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ON PRIMARY IDEALS AND RADICALS OF ORDERED SEMIGROUPS

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Abstract: In this paper, we introduce the concept of the primary ideal of *po*−semigroups and give some characterizations of primary ideal of *po*−semigroups. In particular, for commutative *po*−semigroups, we also prove that every proper ideal can be expressed as an intersection of finite primary ideals.

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1. Introduction

The concepts of right, left ideals, prime ideals and complete prime ideals of rings have been extended to ordered semigroups by N. Kehayopulu in [1]. Kehayopulu use the terms "weakly prime", "prime" [1], [3] the same with A.H. Clifford- instead of "prime", "completely prime", given by N.H. McCoy [4], [5]. In [1], N. Kehayopulu obtained the four equivalent conditions to be weakly prime for an ideal T of a *po*-semigroup S. In this paper we introduce the concept of the primary ideal of *po*−semigroups and give some characterizations of primary ideal of *po*−semigroups. In particular, for commutative *po*-semigroups, we also prove that every proper ideal can be expressed as an intersection of finite primary ideals. Some of our results are analogous to the results given by

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N. Kehayopulu in [1].

A *po*−semigroup (ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that:

$$a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx, \ \forall a, b, x \in S.$$

Definition 1.1. (see [1]) Let S be a *po*-semigroup and $\emptyset \neq A \subseteq S$. A is called a *left* (resp. *right*) *ideal* of S if:

(1) $SA \subseteq A$ (resp. $AS \subseteq A$).

(2) $a \in A, S \ni b \leq a \Rightarrow b \in A$.

A is called an *ideal* of S if it is both a right and a left ideal of S. An ideal T of a *po*-semigroup S is called *maximal* if $T \neq S$ and there exists no ideal A of S such that $T \subset A \subset S$.

Definition 1.2. (see [3]) Let S be a *po*-semigroup and $\emptyset \neq T \subseteq S$. T is called *prime* if

$$A, B \subseteq S, AB \subseteq T \Rightarrow A \subseteq T \text{ or } B \subseteq T.$$

Equivalent Definition. $a, b \in S, ab \in T \Rightarrow a \in T$ or $b \in T$.

Let S be a *po*-semigroup. For $H \subseteq S$, we denote $(H] := \{t \in S | t \leq h$ for some $h \in H\}$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ $(a \in S)$. We denote by $R(a), L(a), I(a)$ the right ideal, left ideal, ideal of S, respectively, generated by $a (a \in S)$. One can easily prove that: $R(a) = (a \bigcup aS], L(a) = (a \bigcup Sa], I(a) = (a \bigcup aS \bigcup Sa \bigcup SaS] \forall a \in S$, see [1]. We define the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{I}$, $\mathcal{H}$, $\sqrt{\mathcal{L}}$, $\sqrt{\mathcal{R}}$, $\sqrt{\mathcal{I}}$, $\sqrt{\mathcal{H}}$ on S as follows:

$$a\mathcal{L}b \text{ if and only if } L(a) = L(b);$$
$$a\mathcal{R}b \text{ if and only if } R(a) = R(b);$$
$$a\mathcal{I}b \text{ if and only if } I(a) = I(b);$$
$$a\sqrt{\mathcal{L}}b \text{ if and only if } \sqrt{L(a)} = \sqrt{L(b)};$$
$$a\sqrt{\mathcal{R}}b \text{ if and only if } \sqrt{R(a)} = \sqrt{R(b)};$$
$$a\sqrt{\mathcal{I}}b \text{ if and only if } \sqrt{I(a)} = \sqrt{I(b)};$$
$$\sqrt{\mathcal{H}} := \sqrt{\mathcal{H}}, \quad \mathcal{H} := \mathcal{R} \bigcap \mathcal{L}.$$

The relation $\mathcal{L}$ is a right congruence, $\mathcal{R}$ is a left congruence on S. One can easily prove that $\sqrt{\mathcal{L}}$, $\sqrt{\mathcal{R}}$, $\sqrt{\mathcal{I}}$ and $\sqrt{\mathcal{H}}$ are equivalence relations on S.

Definition 1.3. (see [2]) Let S be a *po*-semigroup and T a subsemigroup of S, T is called *Archimedean* if for every $a, b \in T$, there exists $n \in Z^+$ such that $a^n \in (TbT]$, where $Z^+ := \{1, 2, \cdots\}$ is the set of positive integers.

Equivalent Definition. $a^n \leq xby$ for some $x, y \in T$.

