International Journal of Pure and Applied Mathematics
Volume 55 No. 3 2009, 383-404

PARAMETRIC STABILITY OF THE SOLUTIONS OF THE IMPULSIVE DIFFERENTIAL SOLOW EQUATION WITH DELAY AND DYNAMIC THRESHOLD EFFECTS

Ivanka M. Stamova[1], Jean-François Emmenegger[2 §]

[1]Department of Mathematics
Bourgas Free University
62, San Stefano Str., Bourgas, 8000, BULGARIA
e-mail: istamova@abv.bg

[2]Department of Quantitative Economics
University of Fribourg
90, Boulevard de Pérolles, Fribourg, 1700, SWITZERLAND
e-mail: jean-francois.emmenegger@unifr.ch

Abstract: The framework of the more than 50 years old Solow growth theory (1956) and Solow's studies on technical change (1957) have not lost their attraction and have been extended widely into modern growth theories. In this paper, the existence of jumps and threshold effects in German capital intensity is identified. For this reason an extension of the initial Solow equation towards a general impulsive Solow differential equation with a delay function is proposed. This extention is aimed to be applied in modern growth theories, for instance to model the German capital intensity. Sufficient conditions for the parametric stability of the solutions of such systems are investigated. The main results are obtained by applying the Lyapunov method.

"As long as we insist on practicing macro-economics
we shall need aggregate relationships."
Robert M. Solow (1957)

Received: June 30, 2009

[§]Correspondence author

AMS Subject Classification: 34K45, 91B84
Key Words: stability, Lyapunov-Razumikhin function, impulsive functional differential Solow equation, macroeconomic time series, capital intensity, random walk, unit root, variance ratio statistic

1. Introduction

Real world phenomenons, subject to short-term perturbations whose duration is negligible compared to the duration of the process, are accurately described with impulsive differential equations [2, 3, 11, 12, 16, 17, 26, 31, 32].

Emmenegger and Stamova considered in an earlier paper [11] the impulsive Solow equation

$$\begin{aligned} \dot{K}(t) &= sF(K(t), L_0 e^{nt}),\ t > t_0,\ t \neq t_k, \\ \Delta K(t_k) &= K(t_k + 0) - K(t_k) = I_k(K(t_k)),\ t_k > t_0,\ k = 1, 2, ..., \end{aligned} \tag{1}$$

where a single factor of production, the physical capital stock K, together with a given labour force of growth rate $n > 0$ of the labour force, $L(t) = L_0 e^{nt}$, to produce the output Y are considered. The Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$, with the technological constant $A = 1$ and the production elasticity of capital α, $0 \leq \alpha < 1$, $0 \leq t_0$, $t_1, t_2, ...$ ($t_0 < t_1 < t_2 < ...$) are the moments of impulsive perturbations due to which the capital K changes from the positions $K(t_k)$ into the position $K(t_k + 0)$ and the functions I_k characterizing the magnitude of the impulse effect at the moments t_k, $k = 1, 2, ...$ have been chosen.

The initial value problem (1) had been implemented into German macroeconomic data: German capital stock, labour force, capital intensity and GDP. Based on the considerations of Nelson and Plosser (1982), that for most macroeconomic time series the hypothesis of the presence of unit roots cannot be rejected, this analyses had been started with investigating for unit roots in these four German macroeconomic time series. It is well known that Cochrane (1988) had relaxed the thesis of Nelson and Plosser and had shown that macroeconomic time series can contain a difference stationary (DS) process, also called a random walk part, and a trend stationary (TS) process, simultaneously. In their previous analysis Emmenegger and Stamova [11] came to the analogous conclusion in the case of German capital stock. Secondly an consequently, a generalization of the Solow growth equation of capital stock and labour force [28] had been realized, extending the Solow equation to an impulsive differential equation. Particularly, the case $\Delta K(t_k) < 0$ corresponding to instantaneous reduction of the capital at times $t_k,\ k = 1, 2, ...,$ while the case $\Delta K(t_k) > 0$, describing heavy intensification of the capital, were treated. Such models are useful when the total capital stock $K(t)$ is subject to shock effects.

