International Journal of Pure and Applied Mathematics
Volume 55 No. 3 2009, 343-348

PERSISTENCE OF THE PREDATOR-PREY MODEL WITH MODIFIED LESLIE-GOWER HOLLING-TYPE II SCHEMES AND IMPULSE

Hanhui Liu
Department of Applied Mathematics
Xinjiang University of Finance and Economics
Urumqi, 830012, P.R. CHINA
e-mail: hjlhh1999@yahoo.com.cn

Abstract: A modified Leslie-Gower and Holling-type II functional response predator-prey system with impulsive effect is proposed. The sufficient conditions for the uniform persistence are given by applying theories of abstract persistence, asymptotically autonomous semiflows, and the comparison theorem.

AMS Subject Classification: 34K99, 92D25, 92C15
Key Words: predator-prey system, uniform persistence, impulsive effect

1. Introduction

Impulsive differential equations are suitable for the mathematical simulation of evolutionary processes whose states are subject to sudden changes at certain moments. A central problem for such systems is to study uniform persistence of species. There have been extensive studies on uniform persistence of species for differential equations and dynamical systems [1], [2].

In this paper, we consider the predator-prey model with modified Leslie-Gower and Holling II schemes with impulse:

Received: July 21, 2008

$$\begin{cases} \left.\begin{array}{rcl} \dot{x}(t) &=& \big(a_1(t)-b_1(t)x(t)-\dfrac{c_1(t)y(t)}{x(t)+k_1(t)}\big)x(t), \\ \dot{y}(t) &=& (a_2(t)-\dfrac{c_2(t)y(t)}{x(t)+k_2(t)})y(t), \end{array}\right\} & \begin{array}{l} t\neq t_k, \\ k\in Z_+, \end{array} \\ \left.\begin{array}{rcl} x(t_k^+) &=& (1+h_k)x(t_k), \\ y(t_k^+) &=& (1+g_k)y(t_k), \end{array}\right\} \quad t=t_k, \quad k\in Z_+, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ present the densities of prey and predator at time t, respectively, and $a_i(t), b_1(t), c_i(t), k_i(t) (i = 1, 2)$ are continuous ω-periodic functions and only positive values. Assume that $h_k, g_k (k \in Z_+)$ are constants and there exists an integer $q > 0$ such that $h_{k+q} = h_k, g_{k+q} = g_k, t_{k+q} = t_k + \omega$. A natural constraint is $1 + h_k > 0, 1 + g_k > 0$ for all $k \in Z_+$.

2. Preliminaries

Let (X, d) be a Banach space with metric d. Suppose that $T(t) : X \to X, t \geq 0$, is a C^0 semiflow on X, that is, $T(0) = E, T(t + s) = T(t)T(s)$ for $t, s \geq 0$, and $T(t)x$ is continuous in t and x. $T(t)$ is said to be point dissipative in X if there is a bounded nonempty set B in X such that, for any $x \in X$, there is a $t_0 = t_0(x, B) > 0$ such that, $T(t)x \in B$ for $t \geq t_0$.

Definition 2.1. Assume that $X = X_0 \cup \partial X_0$ and $X_0 \cap \partial X_0 = \emptyset$ with X_0 being open in X. The semiflow $T(t) : X \to X$ is said to be of uniform persistence with respect to $(X_0, \partial X_0)$ if there exists an $\eta > 0$ such that, for any $x \in X_0$, $\liminf_{t\to\infty} d(T(t)x, \partial X_0) \geq \eta$.

Let $\omega(x)$ denote the ω-limit set of $x \in X$ for semiflow $T(t) : X \to X$ and let $\bar{A}_\partial = \cup_{x\in\partial X_0}\omega(x)$. The set $\bar{A}_\partial$ is said to be acyclic if there exists an isolated covering $\cup_{i=1}^{k} M_i$ of $\bar{A}_\partial$ such that no subset of the M_i's forms a cycle.

