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EXISTENCE OF PERIODIC SOLUTIONS FOR n-TH ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract: By employing the coincidence degree theory of Mawhin, we study the existence of periodic solutions for n-th order differential equations with deviating argument $x^{(n)}(t)+\sum\limits_{i=2}^{n-1} b_i x^{(i)}(t)+f(x(t))x'(t)+g(t,x(t),x(t-\tau(t))) = p(t)$. Some new results on the existence of periodic solutions of the equations are obtained. Our work generalizes the known results.

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1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the n-th order differential equations with deviating arguments

$$x^{(n)}(t)+\sum_{i=2}^{n-1} b_i x^{(i)}(t)+f(x(t))x'(t)+g(t,x(t),x(t-\tau(t))) = p(t), \tag{1.1}$$

where b_i $(i = 2, \cdots, n-1)$ are constants, $f \in C(R,R)$, $g \in C(R^3,R)$ and $g(t+2\pi, x, y) = g(t, x, y)$ for $\forall(x,y) \in R^2$, $p \in C(R,R), \tau \in C(R,R)$ with $p(t+2\pi) = p(t), \tau(t+2\pi) = \tau(t)$.

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In recent years, there are many papers studying the existence of periodic solutions of the higher order differential equations of the form

$$x^{(2n)}(t) + \sum_{j=1}^{n-1} a_j x^{(2j)}(t) + (-1)^{(k+1)} g(t,x) = 0, \tag{1.2}$$

$$x^{(2n+1)}(t) + \sum_{j=1}^{n-1} a_j x^{(2j+1)}(t) + g(t,x) = 0. \tag{1.3}$$

The authors obtain the existence of periodic solutions, see [1], [2], [10], [9], [8], [3], [6], [17], [11]. For example, in [9], Liu studied the existence of periodic solutions of the following differential equations.

$$x^{(n)}(t) + \sum_{i=2}^{n-1} a_i x^{(i)}(t) + f_1(x)|x^{'}(t)|^2 + f_2(x)x^{'}(t) + g(t,x(t)) = e(t). \tag{1.4}$$

The author established the theorems of the existence of periodic solutions of equation (1.4), and the theorems were related to $f_1(x)$ and $g(t,x)$.

In the present paper, by using Mawhin's Continuation Theorem, we will establish some theorems on the existence of periodic solutions of equation (1.1). The results are related to not only b_i and the x, y of $g(t,x,y)$ but also the deviating argument $\tau(t)$. In particular, in case $n = 2, g(t,x,y) = g(x)$ equation (1.1) is reduced to second order Liénard equations

$$x^{''}(t) + f(x)x^{'}(t) + g(x(t)) = p(t). \tag{1.5}$$

For the existence of periodic solutions of first, second or third order differential equations, the references [16], [12], [13], [18], [5], [7], [20], [19], [4], [21] may be consulted.

2. Some Lemmas

We investigate the theorems based on the following lemmas.

Lemma 2.1. (see [15]) *Suppose $x \in C^n(R,R)$, and $x(t+2\pi) = x(t)$, then*

$$\left(\int_0^{2\pi} |x^{'}(t)|^2 dt\right)^{\frac{1}{2}} \le \left(\int_0^{2\pi} |x^{''}(t)|^2 dt\right)^{\frac{1}{2}} \le \cdots \le \left(\int_0^{2\pi} |x^{(n)}(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.1}$$

Lemma 2.2. (see [13]) *Let $\alpha \in [0, +\infty)$ be constants, $s \in C(R,R)$ with $s(t+2\pi) = s(t)$, and $s(t) \in [-\alpha, \alpha], \forall t \in [0, 2\pi]$. Then for $\forall x \in C^1(R,R)$ with $x(t+2\pi) = x(t)$, we have*

$$\int_0^{2\pi} |x(t) - x(t-s(t))|^2 dt \le 2\alpha^2 \int_0^{2\pi} |x^{'}(t)|^2 dt. \tag{2.2}$$

We first introduce Mawhin's Continuation Theorem.

Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero, here $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y$ be projectors such that

$$ImP = KerL, KerQ = ImL, X = KerL \oplus KerP, Y = ImL \oplus ImQ.$$

It follows that

$$L|_{D(L)\cap KerP} : D(L) \cap KerP \to ImL$$

is invertible, we denote the inverse of that map by K_p. Let Ω be an open boumded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.3. (see [14]) *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $\quad Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$.

(ii) $\quad QNx \neq 0, \forall x \in \partial\Omega \cap KerL$.

(iii) $\quad deg\{QNx, \Omega \bigcap KerL, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap D(L)$.