2. An Impulsive Solow Equation with Delay

The framework of the old Solow growth theory [28] and his studies on technical change ("slowdowns, speedups, improvement of education in the labour force", [29], p. 312) are far from loosing their attraction and have been extended widely into modern growth theories [1].

Also in certain circumstances, the future state of the growth equation of capital depends not only on the present state but also on its past history. Thus incorporating delay in the differential equations ensures a better model of the process involved. Indeed, in many situations, the growth equation is designed so that stable equilibrium points represent stored information. In this framework, relevant information is retrieved by initializing the solution of the growth equation at a point within the basin of attraction of the corresponding stable equilibrium point, and allowing the system to evolve to its stationary state. Therefore, the Solow growth equation of capital *with delay* and *impulsive effects* should be more accurate to describe the process of the systems. Since delays and impulses can affect the dynamical behaviour of the system, it is necessary to investigate both *delay* and *impulsive effects* on the stability of business cycles.

In this paper we focus on the Solow models, developed with a *time delay function* and first give a list of models of this type.

α) Brock and Taylor (*Handbook of Economic Growth*, Chapter 28 [1], 2005) develop the so called Green Solow benchmark ([6], p. 1772), where the initial Solow equation with constant rate technological progress is considered.

β) "Knowledge and learning are the key elements of modern theories of economic growth", see [33]. Romer in 1996 stated [24] that "the classic Solow model of growth introduced an exogenously defined exogenous variable, called effectiveness of labour. Then the changes of the physical capital per worker and effectiveness of labour are sources of growth. The effectiveness of labour is treated here as an abstract knowledge or technology" [33].

γ) Szydlowski and Krawiec [33] have analysed in 2001 the growth in scientific results of natural science and have proposed "to use functional differential equations to model the evolution of science in its sociological aspects." They

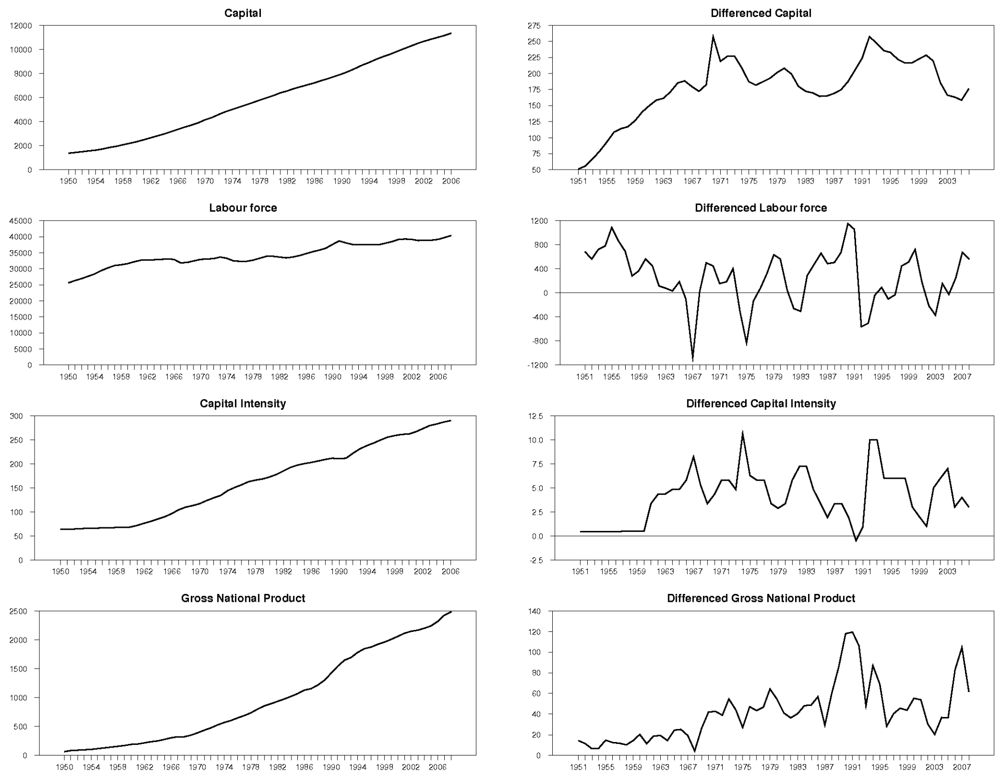

Figure 1: Levels and differences 1950-2008. German capital stock, labour force, capital in tensity, GDP