Firstly, we consider some properties of the solution operator $T(t)$ of the following ordinary differential equations with impulse:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \quad t \neq t_k, \\ \triangle x(t)|_{t=t_k} = x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, \\ x(0) = \phi, \end{cases} \tag{2.1}$$

where $f \in C(J \times R^n, R^n), J = [0, +\infty)$, $f(t + \omega, x) = f(t, x), \omega > 0, I_k \in C(R^n, R^n), k = 1, 2, \ldots$, there exists a positive integer q such that $I_{t+q}(\cdot) = I_k(\cdot)$, and $t_{k+q} = t_k + \omega, \phi \in R^n, 0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < \ldots, \lim_{m\to\infty} t_m = +\infty$.

In [2], the solution operator $T(t)$ of system (2.1) can be expressed as

$$T(t)x = xe^{-Mt} + \int_0^t e^{-M(t-s)}[f(s,T(s)x) + MT(s)x]ds \\ + \sum_{0<t_k<t} e^{-M(t-t_k)} I_k(T(t_k)x).$$

Lemma 2.1. *The operator $T(\omega)$ is a complete operator.*

Lemma 2.2. $T(0) = E, T(t+\omega) = T(t)T(\omega)$.

The proofs of Lemma 2.1 and Lemma 2.2 are given in [3], [4], respectively.

Let $S = T(\omega), S^2 = S \circ S = T(\omega)(T(\omega)) = T(2\omega)$. Since $T(\omega)$ is a complete operator, so is S. The following theorems can be obtained using [5].

Theorem 2.1. *Let $T(t)$ be a ω-periodic semiflow on X with $T(t)X_0 \subset X_0, t \geq 0$. Assume that $S = T(\omega)$ satisfies the following conditions:*

(1) S is compact;

(2) S is point dissipative in X.

Then the uniform persistence of S with respect to $(X_0, \partial X_0)$ implies that of $T(t) : X \to X, t \geq 0$.

Let A_∂ be the maximal compact invariant set of S in ∂X_0, the $\bar{A}_\partial = \cup_{x\in A_\partial}\omega(x)$ is said to has an isolated and acyclic covering $\cup_{i=1}^k M_i$ in ∂X_0, that is, $\bar{A}_\partial \subset \cup_{i=1}^k M_i$, where $M_1, M_2, \ldots, M_k$ are pairwise disjoint, compact and invariant sets of S in ∂X, and no subset of the M_i's forms a cycle for $S|_\partial = S|_{A_\partial}$ in A_∂.

Theorem 2.2. *Let $S : X \to X$ be a continuous map with $S(X_0) \subset X_0$. Assume that:*

(1) $S : X \to X$ is point dissipative;

(2) S is compact;

(3) $\bar{A}_\partial = \cup_{x\in A_\partial}\omega(x)$ has an isolated and acyclic covering $\cup_{i=1}^k M_i$ in ∂X_0.

Then S is of uniform persistence with respect to $(X_0, \partial X_0)$ if and only if, for each $M_i, i = 1, 2, \ldots, k, W^s(M_i) \cap X_0 = \emptyset$, where $W^s(M_i) = \{x : x \in X, \omega(x) \neq \emptyset,$ and $\omega(x) \subset M_i\}$ is the stable set of M_i.

We consider the following logistic equation with impulsive effect:

$$\begin{cases} \dot{\omega}(t) = \alpha(t)\omega(t) - \beta(t)\omega^2(t), & t \neq t_k, \quad k \in Z_+, \\ \omega(t_k^+) = (1+e_k)\omega(t_k), & t = t_k, \quad k \in Z_+, \end{cases} \tag{2.2}$$

where $\alpha(t), \beta(t)$ are ω-periodic continuous function, and $\beta(t) > 0$. The following lemma is necessary (see [6]).

