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EXISTENCE OF TRAVELING WAVES IN LOTKA-VOLTERRA COMPETITION MODELS WITH DISTRIBUTED MATURATION DELAY

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Abstract: This paper is concerned with a Lotka-Volterra competition model with distributed maturation delays. The stability of uniform steady states are firstly studied by investigating the eigenvalues of the linearization in every equilibrium and then, the existence of traveling wave solution connecting two boundary equilibriums is established by using monotone iteration together with upper and lower solutions method when the coexistence equilibrium absents. The results presented in this paper expand a number of existing ones.

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1. Introduction

Population models with stage structure are of current interest in mathematical biology. They can exhibit phenomena similar to those of partial differential equations and many important physiological parameters can be incorporated. Moreover, they are often much simpler than the corresponding models governed by partial differential equation. There has been much work on modelling stage-structured population models (see, for example, [1], [5], [3], [4], [7], [8], [2], [6], [9]).

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In this paper, we will discuss the existence of travelling wave fronts of the equations describing the competition between the adult members of the two species, i.e., we study the following subsystem:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \triangle u_1 + \alpha_1 \displaystyle\int_0^\infty \int_{-\infty}^\infty g_1(s) e^{-\gamma_1 s} \frac{1}{\sqrt{4\pi d_1 s}} e^{\frac{-(x-y)^2}{4d_1 s}} u_1(t-s,y) dy ds \\ \qquad - \beta_1 u_1^2(t,x) - E_1 u_1(t,x) - a_1 u_1(t,x) u_2(t,x), \\ \dfrac{\partial u_2}{\partial t} = D_2 \triangle u_2 + \alpha_2 \displaystyle\int_0^\infty \int_{-\infty}^\infty g_2(s) e^{-\gamma_2 s} \frac{1}{\sqrt{4\pi d_2 s}} e^{\frac{-(x-y)^2}{4d_2 s}} u_2(t-s,y) dy ds \\ \qquad - \beta_2 u_2^2(t,x) - E_2 u_2(t,x) - a_2 u_1(t,x) u_2(t,x). \end{cases} \tag{1.1}$$

This paper is organized as follows. In the next section, we shall summarize those aspects of the work of Wang and Li and Ruan [6]. In Section 3, we introduce some notations and terminology, and show the existence of a travelling wave solution to the two coupled equations of the mature populations by using the technique developed by Wang and Li and Ruan [6].

2. Theory of Wang and Li and Ruan

The theory is for reaction diffusion systems of the form

$$\frac{\partial u(t,x)}{\partial t} = D \frac{\partial^2 u(t,x)}{\partial^2 x} + f(u(t,x), (g * u)(t,x)),$$

where $t \geq 0$, $x \in R$, $D = diag(d_1, ..., d_n)$, $d_i > 0$, $i = 1, ..., n$, $n \in N$; $u(t,x) = (u_1(t,x), ...u_n(t,x))^T$, $f \in C(R^{2n}, R^n)$, and

$$(g * u)(t,x) = \int_{-\infty}^t \int_{-\infty}^\infty g(t-s, x-y) u(s,y) dy ds,$$

or

$$(g * u)(t,x) = \int_{-\infty}^t g(t-s) u(s,x) ds.$$

A travelling wave front is a solution $u(t,x) = \varphi(x+ct)$, where $c > 0$ is a given constant and $\varphi \in BC^2(R, R^n)$ is an increasing function satisfying the following functional differential system:

$$-D\varphi''(t) + c\varphi'(t) = f(\varphi(t), (g * \varphi)(t)), \quad \text{a.e. } t \in R$$

and the conditions

$$\varphi(-\infty) = 0, \quad \varphi(+\infty) = K = (K_1, ..., K_n)^T, \quad 0 < K.$$

Now we list some assumptions, which will be used in later.

(H_0^1) $\int_{-\infty}^{+\infty} g(t,x)dx$ is uniformly convergent for $t \in [0,a], a > 0, j = 1,...m.$ In other words, if given $\varepsilon > 0$, then there exists $M > 0$ such that $\int_M^{+\infty} g(t,x)dx < \varepsilon$ for any $t \in [0,a]$.

(H_1^1) There exists a matrix $\gamma = diag(\gamma_1,...,\gamma_n)$, $\gamma_1 > 0$, $i = 1,...,n$, such that

$$f(\varphi_2(t),(g*\varphi_2)(t)) + \gamma\varphi_2(t) \geq f(\varphi_1(t),(g*\varphi_1)(t)) + \gamma\varphi_1(t),$$

where φ_1, $\varphi_2 \in C(R,R^n)$, satisfy $0 \leq \varphi_1(t) \leq \varphi_2(t) \leq K$, $t \in R$.