"showed that *the delay parameter* describing time required to learn and apply past scientific results to new discoveries plays a crucial role in generating cyclic behaviour via the Hopf bifurcation scenario", see ([33], p. 83). The developments of Szydlowski and Krawiec partly rely on the work of Derek de Solla Price [27], who published contemplation of using bibliometric techniques to create maps of scientific literatures. Therefore, they consider economic growth with knowledge accumulation, introducing a *delay* parameter T in the technological variable $A(t)$ of the Solow equation, assuming a constant growing rate g of technology. As usual s is the saving rate. This gives their Solow model with delay T ([33], p. 92)

$$\begin{aligned} \dot{k}(t) &= sk(t)^{\alpha} - (n + g\frac{A(t-T)}{A(t)})k(t), \\ \dot{A}(t) &= gA(t-T). \end{aligned} \tag{2}$$

δ) Benhabib and Rustichini [4] in 1991 considered an economy having a *vintage capital structure*, that means, machines and equipment belonging to separate generations and having different productivity or facing different depreciation schedules.

ϵ) Boucekkine, Licandro and Paul ([7], p. 3) in 1996 have shown that the vintage capital growth model of Solow et al "can be transformed into a system of differential-difference equations of the form

$$\dot{y}(t) = F[t, y(t), y(t - \tau(t, y(t)))], \tag{3}$$

where t is the time index, $y(t)$ is a vector of endogenous variables, $F(.)$ is an appropriated dimensioned vector function, and $\tau(.)$ is a real-valued function. In the case studied here, $\tau(.)$ is positive and corresponds to the lag." The resulting differential system is called a *delay differential equation system*. Note that in equation (3), the lag is not only time-dependent, it is state-dependent in the sense that it depends on the endogenous variables $y(t)$ ([7], p. 3).

3. German Capital, Labour Force, Capital Intensity and GDP

Again, consider the German macroeconomic time series of annual values of the capital stock $K(t)$, the labour force $L(t)$, the capital intensity $r(t) = \frac{K(t)}{L(t)}$ and the Gross Domestic Product $GDP(t)$, but now for the period 1950-2006[1]. All the values of the four series are expressed at constant prices of the year 1991.

Based on the empirical research of Nelson and Plosser [20] in 1982 and of Cochrane [8] in 1988, where the world is in harmony between *continuous* and *discrete* processes, these German macroeconomic time series are reexamined following the technique of Cochrane [8] to discover the presence of unit roots, random walk components and time trend components.

The eight pictures of Figure 1 present in the first column the German macroeconomic time series. In the second column the 1−differences of the series in levels are shown. Being in a context of growth analysis, it is natu-

[1]The "Statistisches Bundesamt" (2008, Volkswirtschaftliche Gesamtrechnung) publishes time series of the former and the reunified *Federal Republic of Germany* (FRG). Consequently, in order to get series of a maximal number of observed values, the time series $1991 - 2000$ of the enlarged FRG have been linked to the time series of the former FRG for $1950 - 1991$. The year 1991 is the breaking period between both time series. For example: In the former FRG, the GDP of the year 1991 was $GDP1_{1991} = 1'419.63$ Mrd Euro, in the reunified FRG, the GDP is $GDP2_{1991} = 1'538.64$ Mrd Euro. Thus, all the values $GDP2_t; t = 1991, 1992, ..., 2000$ have to be multiplied by the scaling factor $c = \frac{1419.63}{1538.64} = 0.9227$, in order to link both GDP series.