Now, we define $Y = \{x \in C(R, R) \mid x(t + 2\pi) = x(t)\}$ with the norm $|x|_\infty = \max_{t\in[0,2\pi]}\{|x(t)|\}$ and $X = \{x \in C^{n-1}(R, R) \mid x(t + 2\pi) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x^{'}|_\infty \cdots, |x^{(n-1)}|_\infty\}$, we can easily see that X, Y are two Banach spaces. we also define the operators L and N as follows:

$$\begin{aligned} &L : D(L) \subset X \to Y, \qquad Lx = x^{(n)},\\ &D(L) = \{x | x \in C^n(R, R), x(t + 2\pi) = x(t)\}, \qquad (2.3)\\ &N : X \to Y,\end{aligned}$$

$$Nx = -\sum_{i=2}^{n-1} b_i x^{(i)}(t) - f(x(t))x^{'}(t) - g(t, x(t), x(t - \tau(t)) + p(t). \qquad (2.4)$$

It is easy to see that equation (1.1) can be converted the abtract equation $Lx = Nx$. Moreover, from the definition of L, we can see $kerL = R$, $dim(kerL) = 1$, $ImL = \{y | y \in Y, \int_0^{2\pi} y(s)ds = 0\}$ is subset, and $dim(Y \setminus ImL) = 1$, we have $codim(ImL) = dim(kerL)$, so L is a Fredholm operator with index zero. Let

$$P : X \longrightarrow KerL, Px = x(0), Q : Y \longrightarrow Y \setminus ImL, Qy = \frac{1}{2\pi}\int_0^{2\pi} y(t)dt$$

and let

$$L|_{D(L)\cap KerP} : D(L) \cap KerP \to ImL.$$

Then $L|_{D(L)\cap KerP}$ has a unique continuous inverse K_p. One can easily find that N is L-compact in $\overline{\Omega}$, where $\overline{\Omega}$ is an open bounded subset of X.

3. Main Result

Theorem 3.1. *Suppose $n = 4k+1$ for a positive integer, and the following conditions hold:*

(H_1) there is a constant $c > 0$, such that $\max\limits_{x\in R} |f(x)| \leq \sigma$,

(H_2) there is a constant $c > 0$, such that $g(t,x,y) > |p(t)|_\infty$, $\forall t \in R$, $x, y > c$,

(H_3) there is a constant $c > 0$, such that $g(t,x,y) < -|p(t)|_\infty$, $\forall t \in R$, $x, y < -c$,

(H_4) the function f has the decomposition

$$g(t,x,y) = h(t,x) + e(t,y), \tag{3.1}$$

such that

$$|h(t,x)| \leq \beta_1 + \beta_2|x|, \tag{3.2}$$

$$|e(t,x) - e(t,y)| \leq \alpha|x-y|, \tag{3.3}$$

and

$$\lim_{x\to\infty} |\tfrac{e(t,x)}{x}| \leq \gamma, \tag{3.4}$$

where $\beta_1, \beta_2, \alpha, \gamma > 0$.

If

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty + \sigma + 2\pi\beta_2 + 2\pi\gamma < B_1, \tag{3.5}$$

where $B_1 = 1 - \sum\limits_{i=1}^{k}|b^-_{4i-3}| - \sum\limits_{i=1}^{k}|b^+_{4i-1}|$, $b_i^+ = \max\{b_i, 0\}$, $b_i^- = \min\{b_i, 0\}$. Then equation (1.1) has at least one 2π-periodic solution.

Proof. Consider the equation

$$Lx = \lambda Nx, \lambda \in (0,1),$$

where L and N are defined by (2.3) and (2.4). Let

$$\Omega_1 = \{x \in D(L)/KerL, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}$$

for $x \in \Omega_1$. We have

$$x^{(n)}(t) + \lambda\sum_{i=2}^{n-1} b_i x^{(i)} + \lambda f(x(t))x^{'}(t) + \lambda g(t, x(t)x(t-\tau(t))) = \lambda p(t),$$

$$\lambda \in (0,1). \quad (3.6)$$

Integrating them on $[0,T]$, we have

$$\int_0^{2\pi} g(t,x(t),x(t-\tau(t)))dt - \int_0^{2\pi} p(t)dt = 0 \quad (3.7)$$

We can prove that there is a $t_1 \in [0,T]$ such that $|x(t_1)| < c$. Indeed, from (3.7), there is a $t_0 \in [0,T]$ such that

$$g(t_0,x(t_0),x(t_0-\tau(t_0))) = \frac{1}{T}\int_0^T p(t)dt \le |p(t)|_\infty. \quad (3.8)$$

If $|x(t_0)| \le c$, then taking $t_1 = t_0$ such that $|x(t_1)| \le c$.

