right).$$

Theorem 2.

$$\frac{1}{4^{2n+1}}\sum_{k=0}^{n} 9^k \binom{2n}{2k} - \frac{1}{4^{2n+1}}\sum_{k=1}^{n} 9^{k-1} \binom{2n}{2k-1}$$
$$= \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k} \binom{i+1+k}{j} \frac{(-1)^{j}}{2^{i+2+k}} + \sum_{i=1}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k} \binom{i+1+k}{j} \frac{(-1)^{j}}{2^{i+2+k}}$$
$$= \frac{1}{6}\left(\frac{1}{4^n} - 1\right).$$

3. Proofs of Main Results

We look for a solution of the form $a_n = c_1 + c_2 n + c_3\alpha^n + c_4\beta^n$. Upon substituting this in the recurrence relation, we get

$$\alpha = \frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{1}{c}} \text{ and } \beta = \frac{b}{2c} - \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{1}{c}},$$

and

$$c_1 = \frac{c-2b-3}{(b+1-c)^2}, \quad c_2 = \frac{-1}{b+1-c},$$

$$c_3 = \frac{3bc + b^3 + b^2c\sqrt{\frac{b^2+4c}{c^2}} + c^2\sqrt{\frac{b^2+4c}{c^2}}}{c^2\sqrt{\frac{b^2+4c}{c^2}}\left(b^2+2c+2c^2-2bc+bc\sqrt{\frac{b^2+4c}{c^2}}-2c^2\sqrt{\frac{b^2+4c}{c^2}}\right)},$$

and

$$c_4 = \frac{-3bc - b^3 + b^2c\sqrt{\frac{b^2+4c}{c^2}} + c^2\sqrt{\frac{b^2+4c}{c^2}}}{c^2\sqrt{\frac{b^2+4c}{c^2}}\left(b^2+2c+2c^2-2bc-bc\sqrt{\frac{b^2+4c}{c^2}}+2c^2\sqrt{\frac{b^2+4c}{c^2}}\right)}.$$

If we rationalize the denominator of c_3, then we can write $c_3 = A_1 + A_2$, and $c_4 = A_1 - A_2$, where

$$A_1 = -\frac{1}{c}\frac{(c+b^2)\left((b-c)^2+(c+1)^2-1\right)+(b^3+3bc)(2c-b)}{\left((b^2+4c)(2c-b)^2-(1-(b-c)^2-(c+1)^2\right)^2},$$

and

$$A_2 = -\frac{(c+b^2)(2c-b)\sqrt{b^2+4c}+(b^3+3bc)\left((b-c)^2+(c+1)^2-1\right)(\sqrt{b^2+4c})^{-1}}{c\left((b^2+4c)(2c-b)^2-(1-(b-c)^2-(c+1)^2\right)^2}.$$

Thus,

$$a_n = \frac{c-2b-3}{(b+1-c)^2} - \left(\frac{1}{b+1-c}\right)n + (A_1+A_2)\sum_{i=0}^{n}\binom{n}{i}\left(\frac{b}{2c}\right)^{n-i}\left(\left(\frac{b}{2c}\right)^2+\frac{1}{c}\right)^{i/2} + (A_1-A_2)\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\frac{b}{2c}\right)^{n-i}\left(\left(\frac{b}{2c}\right)^2+\frac{1}{c}\right)^{i/2}.$$

At this stage we write a_n for even and odd cases separately. For even n (with even and odd terms separated) one gets

$$a_{2n} = \frac{c-2b-3}{(b+1-c)^2} - \left(\frac{1}{b+1-c}\right)2n + 2A_1\sum_{i=0}^{n}\binom{2n}{2i}\left(\frac{b}{2c}\right)^{2(n-i)}\left(\left(\frac{b}{2c}\right)^2+\frac{1}{c}\right)^{i} + 2A_2\sum_{i=1}^{n}\binom{2n}{2i-1}\left(\frac{b}{2c}\right)^{2(n-i)+1}\left(\left(\frac{b}{2c}\right)^2+\frac{1}{c}\right)^{(2i-1)/2}.$$