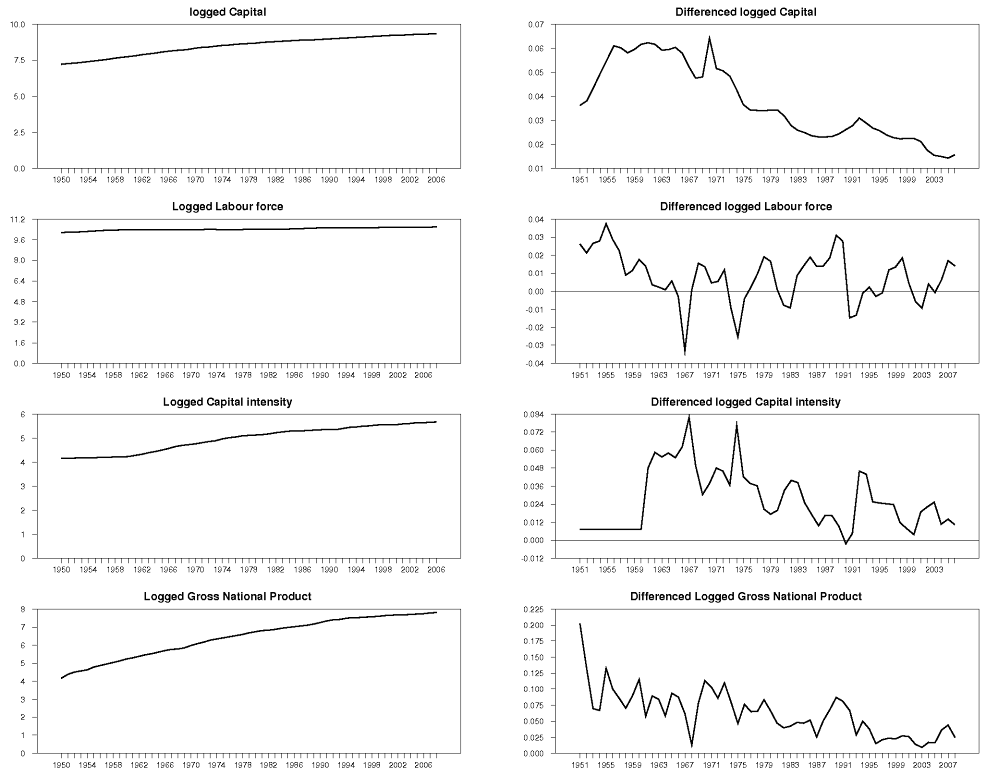

Figure 2: Logged levels and differences 1950-2008. Logged German capital stock, Labour force, capital intensity, GDP

ral also to consider the logged series. Figure 2 presents in the first column the logged German macroeconomic time series. In the second column the first differences of the logged series are shown.

3.1. Unit Roots and the Variance Ratio Statistic

When the macroeconomic time series are described by DS models, there is the necessity to investigate their degree of integration. That means, the presence of unit roots has to be evaluated. The Dickey-Fuller (DF), the augmented Dickey-Fuller (ADF) or the Phillips-Perron (PP) tests are frequently used for unit root tests, based on the Box-Jenkins methodology. Following this strategy, unit root tests are performed for all the four German macroeconomic time series (in the levels and logged)[2]. Nelson and Plosser (1982) applied the DF tests to many US time series and showed that *most* macroeconomic variables are well described by ARIMA models with one unit root, see Maddala and In-Moo-Kim ([18], p. 47). To some extent, it is the result got here. All these four German macroeconomic time series are at least integrated I(1), some unit root test even indicate I(2).

For these reasons, a Box-Jenkins analysis has been performed.

The four German macroeconomic times series, in the levels and logged, are noted as y_t. With $i = 1, 2$ and the i-order difference operators $\Delta^i y_t$ one describes ARIMA(1,1,0) and ARIMA(1,2,0) models[3]. The equations (4)

$$(1 - \underset{(\sigma_1)}{\Phi_1} L)\Delta^i y_t = \varepsilon_t \; ; \quad \varepsilon_t \sim i.i.d. \quad (0, \sigma^2), \quad i = 1, 2. \tag{4}$$

present those ARIMA(1,i,0) models. The main estimated coefficients Φ_1 are presented with their standard errors, see Table 1.