Definition 1.4. If T is subset of a *po*-semigroup S, then the subset of S

$$\{x \in S | x^n \in (T] \ \text{ for some } \ n \in Z^+\}$$

is called the *radical* of T denote by $\sqrt{T}$.

It is easily to see that $T \subseteq \sqrt{T}$ and $(\sqrt{T}] = \sqrt{T}$.

Definition 1.5. Let S be a *po*-semigroup and $\emptyset \neq T \subseteq S$. T is called a *primary* of S if for every $a, b \in S$ such that $ab \in T$, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.

Equivalent Definition. $a, b \in S, ab \in T \Rightarrow a \in \sqrt{T}$ or $b \in \sqrt{T}$.

A set T of S is called a primary ideal if it is both a ideal and a primary of S.

Clearly, primary ideals are a generalization of the concept of prime ideals and each prime ideal of S is a primary ideal of S.

Lemma 1.1. (see [1]) *Let S be a po-semigroup. Then we have:*

(1) $A \subseteq (A], \forall A \subseteq S$.

(2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*

(3) $(A](B] \subseteq (AB], \forall A, B \subseteq S$.

(4) $((A]] = (A], \forall A \subseteq S$.

(5) *For every right ideal, left ideal, ideal T of S, we have $(T] = T$.*

(6) *If A, B are ideals of S, then $(AB], A \bigcap B, A \bigcup B$ are ideals of S.*

(7) *$(Sa]$ (resp. $(aS]$) is a left (resp. right) ideal of S, $(SaS]$ is an ideal of S for every $a \in S$.*

2. Main Results

Theorem 2.1. *Let S be a po-semigroup. Then the following conditions are equivalent:*

(1) *T is primary ideal.*

(2) *If for every $a, b \in S$ such that $(aSb] \subseteq T$, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.*

(3) *If for every $a, b \in S$ such that $(I(a)I(b)] \subseteq T$, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.*

(4) *If for every $a, b \in S$ such that $(R(a)R(b)] \subseteq T$, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.*

(5) *If for every* $a, b \in S$ *such that* $(L(a)L(b)] \subseteq T$, *then there exists* $n \in Z^+$ *such that* $a^n \in T$ *or* $b^n \in T$.

(6) *If for every* $a, b \in S$ *such that* $(R(a)L(b)] \subseteq T$, *then there exists* $n \in Z^+$ *such that* $a^n \in T$ *or* $b^n \in T$.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in S$ such that $(aSb] \subseteq T$. By Lemma 1.1, we have

$$\begin{aligned}(I(b))^3 &= (b\bigcup Sb\bigcup bS\bigcup SbS]^3\\ &\subseteq ((b\bigcup Sb\bigcup bS\bigcup SbS)(b\bigcup Sb\bigcup bS\bigcup SbS)](b\bigcup Sb\bigcup bS\bigcup SbS]\\ &\subseteq (Sb\bigcup SbS](b\bigcup Sb\bigcup bS\bigcup SbS]\\ &\subseteq ((Sb\bigcup SbS)(b\bigcup Sb\bigcup bS\bigcup SbS)] \subseteq (SbS],\end{aligned}$$

and

$$\begin{aligned}I(a)(I(b))^3 &\subseteq (a\bigcup Sa\bigcup aS\bigcup SaS](SbS] \subseteq ((a\bigcup Sa\bigcup aS\bigcup SaS)(SbS)]\\ &\subseteq (aSbS\bigcup SaSbS] \subseteq (TS\bigcup STS] \subseteq (T] = T.\end{aligned}$$

Since T is a primary ideal and $ab^3 \in I(a)I(b)^3 \subseteq T$, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^{3n} \in T$.

(2)$\Rightarrow$(3) Let $a, b \in S$ such that $(I(a)I(b)] \subseteq T$. By Lemma 1.1, we have

$$T \supseteq ((a\bigcup Sa\bigcup aS\bigcup SaS](b\bigcup Sb\bigcup bS\bigcup SbS]] \supseteq ((a](Sb]] \Rightarrow T \supseteq ((a](Sb]].$$

On the other hand,

$$aSb \subseteq (a](Sb] \Rightarrow (aSb] \subseteq ((a](Sb]] \subseteq (aSb] \Rightarrow ((a](Sb]] = (aSb].$$

Thus, we have $(aSb] = ((a](Sb]] \subseteq T$. By (2), there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.