Substituting $b = -1, c = 2$ in a_{2n} and simplifying gives

$$a_{2n} = \frac{1}{4} + n + \frac{1}{4^{2n+1}} \sum_{i=0}^{n} 9^i \binom{2n}{2i} - \frac{1}{4^{2n+1}} \sum_{i=1}^{n} 9^{i-1} \binom{2n}{2i-1}. \tag{2}$$

A similar approach for the case when n is odd yields

$$a_{2n+1} = \frac{3}{4} + n + \frac{1}{4^{2n+2}} \sum_{i=0}^{n} \left[\binom{2n+1}{2i+1} - \binom{2n+1}{2i} \right] 9^i. \tag{3}$$

We now proceed to derive the same identities using a different approach. We denote the coefficients a_n related to $F_j(x)$ by $a_{0,n}$ and those related to $F'_j(x)$ by $a_{1,n}$. It is known that

$$a_{1,2n} = \sum_{i=0}^{2n} a_{0,2i}, \quad a_{1,2n+1} = \sum_{i=0}^{2n+1} a_{0,2i}, \tag{4}$$

where for all $n \geq 0$,

$$\begin{aligned} -a_{0,n} - ba_{0,n+1} + ca_{0,n+2} &= 1 \text{ with} \\ a_{0,0} &= \frac{1}{c}, \\ a_{0,1} &= \frac{b}{c^2} + \frac{1}{c}, \\ a_{0,2} &= \frac{b^2}{c^3} + \frac{b+1}{c^2} + \frac{1}{c}, \\ a_{0,3} &= \frac{b^3}{c^4} + \frac{b^2+2b}{c^3} + \frac{b+1}{c^2} + \frac{1}{c}. \end{aligned} \tag{5}$$

Note that

$$a_{1,2n} = \sum_{i=0}^{n} a_{0,2i} + \sum_{i=0}^{n-1} a_{0,2i+1} = a_{1,2n+1} - a_{0,2n+1}.$$

From (4) and (5) we get

$$\begin{aligned} a_{1,0} &= \frac{1}{c}, \\ a_{1,1} &= \frac{b}{c^2} + \frac{2}{c}, \\ a_{1,2} &= \frac{b}{c}\left(\frac{b}{c^2} + \frac{2}{c}\right) + \frac{1}{c^2} + \frac{3}{c}, \\ a_{1,3} &= \frac{b}{c}\left(\frac{3}{c} + \frac{1}{c^2} + \frac{b}{c}\left(\frac{b}{c^2} + \frac{2}{c}\right)\right) + \frac{1}{c}\left(\frac{2}{c} + \frac{b}{c^2}\right) + \frac{4}{c}. \end{aligned}$$

It is known that [8],

$$a_{0,2n} = \frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{n+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \frac{1}{c^{n+2}}\sum_{k=0}^{n-1}\sum_{i=2k+2}^{n+1+k}\binom{n+1+k}{i}\frac{b^i}{c^k},$$

$$a_{0,2n+1} = \frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{n+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \frac{1}{c^{n+2}}\sum_{k=0}^{n}\sum_{i=2k+1}^{n+1+k}\binom{n+1+k}{i}\frac{b^i}{c^k}.$$

Thus, we get

$$\begin{aligned} a_{1,2n} &= \sum_{i=0}^{n}\frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{i+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \sum_{i=0}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \\ &+ \sum_{i=0}^{n-1}\frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{i+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \sum_{i=0}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \\ &= 2\sum_{i=0}^{n-1}\frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{i+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{n+1}}{1-\left(\frac{1+b}{c}\right)}\right) \\ &+ \sum_{i=0}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \\ &+ \sum_{i=0}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}}. \end{aligned}$$

Similarly,

$$\begin{aligned} a_{1,2n+1} &= 2\sum_{i=0}^{n}\frac{1}{c}\left(\frac{1-\left(\frac{1+b}{c}\right)^{i+1}}{1-\left(\frac{1+b}{c}\right)}\right) + \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \\ &+ \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}}. \end{aligned}$$