Lemma 2.3. *System (2.2) admits a unique positive ω-periodic solution if and only if $\int_0^\omega \alpha(s)ds > \ln \prod_{k=1}^q \frac{1}{1+e_k}$, which, moreover, is globally attracting.*

Let $\theta_{[\alpha,\beta]}$ denote the unique positive periodic solution of (2.2) and define average of y to be $[y] = \omega^{-1}\int_0^\omega y(s)ds$.

3. Main Results

By Lemma 2.3, we know that the solution operator $T(t)$ of system (1.1) is dissipative, so is S. Let

$$X_i^+ = \{x_i : x_i \in R, x_i \geq 0\}, i = 1, 2; \quad X_{i0}^+ = \{x_i : x_i \in R, x_i > 0\},$$
$$X = X_1^+ \times X_2^+; \qquad X_0 = X_{10}^+ \times X_{20}^+; \qquad \partial X_0 = X \backslash X_0, \quad i = 1, 2.$$

Lemma 3.1. *Suppose that* $[a_2] > \omega^{-1}\ln\prod_{k=1}^q \frac{1}{1+g_k}$, *and*

$$[a_1] > \omega^{-1}\ln\prod_{k=1}^q \frac{1}{1+h_k} + [\frac{c_1}{k_1}\theta_{[a_2,\frac{c_2}{k_2}]}].$$

Then there exists a $\delta > 0$ *such that for any* $\phi \in X_0$,

$$\lim_{t\to\infty}\sup x(0,\phi)(t) > \delta, \quad \lim_{t\to\infty}\sup y(0,\phi)(t) > \delta. \tag{3.1}$$

Proof. Denoting $\lambda_0 = [a_1] - \omega^{-1}\ln\prod_{k=1}^q \frac{1}{1+h_k} - [\frac{c_1\theta_{[a_2,\frac{c_2}{k_2}]}}{k_1}]$, and $\lambda_0 > 0$. If (3.1) cannot hold, then

$$\lim_{t\to\infty}\sup x(0,\phi)(t) < \frac{\delta^*}{4}, \tag{3.2}$$

where $\delta^* > 0$ satisfies $\int_0^\omega b(t)\delta^* dt < \frac{\lambda_0}{4}$.

Denote $x(t) = x(0,\phi)(t)$, $y(t) = y(0,\phi)(t)$. Then there exists an integer $M_1 > 0$, such that $x(t) < \delta^*$ when $t > M_1\omega$. Hence we have

$$\begin{cases} \dot{y}(t) < y(t)(a_2(t) - \dfrac{c_2(t)y(t)}{\delta^* + k_2(t)}), & t \neq t_k, \quad k \in Z_+, \\ y(t_k^+) = (1+g_k)y(t_k), & t = t_k, \quad k \in Z_+. \end{cases}$$

Consider the following equation

$$\begin{cases} \dot{z}(t) = z(t)(a_2(t) - \dfrac{c_2(t)z(t)}{\delta^* + k_2(t)}), & t > M_1\omega, \\ z(t_k^+) = (1+g_k)z(t_k), & t_k > M_1\omega, \quad k \in Z_+. \\ z(0) = y(M_1\omega). \end{cases} \tag{3.3}$$

Since $[a_2] > \omega^{-1}\ln\prod_{k=1}^q \frac{1}{1+g_k}$, system (3.3) admits a globally attractive positive ω-periodic solution $z^*(t)$, that is, for arbitrary $\varepsilon > 0$, there exists a $T_0 > 0, z^*(t) + \frac{\varepsilon}{2} > z(t) > z^*(t) - \frac{\varepsilon}{2}$, when $t > T_0$. By the continuous dependence theorem with respect to parameters of a differential equation with impulse, for the above $\varepsilon > 0$, there exists $\delta^{**} > 0$, when $0 < \delta^* \leq \delta^{**}$, we have $z^*(t) <$