If $|x(t_0)| > c$, it follows from assumptions (H_2) and (H_3) that $|x(t_0 - \tau(t_0))| \le c$. Since $x(t)$ is continuous for $t \in R$ and $x(t+2\pi) = x(t)$. So there must be an integer k and a point $t_1 \in [0,T]$ such that $t_0 - \tau(t_0) = 2\pi k + t_1$. So $|x(t_1)| = |x(t_0 - \tau(t_0))| < c$, which implies

$$|x(t)|_\infty \le c + \int_0^{2\pi} |x^{'}(t)|dt. \quad (3.9)$$

Then, multiplying both sides of (3.6) by $x^{(n)}(t)$, and integrating them on $[0,T]$, we have for $\lambda \in (0,1)$

$$\int_0^{2\pi} |x^{(n)}(t)|^2dt + \lambda\sum_{i=2}^{n-1} b_i \int_0^{2\pi} x^{(i)}(t)x^{(n)}(t)dt + \lambda\int_0^{2\pi} f(x(t))x^{'}(t)x^{(n)}(t)dt$$
$$+ \lambda\int_0^{2\pi} g(t,x(t),x(t-\tau(t)))x^{(n)}(t)dt = \lambda\int_0^{2\pi} p(t)x^{(n)}(t)dt. \quad (3.10)$$

It is easy to see that, for positive integer i and $n = 4k+1$,

$$\int_0^{2\pi} x^{(2i)}(t)x^{(n)}(t)dt = 0, \int_0^{2\pi} x^{(2i-1)}(t)x^{(n)}(t)dt$$
$$= (-1)^{2k-i+1}\int_0^{2\pi}[x^{(2k+i)}(t)]^2dt. \quad (3.11)$$

From (3.10), we have

$$\int_0^{2\pi}[x^{(n)}(t)]^2dt = -\lambda\sum_{i=1}^{2k}(-1)^{2k-i+1}b_{2i-1}\int_0^{2\pi}[x^{(2k+i)}(t)]^2dt$$
$$-\lambda\int_0^{2\pi} f(x(t))x^{'}(t)x^{(n)}(t)dt - \lambda\int_0^{2\pi} g(t,x(t),x(t-\tau(t)))x^{(n)}(t)dt$$
$$+\lambda\int_0^{2\pi} p(t)x^{(n)}(t)dt = -\lambda\sum_{i=1}^{k} b_{4i-3}\int_0^{2\pi}[x^{(2k+2i-1)}(t)]^2dt$$

$$+\lambda\sum_{i=1}^{k}b_{4i-1}\int_0^{2\pi}[x^{(2k+2i)}(t)]^2dt+\lambda\int_0^{2\pi}f(x(t))x^{'}(t)x^{(n)}(t)dt$$
$$+\lambda\int_0^{2\pi}g(t,x(t),x(t-\tau)))x^{(n)}(t)dt+\lambda\int_0^{2\pi}p(t)x^{(n)}(t)dt$$
$$\leq\sum_{i=1}^{k}|b_{4i-3}^{-}|\int_0^{2\pi}[x^{(2k+2i-1)}(t)]^2dt+\sum_{i=1}^{k}|b_{4i-1}^{+}|\int_0^{2\pi}[x^{(2k+2i)}(t)]^2dt$$
$$+\int_0^{2\pi}|f(x(t))||x^{'}(t)|x^{(n)}(t)|dt\int_0^{2\pi}|f(t,x(t),x(t-\tau(t)))||x^{'}(t)|dt$$
$$+\int_0^{2\pi}|p(t)||x^{(n)}(t)|dt\leq\sum_{i=1}^{k}|b_{4i-3}^{-}|\int_0^{2\pi}[x^{(2k+2i-1)}(t)]^2dt$$
$$+\sum_{i=1}^{k}|b_{4i-1}^{+}|\int_0^{2\pi}[x^{(2k+2i)}(t)]^2dt+\sigma\int_0^{2\pi}|x^{'}(t)||x^{(n)}(t)|dt$$
$$+\int_0^{2\pi}|h(t,x(t))||x^{(n)}(t)|dt+\int_0^{2\pi}|e(t,x(t-\tau(t)))||x^{(n)}(t)|dt+\int_0^{2\pi}|p(t)||x^{(n)}(t)|dt$$
$$\leq(\sum_{i=1}^{k}|b_{4i-3}^{-}|+\sum_{i=1}^{k}|b_{4i-1}^{+}|)\int_0^{2\pi}[x^{(n)}(t)]^2dt+\sigma\int_0^{2\pi}|x^{'}(t)||x^{(n)}(t)|dt$$
$$+\int_0^{2\pi}|h(t,x(t))||x^{(n)}(t)|dt+\int_0^{2\pi}|e(t,x(t-\tau(t)))-e(t,x(t))||x^{(n)}(t)|dt$$
$$+\int_0^{2\pi}|e(t,x(t))||x^{(n)}(t)|dt+\int_0^{2\pi}|p(t)||x^{'}(t)|dt. \quad (3.12)$$