If we let $r = \frac{1+b}{c}$ and use the geometric sum formula to simplify these expressions of $a_{1,2n}$ and $a_{1,2n+1}$, we get

$$\begin{aligned} a_{1,2n} &= \frac{2}{c^2(1-r)^2}\left(cn-(1+b)(n+1)+r^{n+1}\right) + \frac{1-r^{n+1}}{c(1-r)} \\ &+ \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \end{aligned}$$

$$+ \sum_{i=0}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}}, \tag{6}$$

$$\begin{aligned} a_{1,2n+1} &= \frac{2}{c^2(1-r)^2}\left(c(n+1)(1-r)\right) - \frac{2}{c^2(1-r)^2}\left((1+b)-r^{n+2}\right) \\ &+ \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}} \\ &+ \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{b^j}{c^{i+2+k}}. \end{aligned} \tag{7}$$

As before, if we let $c = 2,\ b = -1$ in (6) and (7), then we get

$$\begin{aligned} a_{1,2n} &= n + \frac{1}{2} + \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}} \\ &+ \sum_{i=0}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} a_{1,2n+1} &= n + 1 + \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}} \\ &+ \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}. \end{aligned} \tag{9}$$

Note that the first term in the second triple sum (i.e. the term corresponding to $i = 0$) of (8) and (9) is $-1/4$. Hence if the summation index i starts at 1 in each case, we get

$$\begin{aligned} a_{1,2n} &= n + \frac{1}{4} + \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}} \\ &+ \sum_{i=1}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}, \end{aligned} \tag{10}$$

and

$$a_{1,2n+1} = n + \frac{3}{4} + \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}$$

$$+ \sum_{i=1}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}. \tag{11}$$

Comparing the expressions for a_{2n} in (2) and (10) and for a_{2n+1} in (3) and (11) gives

$$\begin{aligned}
&\frac{1}{4^{2(n+1)}}\sum_{k=0}^{n}9^k\binom{2n+1}{2k} - \frac{1}{4^{2(n+1)}}\sum_{k=0}^{n}9^k\binom{2n+1}{2k+1}\\
&= \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^{j+1}}{2^{i+2+k}} - \sum_{i=1}^{n}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}\\
&= \frac{1}{12}\left(1-\frac{1}{4^n}\right),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\frac{1}{4^{2n+1}}\sum_{k=0}^{n}9^k\binom{2n}{2k} - \frac{1}{4^{2n+1}}\sum_{k=1}^{n}9^{k-1}\binom{2n}{2k-1}\\
&= \sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=2k+2}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}} + \sum_{i=1}^{n-1}\sum_{k=0}^{i}\sum_{j=2k+1}^{i+1+k}\binom{i+1+k}{j}\frac{(-1)^j}{2^{i+2+k}}\\
&= \frac{1}{6}\left(\frac{1}{4^n}-1\right).
\end{aligned} \tag{13}$$

Remark. There are known results for representation of real numbers of type (12) and (13) using single sums. For example, using a computer algebra system such as *Maple* and the results in [1], it is well-known that

$$\begin{aligned}
\sum_{k=0}^{n}\frac{1}{a^k}\binom{n+k}{2k} &= \frac{1}{(2a)^{n+1}}\frac{a}{\sqrt{4a+1}}[(2a+1+\sqrt{4a+1})^n(\sqrt{4a+1}+1)\\
&+ (2a+1-\sqrt{4a+1})^n(\sqrt{4a+1}-1)],
\end{aligned} \tag{14}$$

$$\sum_{k=0}^{n-1}\frac{1}{a^k}\binom{n+k}{2k+1} = \frac{1}{(2a)^n}\frac{a}{\sqrt{4a+1}}(\alpha^n-\beta^n), \tag{15}$$

where $\alpha = 2a+1+\sqrt{4a+1}, \beta = 2a+1-\sqrt{4a+1}$. By substituting $a = 2$ and multiplying both sides of (14) and (15) by $2^{-(n+2)}$, one recovers the results discussed in [8]

$$\frac{1}{2^{n+2}}\sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{2k} = \frac{1}{4^{2n+1}}\sum_{k=0}^{n}9^k\binom{2n+1}{2k+1} = \frac{1}{6}\left(1+\frac{2}{4^{n+1}}\right)$$

and

$$\frac{1}{2^{n+2}}\sum_{k=0}^{n-1}\frac{1}{2^k}\binom{n+k}{2k+1}=\frac{1}{4^{2n}}\sum_{k=0}^{n-1}9^k\binom{2n}{2k+1}=\frac{1}{6}\left(1-\frac{1}{4^n}\right).$$

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