(3)$\Rightarrow$(4) Let $a, b \in S$ such that $I(a)I(b) \subseteq T$. By Lemma 1.1, we have

$$\begin{aligned}I(a) &= (a\bigcup Sa\bigcup aS\bigcup SaS] \subseteq (R(a)\bigcup SR(a)\bigcup R(a)S\bigcup SR(a)S]\\ &\subseteq (R(a)\bigcup SR(a)],\end{aligned}$$

$$\begin{aligned}I(b) &= (b\bigcup Sb\bigcup bS\bigcup SbS] \subseteq (R(b)\bigcup SR(b)\bigcup R(b)S\bigcup SR(b)S]\\ &\subseteq (R(b)\bigcup SR(b)].\end{aligned}$$

Then, by Lemma 1.1,

$$\begin{aligned}I(b)I(b) &\subseteq (R(a)\bigcup SR(a)](R(b)\bigcup SR(b)]\\ &\subseteq ((R(a)\bigcup SR(a)(R(b)\bigcup SR(b)]\end{aligned}$$

$$\subseteq ((R(a)R(b)\bigcup R(a)SR(b)\bigcup SR(a)R(b)\bigcup SR(a)SR(b)]$$
$$\subseteq ((R(a)R(b)\bigcup SR(a)R(b)] \subseteq (T\bigcup ST] \subseteq (T] = T.$$

By (3), there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.

(3)⇒(5) The proof is similar to the previous case.

(3)⇒(6) Let $a, b \in S$ such that $R(a)L(b) \subseteq T$. Since

$$I(a) \subseteq (R(a)\bigcup SR(a)], \qquad I(b) \subseteq (L(b)\bigcup R(b)S],$$

we have

$$I(b)I(b) \subseteq (R(a)\bigcup SR(a)](L(b)\bigcup L(b)S]$$
$$\subseteq ((R(a)L(b)\bigcup SR(a)L(b)\bigcup R(a)L(b)S\bigcup SR(a)L(b)S]$$
$$\subseteq (T\bigcup ST\bigcup TS\bigcup STS] \subseteq (T] = T.$$

By (3), there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$.

(4), (5), (6)⇒(1) They are obvious. □

Theorem 2.2. *Let S be a po-semigroup. Then the following conditions are equivalent*:

(1) *T is primary ideal.*

(2) *If A, B are ideals of S such that $AB \subseteq T$, then $\sqrt{A} \subseteq \sqrt{T}$ or $\sqrt{B} \subseteq \sqrt{T}$.*

(3) *If A, B are right ideals of S such that $AB \subseteq T$, then $\sqrt{A} \subseteq \sqrt{T}$ or $\sqrt{B} \subseteq \sqrt{T}$.*

(4) *If A, B are left ideals of S such that $AB \subseteq T$, then $\sqrt{A} \subseteq \sqrt{T}$ or $\sqrt{B} \subseteq \sqrt{T}$.*

(5) *If A is a right ideal, B is left ideal of S such that $AB \subseteq T$, then $\sqrt{A} \subseteq \sqrt{T}$ or $\sqrt{B} \subseteq \sqrt{T}$.*

Proof. (1) ⇒(2) Let A, B are ideals of S such that $AB \subseteq T$. Let $a \in \sqrt{A}$, $b \in \sqrt{B}$, then there exists $n, m \in Z^+$ such that $a^n \in A$, $b^m \in B$. we have $I(a^n) \subseteq A$, $I(b^m) \subseteq B$, so we get $I(a^n)I(b^m) \subseteq AB \subseteq T$. By Theorem 2.1, then there exists $k \in Z^+$ such that $a^{kn} \in T$, or $b^{km} \in T$. That is $\sqrt{A} \subseteq \sqrt{T}$ or $\sqrt{B} \subseteq \sqrt{T}$.

(2) ⇒(1) Let $a, b \in S$ such that $I(a)I(b) \subseteq T$. By (2), then $\sqrt{I(a)} \subseteq \sqrt{T}$ or $\sqrt{I(b)} \subseteq \sqrt{T}$. If $\sqrt{I(a)} \subseteq \sqrt{T}$, then $a \in \sqrt{T} \Rightarrow a^n \in T$ for some $n \in Z^+$. If $\sqrt{I(b)} \subseteq \sqrt{T}$, then $b \in \sqrt{T} \Rightarrow b^m \in T$ for some $m \in Z^+$. By Theorem 2.1, we have T is primary.