(H_2^1) $f(\mu,...,\mu) \neq 0 \quad 0 < \mu < K$.

(H_3^1) $f(\mu,...,\mu) = 0$, $\mu = 0$ or $\mu = K$.

Definition 2.1. A continuous function $\varphi : R \longrightarrow R^n$ is called *an upper solution* if $\varphi^{'}$ and $\varphi^{''}$ exist almost everywhere and are essentially bounded on R, and φ satisfies

$$-D\varphi^{''}(t) + c\varphi^{'}(t) \geq f(\varphi(t),(g*\varphi)(t)), \quad \text{a.e. } t \in R. \tag{2.1}$$

Definition 2.2. A continuous function $\varphi : R \longrightarrow R^n$ is called *A lower solution* if $\varphi^{'}$ and $\varphi^{''}$ exist almost everywhere and are essentially bounded on R, and φ satisfies

$$-D\varphi^{''}(t) + c\varphi^{'}(t) \leq f(\varphi(t),(g*\varphi)(t)), \quad \text{a.e. } t \in R. \tag{2.2}$$

Definition 2.3. We definite the set of profiles of the corresponding wave system,

$$\Gamma = \left\{ \begin{array}{ll} (i) & \phi(-\infty) = 0,\ \phi(\infty) = K; \\ (ii) & \phi(z) \text{ in } R \text{ increasing}; \\ (iii) & \phi \in Y,\ Y = \left\{x \in BC(R,R^n) : x', x'' \in L^\infty(R,R^n)\right\} \end{array} \right\}.$$

3. Existence of Travelling Wave

System (1.1) always has a trivial equilibrium $E_0 = (0,0)$. If

$$\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds - E_1 > 0, \quad \alpha_2 \int_0^\infty g_2(s)e^{-\gamma_2 s}ds - E_2 > 0$$

then it has two semitrivial equilibria

$$E_u = (\frac{\alpha_1}{\beta_1}(\int_0^\infty g_1(s)e^{-\gamma_1 s}ds - E_1), 0) = (0, \frac{\alpha_2}{\beta_2}(\int_0^\infty g_2(s)e^{-\gamma_2 s}ds - E_2)$$

and, if

$$\beta_2(\alpha_1\int_0^\infty g_u(s)e^{-\gamma_u s}ds-E_1)>a_1(\alpha_2\int_0^\infty g_2(s)e^{-\gamma_2 s}ds-E_2)>0,$$
$$\beta_1(\alpha_2\int_0^\infty g_2(s)e^{-\gamma_2 s}ds-E_2)>a_2(\alpha_1\int_0^\infty g_1(s)e^{-\gamma_1 s}ds-E_1)>0,$$

then system has a unique positive equilibrium $E=(\widehat{u},\widehat{v})$, where

$$\widehat{u}=\frac{\beta_2(\alpha_1\int_0^\infty f_u(s)e^{-\gamma_u s}ds-E_1)-a_1(\alpha_2\int_0^\infty f_2(s)e^{-\gamma_2 s}ds-E_2)}{\beta_1\beta_2-a_1a_2},$$
$$\widehat{v}=\frac{\beta_1(\alpha_2\int_0^\infty f_2(s)e^{-\gamma_2 s}ds-E_2)-a_2(\alpha_1\int_0^\infty f_1(s)e^{-\gamma_1 s}ds-E_1)}{\beta_1\beta_2-a_1a_2}.$$

Since represents simple, let

$$k_1=\frac{\alpha_1}{\beta_1}(\int_0^\infty g_1(s)e^{-\gamma_1 s}ds-E_1);$$
$$k_2=\frac{\alpha_2}{\beta_2}(\int_0^\infty g_2(s)e^{-\gamma_2 s}ds-E_2).$$

In the following, we first discuss the local stability of the nonnegative equilibria E_u, E_v of system (1.1).

Lemma 3.1. *If the following statements hold:*

(i) $\beta_2(\alpha_1\int_0^\infty g_1(s)e^{-\gamma_u s}ds-E_1)>a_1(\alpha_2\int_0^\infty g_2(s)e^{-\gamma_2 s}ds-E_2)$,

(ii) $\beta_1(\alpha_2\int_0^\infty g_2(s)e^{-\gamma_2 s}ds-E_2)<a_2(\alpha_1\int_0^\infty g_1(s)e^{-\gamma_1 s}ds-E_1)$,

then the nonnegative equilibrium $E_u=(\frac{\alpha_u}{\beta_u}\int_0^\infty g_1(s)e^{-\gamma_u s}ds,0)$ *is locally stable and* $E_v=(0,\frac{\alpha_v}{\beta_v}\int_0^\infty g_2(s)e^{-\gamma_v s}ds)$ *is unstable.*