Thus, applying Lemma 2.1, we obtain

$$B_1\int_0^{2\pi}|x^{(n)}(t)|^2dt\leq\sigma\int_0^{2\pi}|x^{'}(t)||x^{(n)}(t)|dt+\int_0^{2\pi}|h(t,x(t))||x^{(n)}(t)|dt$$
$$+\int_0^{2\pi}|e(t,x(t-\tau(t)))-e(t,x(t))||x^{(n)}(t)|dt+\int_0^{2\pi}|e(t,x(t))||x^{(n)}(t)|dt$$
$$+|\int_0^{2\pi}|p(t)||x^{(n)}(t)|dt. \quad (3.13)$$

Choose a constant $\varepsilon>0$ such that

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty+\sigma+2\pi\beta_2+2\pi(\gamma+\varepsilon)<B_1.$$

For the above constant $\varepsilon>0$, from (3.4) we see that there is a constant $\delta>0$ such that

$$|e(t,x)|<(\gamma+\varepsilon)|x|,\quad \text{for}\quad |x|>\delta,t\in[0,2\pi]. \quad (3.14)$$

Denote

$$\Delta_1 = \{t \in [0, 2\pi] : |x(t)| \le \delta\}, \qquad \Delta_2 = \{t \in [0, 2\pi] : |x(t)| > \delta\}. \tag{3.15}$$

Applying Schwarz inequality and Lemma 2.1, from (3.4) it follows that

$$\begin{aligned}
\int_0^{2\pi} |e(t,x(t))||x^{(n)}(t)|dt &\le \int_{\Delta_1} |e(t,x(t))||x^{(n)}(t)|dt + \int_{\Delta_2} |e(t,x(t))||x^{(n)}(t)|dt \\
&\le e_\delta \int_0^{2\pi} |x^{(n)}(t)|dt + (\gamma+\varepsilon)\int_0^{2\pi} |x(t)||x^{(n)}(t)|dt \\
&\le e_\delta \int_0^{2\pi} |x^{(n)}(t)|dt + (\gamma+\varepsilon)(c + \int_0^{2\pi} |x'(t)|dt)\int_0^{2\pi} |x^{(n)}(t)|dt) \\
&= [e_\delta + (\gamma+\varepsilon)c]\int_0^{2\pi} |x^{(n)}(t)|dt + 2\pi(\gamma+\varepsilon)(\int_0^{2\pi} |x'(t)|^2dt)^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}} \\
&\le [e_\delta + (\gamma+\varepsilon)c](\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}}(2\pi)^{\frac{1}{2}} + 2\pi(\gamma+\varepsilon)\int_0^{2\pi} |x^{(n)}(t)|^2dt,
\end{aligned} \tag{3.16}$$

where $e_\delta = \max_{t\in[0,2\pi],|x|\le\delta} |e(t,x)|$. By using Schwarz inequality and Lemma 2.1 and Lemma 2.2, it follows from (3.3) that

$$\begin{aligned}
\int_0^{2\pi} |e(t,x(t)) - e(t,x(t-\tau(t))||x^{(n)}(t)|dt &\le \alpha \int_0^{2\pi} |x(t) - x(t-\tau(t))||x^{(n)}(t)|dt \\
&\le \alpha(\int_0^{2\pi} |x(t) - x(t-\tau(t))|^2dt)^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}} \\
&\le 2^{\frac{1}{2}}\alpha|\tau(t)|_\infty(\int_0^{2\pi} |x'(t)|^2dt)^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}} \le 2^{\frac{1}{2}}\alpha|\tau(t)|_\infty \\
&\quad \times \int_0^{2\pi} |x^{(n)}(t)|^2dt.
\end{aligned} \tag{3.17}$$

From (3.2), (3.9) and Lemma 2.1, we get

$$\begin{aligned}
\int_0^{2\pi} |h(t,x(t))||x^{(n)}(t)|dt &\le \int_0^{2\pi} [\beta_1 + \beta_2|x(t)|]|x^{(n)}(t)|dt \\
&\le \beta_1 \int_0^{2\pi} |x^{(n)}(t)|dt + \beta_2|x(t)|_\infty \int_0^{2\pi} |x^{(n)}(t)|dt \\
&\le (\beta_1 + \beta_2 c)2\pi^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|dt)^{\frac{1}{2}} + \beta_2 2\pi(\int_0^{2\pi} |x'(t)|^2dt)^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}} \\
&\le (\beta_1 + \beta_2 c)(2\pi)^{\frac{1}{2}}(\int_0^{2\pi} |x^{(n)}(t)|^2dt)^{\frac{1}{2}} + 2\pi\beta_2 \int_0^{2\pi} |x^{(n)}(t)|^2dt.
\end{aligned} \tag{3.18}$$