The Presence of Random Walk Components in the Differences. For one difference, $i = 1$, the coefficients Φ_1, col. (1) are all significant, but the values of the Durbin-Watson statistics DW, col. (3), are mainly far from 2, indicating the presence of serial correlation. For second order differences, $i = 2$, all the coefficients Φ_1, col. (4), are *not* significant and can be replaced by $\Phi_1 = 0$. The DW-statistics, col. (5), are nearly 2, indicating that there are no longer serial correlations. For these reasons the model (4) can be rewritten: $(1 - \Phi_1 L)\Delta^2 y_t \cong \Delta^2 y_t = \Delta y_t - \Delta y_{t-1} = \varepsilon_t \; ; \; \varepsilon_t \sim i.i.d. \; (0, \sigma^2)$, exhibiting a *random walk structure* for the differenced series Δy_t,

$$\Delta y_t = \Delta y_{t-1} + \varepsilon_t \; ; \; \varepsilon_t \sim i.i.d. \; (0, \sigma^2). \tag{5}$$

The Presence of Jumps! This is the *main* point. When the *differenced*

[2]The result are summarized as follows: a) The four series in the levels exhibit a unit root, using Dickey-Fuller tests without trend, and different numbers of lags, here 0, 1, 2. b) The four series in the levels exhibit a unit root, using Dickey-Fuller tests with trend, and different numbers of lags, here 0, 1, 2 in 10 of the 12 cases. The exceptions are: capital stock with lag = 0 and capital intensity with lag = 0. c) The four logged series exhibit ambiguous results, using Dickey-Fuller tests without trend, and different numbers of lags, here 0, 1, 2. Only 5 from 12 logged series exhibit unit roots. d) The four logged series exhibit a unit root, using Dickey-Fuller tests with trend, and different numbers of lags, here 0, 1, 2 in 10 of 12 cases. The exceptions are: logged labour force with 1 and 2 lags. e) The four first differences of all the series (logged and in the levels) exhibit ambiguous results, using Dickey-Fuller tests with and without trend, and different numbers of lags, here 0, 1, 2. f) The four second differences of all the series (logged and in the levels) exhibit *no* unit root, using Dickey-Fuller tests with and without trend, and different numbers of lags, here 0, 1, 2.

[3]All the calculations have been established with the software package *WINRATS 6.3* of Estima, Illinois, USA [10].

	Φ_1 (1)	DW (2)	p-value (3)	Φ_1 (4)	DW (5)	p-value (6)
i	1	1	1	2	2	2
KS	1.0036 (0.0122)	1.740	0.000	0.1259 (0.1376)	1.987	0.3647
$Labour$	0.6659 (0.0988)	1.681	0.000	0.0343 (0.1347)	1.966	0.8001
KI	0.9008 (0.0122)	1.740	0.000	-0.0274 (0.1377)	2.012	0.8430
GDP	0.9435 (0.0491)	2.012	0.000	-0.0933 (0.1411)	1.959	0.5111
$log(KS)$	0.9899 (0.0126)	1.818	0.000	0.0893 (0.1366)	1.995	0.5161
$log(Labour)$	0.6912 (0.0933)	1.708	0.000	0.0330 (0.1346)	1.963	0.8071
$log(KI)$	0.9169 (0.0545)	2.027	0.000	-0.0565 (0.1372)	2.030	0.6823
$log(GDP)$	0.8662 (0.0459)	2.009	0.000	-0.0670 (0.1271)	2.171	0.6003

Table 1: Box-Jenkins ARMA(1,1,0) and ARMA(1,2,0) models of the four German macroeconomic time series (in levels and logged)

four German macroeconomic time series (in levels or logged) exhibit a *random walk structure*, then the existence of jumps at some time points cannot be excluded, because the time series (in levels or logged) exhibit at least at the critical moments the jumps present in the differences at these moments.