Proof. For the nonnegative equilibrium $E_u=(\frac{\alpha_u}{\beta_u}\int_0^\infty g_1(s)e^{-\gamma_u s}ds,0)$, after some algebra, the associated linearized system has nontrivial solution of the form $(c_1,c_2)\exp(\sigma t+ikx)$ if and only if the following holds:

$$g_1(\sigma,k^2)g_2(\sigma,k^2)=0,$$

where

$$g_1(\sigma,k^2)=\sigma-\alpha_1\int_0^\infty g_1(s)e^{-s(\gamma_1+\sigma+d_1k^2)}ds+D_1k^2 + 2\alpha_1\int_0^\infty f_1(s)e^{-\gamma_1 s}ds-E_1,$$

$$g_2(\sigma,k^2)=\sigma-\alpha_2\int_0^\infty g_2(s)e^{-s(\gamma_2+\sigma+d_2k^2)}ds+D_2k^2+E_2 + \frac{\alpha_2}{\beta_1}(\alpha_1\int_0^\infty g_1(s)e^{-\gamma_1 s}ds-E_1).$$

First, if

$$\sigma + 2\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds + D_2k^2 = \alpha_1 \int_0^\infty g_1(s)e^{-s(\gamma_1+\sigma+d_1k^2)}ds + E_1,$$

then Re$\sigma < 0$. Suppose that σ^*, Re$\sigma^* > 0$ such that

$$|\sigma^* + 2\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds + D_1k^2| = |\alpha_1 \int_0^\infty g_1(s)e^{-s(\gamma_1+\sigma^*+d_1k^2)}ds + E_1|$$

$$\leq \alpha_1 \int_0^\infty g_1(s)e^{-s(\mathrm{Re}\sigma^*)}e^{-s(\gamma_1+D_1k^2)}ds + E_1 \leq \alpha_1 \int_0^\infty g_1(s)e^{-s\gamma_1}ds + E_1$$

$$\leq 2\alpha_1 \int_0^\infty g_1(s)e^{-s\gamma_1}ds.$$

Assume

$$R = \alpha_1 \int_0^\infty g_1(s)e^{-s\gamma_1}ds,$$

then

$$|\sigma^* + 2R + d_1k^2| \leq 2R,$$

naturally, Re$\sigma < 0$.

Second, we prove: if $g_2(\sigma, k^2) = 0$ then Re$\sigma < 0$. Suppose there exists a σ^*, such that Re$\sigma^* > 0$. Let

$$|\sigma^* + D_2k^2 + E_2 + \frac{\alpha_2}{\beta_1}(\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds - E_1)|$$

$$\leq |\alpha_2 \int_0^\infty g_2(s)e^{-s(\gamma_2+\sigma^*+d_2k^2)}ds| < \alpha_2 \int_0^\infty g_2(s)e^{-s\gamma_2}ds,$$

by assumption,

$$\beta_1(\alpha_2 \int_0^\infty g_2(s)e^{-\gamma_2 s}ds - E_2) < a_2(\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds - E_1),$$

we have

$$-D_2k^2 - E_2 - \frac{\alpha_2}{\beta_1}(\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds - E_1) + \alpha_2 \int_0^\infty g_2(s)e^{-s\gamma_2}ds < 0.$$

It is not possible, then Re $\sigma < 0$.

Similarly we can prove: if

$$\beta_2(\alpha_1 \int_0^\infty g_u(s)e^{-\gamma_u s}ds - E_1) > a_1(\alpha_2 \int_0^\infty g_2(s)e^{-\gamma_2 s}ds - E_2)$$

holds, then

$$E_v = (0, \frac{\alpha_v}{\beta_v} \int_0^\infty g_v(s)e^{-\gamma_v s}ds)$$

is unstable.

Now, we begin to discuss the existence of travelling wave front of (1.1) by using the upper-lower solution method and an iteration scheme developed by Wang and Li and Ruan [6].