By using Schwarz inequality and Lemma 2.1, we obtain

$$\int_0^{2\pi}|p(t)||x^{(n)}(t)|dt \le |p(t)|_\infty \int_0^{2\pi}|x^{(n)}(t)|dt$$
$$\le |p(t)|_\infty (2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(n)}(t)|^2dt)^{\frac{1}{2}} \quad (3.19)$$

and

$$\int_0^{2\pi}|x^{'}(t)||x^{(n)}(t)|dt$$
$$\le (\int_0^{2\pi}|x^{'}(t)|^2dt)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(n)}(t)|^2dt)^{\frac{1}{2}} \le \int_0^{2\pi}|x^{(n)}(t)|^2dt. \quad (3.20)$$

Substituting the above formula, (3.16), (3.17), (3.18) and (3.19) into (3.13), we have

$$B_1\int_0^{2\pi}|x^{(n)}(t)|^2dt \le [\beta_1+\beta_2c+(e_\delta+\gamma c+\varepsilon c)+|p(t)|_\infty](2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(n)}(t)|^2dt)^{\frac{1}{2}}$$
$$+ [2^{\frac{1}{2}}\alpha|\tau(t)|_\infty + \sigma + 2\pi\beta_2 + 2\pi(\gamma+\varepsilon)]\int_0^{2\pi}|x^{(n)}(t)|^2dt. \quad (3.21)$$

Hence

$$[B_1 - 2^{\frac{1}{2}}\alpha|\tau(t)|_\infty - \sigma - 2\pi\beta_2 - 2\pi(\gamma+\varepsilon)]\int_0^{2\pi}[x^{(n)}(t)]^2dt$$
$$\le [\beta_1+\beta_2c+(e_\delta+\gamma c+\varepsilon c)+|p(t)|_\infty](2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(n)}(t)|^2dt)^{\frac{1}{2}}. \quad (3.22)$$

Hence there is a constant $M > 0$ such that

$$\int_0^{2\pi}|x^{(n)}(t)|^2dt \le M. \quad (3.23)$$

We claim that

$$|x^{(i)}(t)| \le 2\pi^{n-i-1}\int_0^{2\pi}|x^{(n)}(t)|dt \qquad (i = 1, 2, \cdots, n-1). \quad (3.24)$$

In fact, noting that $x^{(n-2)}(0) = x^{(n-2)}(2\pi)$, there must be a constant $\xi_1 \in [0, 2\pi]$ such that $x^{(n-1)}(\xi_1) = 0$, we obtain

$$|x^{(n-1)}(t)| = |x^{(n-1)}(\xi_1) + \int_{\xi_1}^t x^{(n)}(s)ds| \le |x^{(n-1)}(\xi_1)|$$
$$+ \int_0^{2\pi}|x^{(n)}(t)|dt = \int_0^{2\pi}|x^{(n)}(t)|dt. \quad (3.25)$$

Similarily, since $x^{(n-3)}(0) = x^{(n-3)}(2\pi)$, there must be a constant $\xi_2 \in [0, 2\pi]$ such that $x^{(n-2)}(\xi_2) = 0$, from (3.25) we get

$$|x^{(n-2)}(t)| = |x^{(n-2)}(\xi_2) + \int_{\xi_2}^{t} x^{(n-1)}(s)ds| \leq \int_0^{2\pi} |x^{(n-1)}(t)|dt \leq 2\pi \int_0^{2\pi} |x^{(n)}(t)|dt. \quad (3.26)$$

By induction, we have

$$|x^{(i)}(t)| \leq (2\pi)^{n-i-1} \int_0^{2\pi} |x^{(n)}(t)|dt \quad (i = 1, 2, \cdots, n-1). \quad (3.27)$$

Furthermore, we have

$$|x^{(i)}(t)|_\infty \leq (2\pi)^{n-i-1} \int_0^{2\pi} |x^{(n)}(t)|dt \leq (2\pi)^{n-i-\frac{1}{2}} M^{\frac{1}{2}} \quad (i = 1, 2, \cdots, n-1) \quad (3.28)$$

and from (3.9), we obtain

$$|x(t)|_\infty \leq c + (2\pi)^{\frac{1}{2}} (\int_0^{2\pi} |x^{'}(t)|^2 dt)^{\frac{1}{2}} \leq c + (2\pi)^{\frac{1}{2}} (\int_0^{2\pi} |x^{(n)}(t)|^2 dt)^{\frac{1}{2}} \leq c + (2\pi)^{\frac{1}{2}} M^{\frac{1}{2}}. \quad (3.29)$$