$\theta_{[a_2,\frac{c_2}{k_2}]}(t+M_1\omega)+\frac{\varepsilon}{2}, t\in[0,\omega]$. By the comparison theorem, there exists an integer M_2, and $M_2\omega>\max\{M_1\omega,T_0\}$, such that $y(t)<\theta_{[a_2,\frac{c_2}{k_2}]}+\varepsilon$, as $t>M_2\omega$. Letting $\varepsilon>0$ such that

$$[a_1]-\frac{1}{\omega}\ln\prod_{k=1}^{q}\frac{1}{1+h_k}-[\frac{c_1}{k_1}(\theta_{[a_2,\frac{c_2}{k_2}]}+\varepsilon)]>\frac{\lambda_0}{2},$$

then

$$\begin{aligned} x(t) &= x(M_2\omega)e^{\int_{M_2\omega}^{t}[a_1(s)-b_1(s)x(s)-\frac{c_1(s)y(s)}{x(s)+k_1(s)}]ds-\ln\prod_{M_2\omega<t_k<t}\frac{1}{1+h_k}} \\ &> x(M_2\omega)e^{\int_{M_2\omega}^{t}(a_1(s)-\frac{c_1(s)}{k_1(s)}(\theta_{[a_2,\frac{c_2}{k_2}]}+\varepsilon))ds-\int_{M_2\omega}^{t}b_1(s)\delta^* ds-\ln\prod_{M_2\omega<t_k<t}\frac{1}{1+h_k}}. \end{aligned}$$

It is easy to know that $x(M_2\omega+l\omega)\geq x(M_2\omega)e^{\frac{\lambda_0}{4}l}$ and $x(M_2\omega+l\omega)\to+\infty$ as $l\to+\infty$. This contradicts (3.2). In the same way, we can prove $\limsup_{t\to\infty} y(0,\phi)(t)>\delta$. This completes the proof. □

Theorem 3.1. *If* $[a_2]>\omega^{-1}\ln\prod_{k=1}^{q}\frac{1}{1+g_k}$, *then system (1.1) is of uniform persistence provided that* $[a_1]>\omega^{-1}\ln\prod_{k=1}^{q}\frac{1}{1+h_k}+[\frac{c_1}{k_1}\theta_{[a_2,\frac{c_2}{k_2}]}]$.

Proof. By Lemma 2.3, we can obtain

$$\bar{A}_\partial=\cup_{\phi\in\partial X_0}\omega(\phi)=\{(0,0),(\theta_{[a_1,b_1]},0),(0,\theta_{[a_2,\frac{c_2}{k_2}]})\}.$$

Let $M_1=(0,0), M_2=(\theta_{[a_1,b_1]},0), M_3=(0,\theta_{[a_2,\frac{c_2}{k_2}]})$. Then $\bar{A}_\partial=M_1\cup M_2\cup M_3$, and M_i and $M_j (i\neq j)$ are disjoint, compact and isolated invariant sets for $S_\partial=S|_{\partial X_0}$. Since $[a_1]>\omega^{-1}\ln\prod_{k=1}^{q}\frac{1}{1+h_k}, [a_2]>\omega^{-1}\ln\prod_{k=1}^{q}\frac{1}{1+g_k}$, Lemma 3.1 implies that each $M_i (i=1,2,3)$ is isolated for S in X_0, since M_i is isolated for S_∂ in ∂X_0 and $S:X_0\to X_0, S_\partial:\partial X_0\to\partial X_0$.

By Lemma 2.3, M_1, M_2, M_3 are acyclic in ∂X_0. Therefore, $M_1\cup M_2\cup M_3$ is an isolated and acyclic covering of A_∂ in ∂X_0. By Lemma 3.1, $W^s(M_i)\cap X_0=\emptyset, i=1,2,3$. Hence, by Theorem 2.2, S is of uniform persistence with respect to $(X_0,\partial X_0)$. By Theorem 2.1 , $T(t)$ is of uniform persistence with respect to $(X_0,\partial X_0)$. This completes the proof. □

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