To seek a travelling wave front solution of system (1.1), we set $u(t,x) = \phi_1(z), v(t,x) = \phi_2(z), z = x + ct$, and $c > 0$, c is the wave speed. Then system (1.1) becomes

$$\begin{cases} D_1\phi_1^{''}(z) - c\phi_1^{'}(z) + \alpha_1 \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_1 s}} e^{\frac{-y^2}{4d_1 s}} \phi_1(z - cs - y) g_1(s) e^{-\gamma_1 s} dyds \\ \qquad - \beta_1\phi_1^2(z) - E_1\phi_1(z) - a_1\phi_1(z)\phi_2(z) = 0, \\ D_2\phi_2^{''}(z) - c\phi_2^{'}(z) + \alpha_2 \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_2 s}} e^{\frac{-y^2}{4d_2 s}} \phi_2(z - cs - y) g_2(s) e^{-\gamma_2 s} dyds \\ \qquad - \beta_2\phi_2^2(z) - E_2\phi_2(z) - a_2\phi_1(z)\phi_2(z) = 0, \end{cases} \tag{3.1}$$

which will be solved subject to the following conditions:

$$\begin{aligned} &\lim_{z\to-\infty} \phi_1(z) = 0, \quad \lim_{z\to\infty} \phi_1(z) = \frac{\alpha_u}{\beta_u}(\int_0^\infty g_1(s)e^{-\gamma_u s}ds - E_1), \\ &\lim_{z\to-\infty} \phi_2(z) = \frac{\alpha_v}{\beta_v}(\int_0^\infty g_2(s)e^{-\gamma_v s}ds - E_2), \quad \lim_{z\to\infty} \phi_2(z) = 0, \end{aligned} \tag{3.2}$$

The solution of (3.1) and (3.2) corresponds to the wavelling wave fronts of (1.1) connecting the two boundary equilibria. And also, for ecological realism, $\phi_1(z), \phi_2(z) \geq 0, z \in (-\infty, \infty)$ for all $z \in (-\infty, \infty)$. The latter is only possible for c exceeding a certain minimum value, as can be seen from the following linearized analysis. As $z \longrightarrow -\infty$, the system (3.1) becomes, approximately,

$$\begin{cases} D_1\phi_1^{''}(z) - c\phi_1^{'}(z) + \alpha_1 \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_1 s}} e^{\frac{-y^2}{4d_1 s}} \phi_1(z - cs - y) g_1(s) e^{-\gamma_1 s} dyds \\ \qquad - E_1 - a_1 k_2 = 0, \\ D_2\phi_2^{''}(z) - c\phi_2^{'}(z) + \alpha_2 \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi d_2 s}} e^{\frac{-y^2}{4d_2 s}} \phi_2(z - cs - y) g_2(s) e^{-\gamma_2 s} dyds \\ \qquad - E_2 - a_2 k_1 = 0. \end{cases} \tag{3.3}$$

Seeking solutions of this proportional to $e^{\lambda z}$ for the the first equation of system (3.3), one finds that

$$\Delta_1(\lambda, c) = 0,$$

where

$$\Delta_1(\lambda, c) = \alpha_1 \int_0^\infty g_1(s) e^{-\gamma_1 s + d_1 s\lambda^2 - cs\lambda} ds - (E_1 + a_1 k_2 + c\lambda - D_1\lambda^2).$$

Similarly, if we let $z \to \infty$ and approximate the second equation of (3.3) suitably, trial solutions of the form $e^{\lambda z}$ exist when

$$\Delta_2(\lambda, c) = 0,$$

where

$$\Delta_2(\lambda, c) = \alpha_2 \int_0^\infty g_2(s) e^{-\gamma_2 s + d_2 s \lambda^2 - cs\lambda} ds - (E_2 + a_2 k_1 + c\lambda - D_2 \lambda^2).$$

Since positive and monotone solutions are sought, any decay to zero as $z \longrightarrow \pm\infty$ must be monoscillatory. Now, $\Delta_1(\lambda, c) = 0$ relates to the situation as $z \longrightarrow -\infty$. So it is necessary that this equation has at least one real positive root; while $\Delta_2(\lambda, c) = 0$, should have a real negative root since the latter equation is for $z \longrightarrow \infty$. □

Lemma 3.2. *Assume that the condition of Lemma 3.1 holds true, g_1 situated (K_1) or (K_2) and g_2 situated (K_1) or (K_2), $\Delta_1(\lambda, c)\Delta_2(\lambda, c)$ as defined then exists $c^* > 0$, $\lambda^* > 0$:*

(i) $\Delta_1(\lambda^*, c^*) = 0$, $\frac{\partial}{\partial \lambda}\Delta_1(\lambda, c^*) \mid_{\lambda=\lambda^*} = 0$;

(ii) for any $c > c^*$, the equation $\Delta_1(\lambda, c) = 0$ has two positive solutions $0 < \lambda_1 < \lambda_2 < \lambda(c)$ and

$$\Delta_1(\lambda, c) \begin{cases} > 0, & \lambda \in (0, \lambda_1), \\ < 0, & \lambda \in (\lambda_1, \lambda_2), \\ > 0, & \lambda \in (\lambda_2, \lambda(c)). \end{cases}$$

When g_1 satisfied (K_1), there is $\lambda(c) = \frac{c+\sqrt{c^2+4d_1(\gamma_1+\rho_1)}}{2d_1}$, however, when g_1 satisfies (K_2), $\lambda(c) = \infty$.