It follows that there is a constant $B > 0$ such that $\|x\| \leq B$. Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in KerL, QNx = 0\}$. Suppose $x \in \Omega_2$, then $|x(t)| = d \in R$ and satisfies

$$QNx = \frac{1}{2\pi} \int_0^{2\pi} [-g(t, d, d) + p(t)]dt = 0. \quad (3.30)$$

From (3.30) and assumptions (H_2) and (H_3), we have $d \leq c$, which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2}$. We can easily see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument we have:

(i) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$.

(ii) $QNx \neq 0$, $\forall x \in \partial\Omega \cap KerL$.

At last we will prove (iii) of Lemma 2.3 is satisfied. We take

$$\begin{aligned} &H(x, \mu) : \overline{\Omega} \times [0, 1] \to X \\ &H(x, \mu) = -\mu x + \frac{1-\mu}{2\pi} \int_0^{2\pi} [-g(t, x(t), x(t-\tau(t))) + p(t)]dt. \end{aligned} \quad (3.31)$$

From assumptions (H_1) and (H_2), we can easily obtain $H(x, \mu) \neq 0, \forall (x, \mu) \in \Omega \cap KerL \times [0, 1]$,which results in

$$deg\{QNx,\Omega\cap KerL,0\}=deg\{H(x,0),\Omega\cap KerL,0\}$$
$$=deg\{H(x,1),\Omega\cap KerL,0\}\neq 0. \quad (3.32)$$

Hence, by using Lemma 2.3, we know that equation (1.1) has at least one T-periodic solution. □

Theorem 3.2. *Suppose* $n=4k+3$ *for a positive integer, and* $(H_1)-(H_4)$ *hold. If*

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty+\sigma+2\pi\beta_2+2\pi\gamma<B_2, \quad (3.33)$$

where $B_2=1-\sum_{i=1}^{k}|b_{4i-1}^-|-\sum_{i=1}^{k+1}|b_{4i-3}^+|$, *then equation (1.1) has at least one* 2π*-periodic solution.*

Theorem 3.3. *Suppose* $n=4k$ *for a positive integer,* (H_1) *and* (H_4) *hold. If*

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty+\sigma+2\pi\beta_2+2\pi\gamma<B_3, \quad (3.34)$$

where $B_3=1-\sum_{i=1}^{k}|b_{4i-2}^+|-\sum_{i=1}^{k-1}|b_{4i}^-|$, *then equation (1.1) has at least one* 2π*-periodic solution.*

Proof. Since assumption $(H_1)-(H_4)$ hold, from the proof of Theorem 3.1, we see that

$$|x(t)|_\infty\leq c+\int_0^T|x^{'}(t)|dt. \quad (3.35)$$

Multiplying the both sides of (3.6) by $x^{(n)}(t)$, similarly to (3.30), we have

$$[B_3-2^{\frac{1}{2}}\alpha|\tau(t)|_\infty-\sigma-2\pi\beta_2-2\pi(\gamma+\varepsilon)]\int_0^{2\pi}[x^{(n)}(t)]^2dt$$
$$\leq[\beta_1+\beta_2c+e_\delta+\gamma c+\varepsilon c+|p(t)|_\infty](2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(n)}(t)|^2dt)^{\frac{1}{2}}. \quad (3.36)$$

Hence there is a constant $M>0$ such that

$$\int_0^{2\pi}|x^{(n)}(t)|^2dt\leq M. \quad (3.37)$$

The remainder can be proved in the same way as in the proof of Theorem 3.1 □

Theorem 3.4. *Suppose* $n=4k+2$ *for a positive integer,* $(H_1)-(H_4)$ *hold, If*

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty+\sigma+2\pi\beta_2+2\pi\gamma<B_4, \quad (3.38)$$

where $B_4=1-\sum_{i=1}^{k}|b_{4i-2}^-|-\sum_{i=1}^{k}|b_{4i}^+|$, *then equation (1.1) has at least one* 2π*-*

periodic solution.