(1) ⇔(3), (4), (5), the proof is similar to (1) ⇔ (2). □

If we consider left (resp. right) regularity of ordered semigroup, we have:

Theorem 2.3. *Let S be a left (resp. right) regular po−semigroup and T a proper ideal of S. Then T is primary if and only if T is prime.*

Proof. Let T be a primary ideal of S, $\forall a, b \in S$ such that $ab \in T$. Since T is a primary ideal of S, then there exists $n \in Z^+$ such that $a^n \in T$ or $b^n \in T$. If $a^n \in T$, by S is left regular, we have $a \le xa^2 \le x^2a^3 \le \cdots \le x^{n-1}a^n \in ST \subseteq T$ for some $x \in S$. By T is an ideal of S, then $a \in T$. If $b^n \in T$, the proof is similar to the previous case. Hence T is a prime ideal. For the converse statement, it is clearly. □

Theorem 2.4. *Let S be Archimedean po−semigroup and T a ideal of S. Then the following statements are true*:

(1) *T is primary if and only if $\sqrt{T}$ is a prime ideal of S.*

(2) *The maximal ideal T of S is primary.*

Proof. (1) ⇒) Let T is primary. Then $\sqrt{T}$ is an ideal of S. In fact: For every $a, b \in \sqrt{T}$, there exist $n, m \in Z^+$ such that $a^n \in T, b^m \in T$. By S is Archimedean *po*-semigroup, we consider ab, a^n, then there exists $k \in Z^+$ such that $(ab)^k \in (Sa^nS] \subseteq (STS] \subseteq (T] = T \Rightarrow ab \in \sqrt{T}$. For every $u \in S$, we consider ua, a^n, then there exists $r \in Z^+$ such that $(ua)^r \in (Sa^nS] \subseteq (STS] \subseteq (T] = T \Rightarrow S\sqrt{T} \subseteq \sqrt{T}$. Similarly, we have $\sqrt{T}S \subseteq \sqrt{T}$. Hence $\sqrt{T}$ is an ideal of S.

Let $a, b \in S$ such that, $ab \in \sqrt{T}$, then there exists $n \in Z^+$ such that $(ab)^n \in T$. S is Archimedean *po*-semigroup, we consider $a, (ab)^n$, then there exists $t \in Z^+$ such that $a^t \in (S(ab)^nS] \subseteq (STS] \subseteq (T] = T \Rightarrow a \in \sqrt{T}$. Similarly, we have $b \in \sqrt{T}$. Therefore $\sqrt{T}$ is a prime ideal of S.

(⇐ Let $\sqrt{T}$ is a prime ideal of S, $a, b \in S$ such that $ab \in T$. Since $T \subseteq \sqrt{T}$, we have $ab \in \sqrt{T}$, then $a \in \sqrt{T}$ or $a \in \sqrt{T}$. Hence there exists $n, m \in Z^+$ such that $a^n \in T$, or $b^m \in T$. So that T is primary.

(2) Let T be a maximal ideal of S such that $I(a)I(b) \subseteq T$ for all $a, b \in S$. Suppose $a, b \notin T$. Since $I(a)\bigcup T, I(b)\bigcup T$ are ideals of S, $I(a)\bigcup T \supset T$, $I(b)\bigcup T \supset T$ and T is maximal, we have $I(a)\bigcup T = S$ and $I(b)\bigcup T = S$. Since S is Archimedean *po*-semigroup, then there exist $n, m \in Z^+$ such that $a^n \in (SbS]$ and $b^m \in (SaS]$. Then, we have

$$a^n \in (SbS] \subseteq ((I(a)\bigcup T)I(b)S] \subseteq (I(a)I(b)S\bigcup TI(b)S] \subseteq (T] = T.$$

$$b^m \in (SaS] \subseteq (SI(a)(I(b)\bigcup T)] \subseteq (SI(a)I(b)\bigcup STI(a)] \subseteq (T] = T.$$

Hence, T is primary. □

Theorem 2.5. *If $\sqrt{T}$ is a prime ideal for every ideal T of S, then T is primary.*

Proof. Let T be an ideal of S, a, $b \in S$ such that $ab \in T$. Since $(SabS]$ is an ideal of S, then $\sqrt{(SabS]}$ is a prime ideal. By $a^2b^2 \in (SabS] \subseteq \sqrt{(SabS]}$, we have $a^2 \in \sqrt{(SabS]}$ or $b^2 \in \sqrt{(SabS]}$. Hence, there exist $n, m \in Z^+$ such that $a^{2n} \in (SabS] \subseteq (STS] \subseteq (T] = T$, or $b^{2m} \in (SabS] \subseteq (STS] \subseteq (T] = T$. Then T is primary. □

Definition 2.6. (see [6]) Let S be a commutative *po*-semigroup, $a \in S$ and T an ideal of S. Then set

$$< a, T >:= \{x \in S | ax \in T\}$$

is called the *extension* of S by a.