(iii) For every $c > c^*$, the equation $\Delta_2(\lambda, c) = 0$ has one negative solution $\lambda_3 > \overline{\lambda}(c)$ and one positive solution $\lambda_4 < \widetilde{\lambda}(c)$ and

$$\Delta_2(\lambda, c) \begin{cases} > 0, & \lambda \in (\overline{\lambda}(c), \lambda_3), \\ < 0, & \lambda \in (\lambda_3, \lambda_4), \\ > 0, & \lambda \in (\lambda_4, \widetilde{\lambda}(c)), \end{cases}$$

when g_2 satisfied (K_1), then $\overline{\lambda}(c) = \frac{c-\sqrt{c^2+4d_2(\gamma_2+\rho_2)}}{2d_2}$, $\widetilde{\lambda}(c) = \frac{c+\sqrt{c^2+4d_2(\gamma_2+\rho_2)}}{2d_2}$, but while g_2 satisfied (K_2), we have $\overline{\lambda}(c) = -\infty$, $\widetilde{\lambda}(c) = \infty$.

Proof. First, if g_1 satisfies (K_1) when $c \geq 0$ is increasing, $\lambda(c)$ is increasing too. For every one $\lambda \in (0, \lambda(c))$ and $c \geq 0$, we have

$$\begin{aligned} \frac{\partial^2}{\partial \lambda^2}\Delta_1(\lambda, c) &= 2d_1\alpha_1 \int_0^\infty s g_1(s) e^{-\gamma_1 s + d_1 s\lambda^2 - cs\lambda} ds \\ &\quad + \alpha_1 (2d_1\lambda - c)^2 \int_0^\infty s^2 g_1(s) e^{-\gamma_1 s + d_1 s \lambda^2 - cs\lambda} ds + 2d_u > 0, \end{aligned}$$

$$\frac{\partial}{\partial c}\Delta_1(\lambda, c) = -\alpha_1 \lambda \int_0^{\infty} s g_1(s) e^{-\gamma_1 s + d_1 s \lambda^2 - cs\lambda} ds - \lambda < 0.$$

Accordingly, for each fixed $c \geq 0$ note that the graph of $y = \Delta(\lambda, c)$ is downwards convex function; and, as $c \geq 0$ is increasing, the graph of $y = \Delta(\lambda, c)$ is monotonically decreasing. And, for every $\lambda > 0$ $c > 0$,

$$\begin{aligned}\Delta_1(\lambda, 0) &= \left(\alpha_1 \int_0^{\infty} g_1(s) e^{-\gamma_1 s + d_1 s \lambda^2} ds - E_1\right) - a_1 k_2 + D_1 \lambda^2 > 0, \\ \Delta_1(0, c) &= \left(\alpha_1 \int_0^{\infty} g_1(s) e^{-\gamma_1 s} ds - E_1\right) - a_1 k_2 > 0.\end{aligned}$$

Let $c_0 = \alpha_1 \int_0^{\infty} g_1(s) e^{-\gamma_1 s} ds - E_1 - a_1 k_2 + 2d_1 + D_1$, at this time $\lambda(c) > 1$, and $\Delta_1(1, c_0) \leq -2d_1 < 0$. Since $\lim_{\lambda \to \lambda(c) - 0} \Delta_1(\lambda, c) = \infty$, then, by these, we can prove (i) and (ii).

The proof of (iii) is similar with the proof of (i) and (ii). This completes the proof. □

Wang and Li and *Ruan's* theory [6] presumes that the equilibria of the travelling front wave equations are 0 and K, where K is a vector with positive components. This is not so in our problem, but can be made so by the change of variables $\overline{\phi_1} = \phi_1$, $\overline{\phi_2} = k_2 - \phi_2$ under which the systems (3.1) and (3.2) are transformed into

$$\begin{cases} D_1 \phi_1''(z) - c\phi_1'(z) + f_1(\phi(z), (g * \phi)(z)) = 0, \\ D_2 \phi_2''(z) - c\phi_2'(z) + f_2(\phi(z), (g * \phi)(z)) = 0, \end{cases} \tag{3.4}$$

and

$$\begin{aligned}&\lim_{z \to -\infty}(\phi_1(z), \phi_2(z)) = (0, 0) = 0, \\ &\lim_{z \to +\infty}(\phi_1(z), \phi_2(z)) = (k_1, k_2) = K,\end{aligned}$$