Theorem 3.5. *Suppose* $n = 2k+1$, $k > 0$ *for a positive integer,* (H_1)-(H_4) *hold. If there is a positive integer* $0 < s \le k$ *such that*

$$\begin{cases} b_{2s} \neq 0, & \text{if} \quad s = k, \\ b_{2s} \neq 0, b_{2s+i} = 0, i = 1, 2, \cdots, 2k-2s, & \text{if} \quad 0 < s < k, \end{cases} \tag{3.39}$$

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty + \sigma + 2\pi\beta_2 + 2\pi\gamma < B_5, \tag{3.40}$$

where $B_5 = b_{2s} - \sum\limits_{i=1}^{s-1} |A_{2i}^-|$, $A_{2i}^- = \min\{(-1)^{s-i}b_{2i}, 0\}$. *Then equation (1.1) has at least one* 2π*-periodic solution.*

Proof. Since assumption (H_1)-(H_4) hold, according to the proof of Theorem 3.1, we see that

$$|x(t)|_\infty \le c + \int_0^{2\pi} |x^{'}(t)|dt. \tag{3.41}$$

Multiplying both sides of (3.6) by $x^{(2s)}(t)$, and integrating them on $[0, 2\pi]$, we have for $\lambda \in (0, 1)$

$$\begin{aligned} \int_0^{2\pi} x^{(n)}(t)x^{(2s)}(t)dt + \lambda \sum_{i=2}^{n-1} b_i \int_0^{2\pi} x^{(i)}(t)x^{(2s)}(t)dt \\ + \lambda \int_0^{2\pi} f(x(t))x^{'}(t)x^{(2s)}(t)dt \\ + \lambda \int_0^{2\pi} g(t, x(t), x(t-\tau(t)))x^{(2s)}(t)dt = \lambda \int_0^{2\pi} p(t)x^{(2s)}(t)dt. \end{aligned} \tag{3.42}$$

It is easy to see that, for positive integer i and $2s$,

$$\begin{aligned} &\int_0^{2\pi} x^{(2i)}(t)x^{(2s)}(t)dt = (-1)^{s-i} \int_0^{2\pi} [x^{(s+i)}(t)]^2 dt, \\ &\int_0^{2\pi} x^{(2i-1)}(t)x^{(2s)}(t)dt = 0. \end{aligned} \tag{3.43}$$

From (3.42), it follows that

$$\begin{aligned} b_{2s} &\int_0^{2\pi} |x^{(2s)}(t)|^2 dt \\ &= -\lambda \sum_{i=1}^{s-1} (-1)^{s-i} b_{2i} \int_0^{2\pi} |x^{(s+i)}(t)|^2 dt - \lambda \int_0^{2\pi} f(x(t))x^{'}(t)x^{(2s)}(t)dt \\ &\quad - \lambda \int_0^{2\pi} g(t, x(t), x(t-\tau(t)))x^{(2s)}(t)dt + \lambda \int_0^{2\pi} p(t)x^{(2s)}(t)dt \end{aligned}$$

$$\leq \sum_{i=1}^{s-1}|A_{2i}^{-}|\int_0^{2\pi}|x^{(s+i)}(t)|^2dt+\int_0^{2\pi}|f(x(t))x^{'}(t)||x^{(2s)}(t)|dt$$
$$+\int_0^{2\pi}|g(t,x(t),x(t-\tau(t)))||x^{(2s)}(t)|dt+\int_0^{2\pi}|p(t)||x^{(2s)}(t)|dt. \quad (3.44)$$

Similarly to (3.22), we obtain

$$[B_5-2^{\frac{1}{2}}\alpha|\tau(t)|_\infty-\sigma-2\pi\beta_2-2\pi(\gamma+\varepsilon)]\int_0^{2\pi}[x^{(2s)}(t)]^2dt$$
$$\leq [\beta_1+\beta_2c+e_\delta+\gamma c+\varepsilon c+|p(t)|_\infty](2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{(2s)}(t)|^2dt)^{\frac{1}{2}}. \quad (3.45)$$