Lemma 2.7. (see [6]) *Let S be a commutative po-semigroup, $a \in S$ and T an ideal of S. Then we have the following*:

(1) $< a, T >$ *is an ideal of S.*

(2) $T \subseteq < a, T > \subseteq < a^2, T >$.

(3) *If $a \in T$, then $< a, T >= S$.*

Definition 2.8. Let S be a *po*-semigroup and T a proper ideal of S, T is called *irreducible* ideal of S if, for any two ideals T_1, T_2, $T = T_1 \bigcap T_2$ implies $T_1 = T$ or $T_2 = T$.

Definition 2.9. A *po*-semigroup S fulfils the *maximal condition* if any nonempty ideal set of S – with respect to the inclusion relation – has a maximal element.

Theorem 2.10. *Let S be a commutative po-semigroup with the maximal condition, then every irreducible ideal of S is primary.*

Proof. Let T be a ideal of S, $a, b \in S$ such that $ab \in T$. By Lemma 2.1, let

$$\mathcal{M} = \{< \dashv^{\|}, \mathcal{T} > |\|| = \infty, \in, \cdots\}, \qquad \mathcal{K} = \{< \lfloor^{\|}, \mathcal{T} > |\|| = \infty, \in, \cdots\}.$$

Since S fulfil the maximal condition, then exist some $n, m \in Z^+$ such that $< a^n, T >$ is a maximal element of $\mathcal{M}$ and $< b^m, T >$ a maximal element of $\mathcal{K}$. So, we have $< a^n, T >=< a^{n+1}, T >$ and $< b^m, T >=< b^{m+1}, T >$. We denote by $T_1 = T \bigcup (Sa^n]$, $T_2 = T \bigcup (Sb^m]$, then T_1, T_2 are ideals of S and T=$T_1 \bigcap T_2$. In fact: $T \subseteq T_1 \bigcap T_2$ is obviously. For any $t \in T_1 \bigcap T_2 = T \bigcup ((Sa^n] \bigcap (Sb^m])$, if $t \notin T$, then $t \in (Sa^n] \bigcap (Sb^m]$. Since $(Sa^n] \bigcap (Sb^m] \neq \emptyset (a^n b^m \in (Sa^n] \bigcap (Sb^m])$, then exists some $u \in S$ such that $ua^n \in (Sb^m]$, so $ua^n \leq vb^m$ for some $v \in S$. Therefore, we have $ua^{n+1} = a \cdot ua^n \leq avb^m = (ab)vb^{m-1} \in T$ implies $ua^{n+1} \in T$. So $u \in < a^{n+1}, T >=< a^n, T >$ implies $t \leq ua^n \in T$. That is, $t \in T$. It is impossible. Thus $T_1 \bigcap T_2 = T$. By T is irreducible ideal of S, then $T = T_1 = T \bigcup (Sa^n]$ or $T = T_2 = T \bigcup (Sb^m]$. If $T = T_1 = T \bigcup (Sa^n]$, then $(Sa^n] \subseteq T$, thus $a^{n+1} \in Sa^n \subseteq (Sa^n] \subseteq T \Rightarrow a^{n+1} \in T$. If $T = T_2 = T \bigcup (Sb^m]$, by the similar method, we have $b^{m+1} \in T$. □

Theorem 2.11. *Let S be a commutative po-semigroup with the maximal condition, then every proper ideal of S is the intersection of finite primary ideals.*

Proof. Let $\mathcal{M}$=$\{T|T$ proper ideal of S and T is not the intersection of finite irreducible ideals$\}$. If S fulfils the maximal condition, then $\mathcal{M}$ has a maximal element, say T_0. Thus T_0 is not the irreducible. Then there exist ideals T_1, T_2 such that $T_0 = T_1 \bigcap T_2, T_0 \neq T_1, T_0 \neq T_2$. By T_0 is a maximal element of $\mathcal{M}$, we have $T_1, T_2 = S$ or T_1, T_2 are the intersection of finite irreducible ideals of S. But, $T_0 \neq S$, thus we have $T_1 \neq S$ or $T_2 \neq S$. Therefore T_0 is the intersection of finite irreducible ideals of S, so $T_0 \in \mathcal{M}$, it is impossible. Thus every proper ideal T of S is the intersection of finite irreducible ideals. By Theorem 2.6, every proper ideal of S is the intersection of finite primary ideals. □

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