where

$$\begin{aligned} f_1(\phi(z), (g * \phi)(z)) = \alpha_1 &\int_0^{\infty} \int_{-\infty}^{\infty} g_1(s) e^{-\gamma_1 s} \frac{1}{\sqrt{4\pi d_1 s}} e^{\frac{-y^2}{4d_1 s}} \phi_1(z - cs - y) dy ds \\ &- \beta_1 \phi_1(z)^2 - E_1 \phi_1(z) - a_1 k_2 \phi_1(z) + a_1 \phi_1(z) \phi_2(z), \\ f_2(\phi(z), (g * \phi)(z)) = \alpha_2 &\int_0^{\infty} \int_{-\infty}^{\infty} g_2(s) e^{-\gamma_2 s} \frac{1}{\sqrt{4\pi d_2 s}} e^{\frac{-y^2}{4d_2 s}} \phi_2(z - cs - y) dy ds \\ &+ \beta_2 \phi_2(z)^2 - (2\beta_2 k_2 + E_2) \phi_2(z) + a_2 k_2 \phi_1(z) - a_2 \phi_1(z) \phi_2(z). \end{aligned}$$

We look for wave front solutions in the following *profile set*:

$$\Gamma = \left\{ \begin{array}{l} (i)\phi(-\infty) = 0,\ \phi(\infty) = K, \\ (ii)\phi(z) \text{is nondecreasing in } R, \\ (iii)\phi \in Y \end{array} \right\}$$

assumed

$$\phi = (\phi_1, \phi_2),$$
$$f(\phi(z), (g * \phi)(z)) = (f_1(\phi(z), (g * \phi)(z)), f_2(\phi(z), (g * \phi)(z))),$$
$$\delta = (\delta_1, \delta_2).$$

Lemma 3.3. *$f(\phi(z), (g * \phi)(z))$ satisfies (H_1^1).*

Proof. We first prove f_1 satisfies (H_1^1). For every one $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2) \in C(R; R^2)$, ϕ, $\psi \in \Gamma$; while $0 \le \psi(z) \le \phi(z) \le K$, $z \in R$. We have:

$$\begin{aligned}
&f_1(\phi(z), (g * \phi)(z)) + \delta_1\phi_1(z) - f_1(\psi(z), (g * \psi)(z)) - \delta_1\psi_1(z)\\
&= \alpha_1 \int_0^\infty \int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}[\phi_1(z - cs - y) - \psi_1(z - cs - y)]dyds\\
&\qquad - \beta_1[\phi_1^2(z) - \psi_1^2(z)] - (E_1 + a_1k_2)[\phi_1(z) - \psi_1(z)]\\
&\qquad + a_1[\phi_1(z)\phi_2(z) - \psi_1(z)\psi_2(z)] + \delta_1[\phi_1(z) - \psi_1(z)]\\
&\ge -\beta_1[\phi_1^2(z) - \psi_1^2(z)] - (E_1 + a_1k_2)[\phi_1(z) - \psi_1(z)] + \delta_1[\phi_1(z) - \psi_1(z)] \ge 0.
\end{aligned}$$

Similarly, we could derive $f_2(\phi(z), (g * \phi)(z)) + \delta_2\phi_2(z) - f_2(\psi(z), (g * \psi)(z)) - \delta_2\psi_2(z) \ge 0$. $f(\phi(z), (g * \phi)(z))$ satisfies (H_1^1). Therefore, $f(\phi(z), (g * \phi)(z))$ is nondecreasing in $z \in R$. □

We now construct and prove that there exists upper-solutions of (3.4).

Let $\overline{\phi}(z) = (\overline{\phi}_1(z), \overline{\phi}_2(z))$ defined by

$$\overline{\phi}_1(z) = \begin{cases} k_1e^{\lambda_1 z}, & z \le 0,\\ k_1, & z > 0,\end{cases} \qquad \overline{\phi}_2(z) = \begin{cases} k_2e^{\lambda_1 z}, & z \le 0,\\ k_2, & z > 0.\end{cases}$$

Lemma 3.4. *$\overline{\phi}$ is a upper solution of (3.4).*

Proof. Obviously $\overline{\phi}(z) \in \Gamma$. (i) $z > 0$, (ii) $z < 0$.