Hence there is a constant $M_1>0$ such that

$$\int_0^{2\pi}|x^{(2s)}(t)|^2dt\leq M_1. \quad (3.46)$$

On the other hand, from(3.6), we have for $\lambda\in(0,1)$

$$\int_0^{2\pi}x^{(n)}(t)dt$$
$$\leq \sum_{i=2}^{2s}|b_i|\int_0^{2\pi}|x^{(i)}(t)|dt+\int_0^{2\pi}|f(x(t))||x^{'}(t)|dt+\int_0^{2\pi}|h(t,x(t))|dt$$
$$+\int_0^{2\pi}|e(t,x(t))|dt+\int_0^{2\pi}|e(t,x(t))-e(t,x(t-\tau(t)))|dt+\int_0^{2\pi}|p(t)|dt$$
$$\leq (2\pi)^{\frac{1}{2}}\sum_{i=2}^{2s}|b_i|(\int_0^{2\pi}|x^{(i)}(t)|^2dt)^{\frac{1}{2}}+(2\pi)^{\frac{1}{2}}\sigma(\int_0^{2\pi}|x^{'}(t)|^2dt)^{\frac{1}{2}}+2\pi\beta_1$$
$$+2\pi\beta_2[c+(2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{'}(t)|^2dt))^{\frac{1}{2}}]+2\pi e_\delta+2\pi(\gamma+\varepsilon)[c+(2\pi)^{\frac{1}{2}}(\int_0^{2\pi}|x^{'}(t)|^2dt))^{\frac{1}{2}}]$$
$$+2^{\frac{1}{2}}\alpha|\tau(t)|_\infty(\int_0^{2\pi}|x^{'}(t)|^2dt)^{\frac{1}{2}}+2\pi|p(t)|_\infty$$
$$\leq [(2\pi)^{\frac{1}{2}}\sum_{i=2}^{2s}|b_i|+(2\pi)^{\frac{1}{2}}\sigma+2\pi\beta_2+(2\pi)^{\frac{3}{2}}(\gamma+\varepsilon)(2\pi)^{\frac{1}{2}}$$
$$+2^{\frac{1}{2}}\alpha|\tau(t)|_\infty](\int_0^{2\pi}|x^{(2s)}(t)|^2dt)^{\frac{1}{2}}+2\pi\beta_1+2\pi\beta_2c+2\pi e_\delta+2\pi(\gamma+\varepsilon)c+2\pi|p(t)|_\infty$$
$$\leq [(2\pi)^{\frac{1}{2}}\sum_{i=2}^{2s}|b_i|+(2\pi)^{\frac{1}{2}}\sigma+2\pi\beta_2+(2\pi)^{\frac{3}{2}}(\gamma+\varepsilon)(2\pi)^{\frac{1}{2}}+2^{\frac{1}{2}}\alpha|\tau(t)|_\infty](M_1)^{\frac{1}{2}}$$
$$+2\pi\beta_1+2\pi\beta_2c+2\pi e_\delta+2\pi(\gamma+\varepsilon)c+2\pi|p(t)|_\infty=M. \quad (3.47)$$

We can finish the remainder proof in the same way as in the proof of Theorem 3.1. □

Theorem 3.6. *Suppose* $n = 2k$, $k > 1$ *for a positive integer,* $(H_1) - (H_4)$ *hold, If there is a positive integer* $0 < s \leq k$ *such that*

$$\begin{cases} b_{2s-1} \neq 0, \quad \text{if} \quad s = k, \\ b_{2s-1} \neq 0, b_{2s+i} = 0, i = 1, 2, \cdots, 2k-2s, \quad \text{if} \quad 0 < s < k, \end{cases} \tag{3.48}$$

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty + \sigma + 2\pi\beta_2 + 2\pi\gamma < B_6, \tag{3.49}$$

where $B_6 = b_{2s-1} - \sum_{i=2}^{s-1} |C_{2i-1}^-|$, $C_{2i+1}^- = \min\{(-1)^{s-i} b_{2i-1}, 0\}$. *Then equation (1.1) has at least one* 2π*-periodic solution.*

Remark. The proofs of Theorems 3.2 and 3.4 are similar to that of Theorem 3.1 and the proof of Theorem 3.6 is similar to that of Theorem 3.5. Here we omit them.

Example 3.1. Consider the following equation

$$x^{(5)}(t) + \frac{1}{2}x^{(4)}(t) + \frac{x(t)}{1+|x(t)|^2}x'(t) + (\frac{1}{30}\cos t)x(t) + (\frac{1}{28}\sin t)x(t - \frac{1}{100}\cos t) = \cos t, \tag{3.50}$$

where $n = 5, b_4 = \frac{1}{2}, f(t,x) = \frac{x}{1+x^2}, \sigma = \frac{1}{2}, h(t,x) = (\frac{1}{30}\cos t)x, e(t,y) = (\frac{1}{28}\sin t)y, p(t) = \cos t, \tau(t) = \frac{1}{100}\cos t$. Thus, $T = 2\pi, \beta_2 = \frac{1}{30}, \alpha = \frac{1}{28}, \gamma = \frac{1}{28}, |\tau(t)|_\infty = \frac{1}{100}, B_1 = 1$. Obviously assumption $(H_2) - (H_4)$ hold and

$$2^{\frac{1}{2}}\alpha|\tau(t)|_\infty + \sigma + 2\pi\beta_2 + 2\pi\gamma = 2^{\frac{1}{2}} \times \frac{1}{28} \times \frac{1}{100} + \frac{1}{2} + \frac{1}{30} \times 2\pi + \frac{1}{28} \times 2\pi < 1 = B_1. \tag{3.51}$$

By Theorem 3.1, we know that equation (3.50) has at least one 2π-periodic solution.

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