(i) When $z > 0$, $\overline{\phi_1}(z) = k_1$, $\overline{\phi_2}(z) = k_2$, accordingly,

$$\begin{aligned}
&D_1\overline{\phi}_1''(z) - c\overline{\phi}_1'(z) + f_1(\overline{\phi}(z), (g * \overline{\phi})(z))\\
&\qquad \le \alpha_1 \int_0^\infty \int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}\overline{\phi}_1(z - cs - y)dyds\\
&- \beta_1k_1^2 - E_1k_1 - a_1k_1k_2 + a_1k_1k_2 \le k_1\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_1 s}ds - \beta_1k_1^2 - E_1k_1 = 0.
\end{aligned}$$

Similarly, we can prove:

$$D_2\overline{\phi}_2''(z) - c\overline{\phi}_2'(z) + f_2(\overline{\phi}(z), (g * \overline{\phi})(z)) \le 0.$$

(ii) $z<0, \overline{\phi}_1(z)=k_1e^{\lambda_1 z}$, $\overline{\phi}_2(z)=k_2e^{\lambda_1 z}$, it is obtained by substituting:

$$\begin{aligned}
&D_1\overline{\phi}_1''(z)-c\overline{\phi}_1'(z)+f_1(\overline{\phi}(z),(g*\overline{\phi})(z))\\
&=\alpha_1\int_0^\infty\int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}\overline{\phi}_1(z-cs-y)dyds\\
&+D_1k_1\lambda_1^2e^{\lambda_1 z}-ck_1\lambda_1e^{\lambda_1 z}-\beta_1k_1^2e^{2\lambda_1 z}-E_1k_1e^{\lambda_1 z}-a_1k_1k_2e^{\lambda_1 z}+a_1k_1k_2e^{2\lambda_1 z}\\
&\le\alpha_1k_1\int_0^\infty\int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}e^{\lambda_1(z-cs-y)}dyds\\
&+D_1k_1\lambda_1^2e^{\lambda_1 z}-ck_1\lambda_1e^{\lambda_1 z}-\beta_1k_1^2e^{2\lambda_1 z}-E_1k_1e^{\lambda_1 z}-a_1k_1k_2e^{\lambda_1 z}+a_1k_1k_2e^{2\lambda_1 z}\\
&=\alpha_1k_1\int_0^\infty g_1(s)e^{-\gamma_1 s}e^{d_1s\lambda_1^2+\lambda_1(z-cs)}ds+D_1k_1\lambda_1^2e^{\lambda_1 z}-ck_1\lambda_1e^{\lambda_1 z}-\beta_1k_1^2e^{2\lambda_1 z}\\
&=k_1e^{\lambda_1 z}[\alpha_1\int_0^\infty g_1(s)e^{-\lambda_1 s+d_1\lambda_1^2 s-c\lambda_1 s}ds\\
&+D_1\lambda_1^2-C\lambda_1-E_1-a_1k_2]+k_1e^{2\lambda_1 z}(a_1k_2-\beta_1k_1)\\
&=k_1e^{\lambda_1 z}\Delta_1(\lambda_1,c)+k_1e^{2\lambda_1 z}(a_1k_2-\beta_1k_1)=k_1e^{2\lambda_1 z}(a_1k_2-\beta_1k_1)\le 0.
\end{aligned}$$

Similarly, we have

$$D_2\overline{\phi}_2''(z)-c\overline{\phi}_2'(z)+f_2(\overline{\phi}(z),(g*\overline{\phi})(z))\le 0.$$

Now, we construct and prove that there exist lower-solutions of (3.4).

Let $\varepsilon>0$ sufficiently small to satisfy

$$\lambda_1<\lambda_1+\varepsilon<\min\{\lambda_2,2\lambda_1\};$$

$M>1$ is constant, and

$$M>\frac{4\beta_1}{-\Delta_1(\lambda_1+\varepsilon,c)}.$$

Assume $\underline{\phi}(z)=(\underline{\phi_1}(z),\underline{\phi_2}(z))$, where

$$\underline{\phi}_1(z)=\begin{cases}(1-Me^{\varepsilon z})e^{\lambda_1 z} & z<z_1\\ 0 & z\ge z_1,\end{cases}\qquad \underline{\phi}_2(z)=0,$$

$z_1=-\frac{1}{\varepsilon}\ln M<0$,

Lemma 3.5. *$\underline{\phi}(z)$ is a lower-solution of (3.4).*

Proof. Obviously, $\underline{\phi}_1(z)\ge 0$, $z\in R$. When $z\ge z_1$, $\underline{\phi}_1(z)\equiv 0$, and

$$\begin{aligned}
&D_1\underline{\phi}_1''(z)-c\underline{\phi}_1'(z)+f_1(\underline{\phi}(z),(g*\underline{\phi})(z))\\
&=\alpha_1\int_0^\infty\int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}\underline{\phi}_1(z-cs-y)dyds\ge 0,
\end{aligned}$$

while $z < z_1$,

$$\underline{\phi}_1(z) = (1 - Me^{\varepsilon z})e^{\lambda_1 z},$$
$$\underline{\phi}_1'(z) = [\lambda_1 - M(\lambda_1 + \varepsilon)e^{\varepsilon z}]e^{\lambda_1 z},$$
$$\underline{\phi}_1''(z) = [\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{\varepsilon z}]e^{\lambda_1 z}.$$

Hence, we have:

$$\begin{aligned}
&D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_1(\underline{\phi}(z), (g * \underline{\phi})(z)) \\
&= D_1[\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{\varepsilon z}]e^{\lambda_1 z} - c[\lambda_1 - M(\lambda_1 + \varepsilon)e^{\varepsilon z}]e^{\lambda_1 z} \\
&\quad + \alpha_1 \int_0^\infty \int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}\underline{\phi}_1(z - cs - y)dyds \\
&\quad - \beta_1[1 - Me^{\varepsilon z}]^2 e^{2\lambda_1 z} - (E_1 + a_1 k_2)(1 - Me^{\varepsilon z})e^{\lambda_1 z},
\end{aligned}$$

and since $z - cs - z_1 < y$, we have

$$\underline{\phi}_1(z - cs - y) = (1 - Me^{\varepsilon(z-cs-y)})e^{\lambda_1(z-cs-y)};$$

where $z - cs - z_1 \geq y$,

$$\underline{\phi}_1(z - cs - y) = 0 \geq (1 - Me^{\varepsilon(z-cs-y)})e^{\lambda_1(z-cs-y)}.$$

By these, we obtain

$$\begin{aligned}
&D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_1(\underline{\phi}(z), (g * \underline{\phi})(z)) \\
&\geq D_1[\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{\varepsilon z}]e^{\lambda_1 z} - c[\lambda_1 - M(\lambda_1 + \varepsilon)e^{\varepsilon z}]e^{\lambda_1 z} \\
&\quad + \alpha_1 \int_0^\infty \int_{-\infty}^\infty g_1(s)e^{-\gamma_1 s}\frac{1}{\sqrt{4\pi d_1 s}}e^{\frac{-y^2}{4d_1 s}}(1 - Me^{\varepsilon(z-cs-y)})e^{\lambda_1(z-cs-y)}dyds \\
&\quad - \beta_1[1 - Me^{\varepsilon z}]^2 e^{2\lambda_1 z} - (E_1 + a_1 k_2)(1 - Me^{\varepsilon z})e^{\lambda_1 z} \\
&= e^{\lambda_1 z}\Delta_1(\lambda_1, c) - Me^{\lambda_1 z}e^{\varepsilon z}\Delta_1(\lambda_1 + \varepsilon, c) - \beta_1[1 - Me^{\varepsilon z}]^2 e^{2\lambda_1 z}.
\end{aligned}$$

Because of $z < z_1 < 0$ and $\varepsilon < \lambda_1$, $e^{\lambda_1 z} < e^{\varepsilon z}$, and

$$(1 - Me^{\varepsilon z})^2 \leq 4.$$

By these, we have

$$\begin{aligned}
D_1\underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + f_1(\underline{\phi}(z), (g * \underline{\phi})(z)) &\geq e^{(\lambda_1+\varepsilon)z}[-M\Delta_1(\lambda_1 + \varepsilon, c) - 4\beta_1] \\
&= -e^{(\lambda_1+\varepsilon)z}\Delta_1(\lambda_1 + \varepsilon, c)[M - \frac{4\beta_1}{-\Delta_1(\lambda_1 + \varepsilon, c)}] \geq 0.
\end{aligned}$$

Similarly we can have

$$D_2\underline{\phi}_2''(z) - c\underline{\phi}_2'(z) + f_2(\underline{\phi}(z), (g * \underline{\phi})(z)) \geq 0.$$

The proof is compelete. □

Through Lemmas 3.1-3.5, we obtain:

Theorem 3.1. *Assume*

(i) $\beta_2(\alpha_1 \int_0^\infty g_1(s)e^{-\gamma_u s}ds - E_1) > a_1(\alpha_2 \int_0^\infty f_2(s)e^{-\gamma_2 s}ds - E_2)$.

(ii) $\beta_1(\alpha_2 \int_0^\infty g_2(s)e^{-\gamma_2 s}ds - E_2) < a_2(\alpha_1 \int_0^\infty f_1(s)e^{-\gamma_1 s}ds - E_1)$.

(iii) $\Delta_2(\lambda_1, c) \leq 2(\beta_2 k_2 - a_2 k_1)$.

(iv) $c > c^*$.

(v) $\delta_1 > a_1 k_1 + 2\beta_1 k_1 + E_1$, $\delta_2 > a_2 k_1 + 2\beta_2 k_2 + E_2$.

Then there exists a travelling wave front for (1.1) with speed c, connecting the equilibria $E_u = (k_1, 0)$ *and* $E_v = (0, k_2)$.

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