How Important is the Random Walk Component? New methods to investigate the presence of unit roots in macroeconomic time series have been developed. Cochrane [8] proposed to use the *variance ratio statistic* to explore the importance of the random walk component within GNP and to compare it to its stationary component. The work of Cochrane relies on previous results of Diebold [9] and Poole [22], who called the *variance ratio* the *variance-time function* and even earlier on Working [34], who called the variance ratio the *error-time relation.* In this paper, the technique of Cochrane is applied to the four German macroeconomic time series (in the levels and logged).

The Variance Ratio Statistics. The VR_p is the *variance of the p-differences* $(y_t - y_{t-p})$ of a time series $\{y_t\}$ divided by p times the *variance of the* 1*-differences* $(y_t - y_{t-1})$ of the time series $\{y_t\}$. Let $\sigma_p^2 = p^{-1}var(y_t - y_{t-p})$ be $1/p$ times the *variance of the p-differences* and $\sigma_{\Delta y}^2 = var(y_t - y_{t-1})$ be the *variance of the* 1*−differences.* Then, the variance ratio statistic is $VR_p = \frac{\sigma_p^2}{\sigma_{\Delta y}^2}$.

Cochrane used the following estimator for the variance ratio

$$\widehat{VR}_p = \frac{var(y_t - y_{t-p})}{p \cdot var(y_t - y_{t-1})} \cdot \left(\frac{T}{T-p+1}\right), \tag{6}$$

defined for each lag p. Consider a series $\{y_t\}$ of T observations, then the *variance ratio statistic* VR_p with a bias ratio corrector $\frac{T}{T-p+1}$ for short time series, measures the variance of the p-differences of the series divided by p to the variance of the 1-differences of the series $\{y_t\}$.

The simplest member of that class of difference stationary processes (DS) is the *random walk* process for which the changes z_t are serially uncorrelated, represented in the following form

$$y_t = \alpha + y_{t-1} + z_t. \tag{7}$$

It is well-known that if $\{y_t\}$ is a pure random walk (7), then $VR_p \to 1$ for $p \to \infty$. If $\{y_t\}$ is trend-stationary, then $VR_p \to 0$ for $p \to \infty$.

Beveridge and Nelson [5] have shown that every 1-difference stationary process $I(1)$ can be decomposed into a *random walk component* and a *stationary component*. Under the assumption that $\{y_t\}$ is a 1-difference stationary process, this decomposition is applied to $\{y_t\}$ and gives

$$y_t = x_t + c_t, \tag{8}$$

where $\{x_t\}$ is the random walk (7) component and $\{c_t\}$ is the stationary component. The innovation variance of the random walk component $\sigma^2_{\Delta x} = var(x_t - x_{t-1})$ is a natural measure of the importance of the random walk component and is put into relation with the variance $\sigma^2_{\Delta y} = var(y_t - y_{t-1})$. Cochrane (1988, p. 906) shows that the limit of σ^2_p is the innovation variance of the random walk component,

$$\lim_{p\to\infty} \sigma^2_p = \lim_{p\to\infty} VR_p \cdot \sigma^2_{\Delta y} = \sigma^2_{\Delta x}. \tag{9}$$

3.2. Empirical Results with the Variance Ratio Test

Cochrane ([8], p. 989) developed the idea that the fluctuations in GNP are partly permanent and partly temporary, which one can model as a combination of a stationary series and a random walk. If the series $\{y_t\}$ represents the $\log(GNP)$, then the plot of $var(y_t - y_{t-p})/p = VR_p \cdot var(y_t - y_{t-1})$ versus p settles down to the variance of the shock to the random walk component $\{x_t\}$ (8) of the series $\{y_t\}$. Cochrane suggested that the innovation variance of the random walk component $\sigma^2_{\Delta x} = var(x_t - x_{t-1})$ is about the part of the variance of year-to-year changes, to which converges the *variance ratio*: Following this

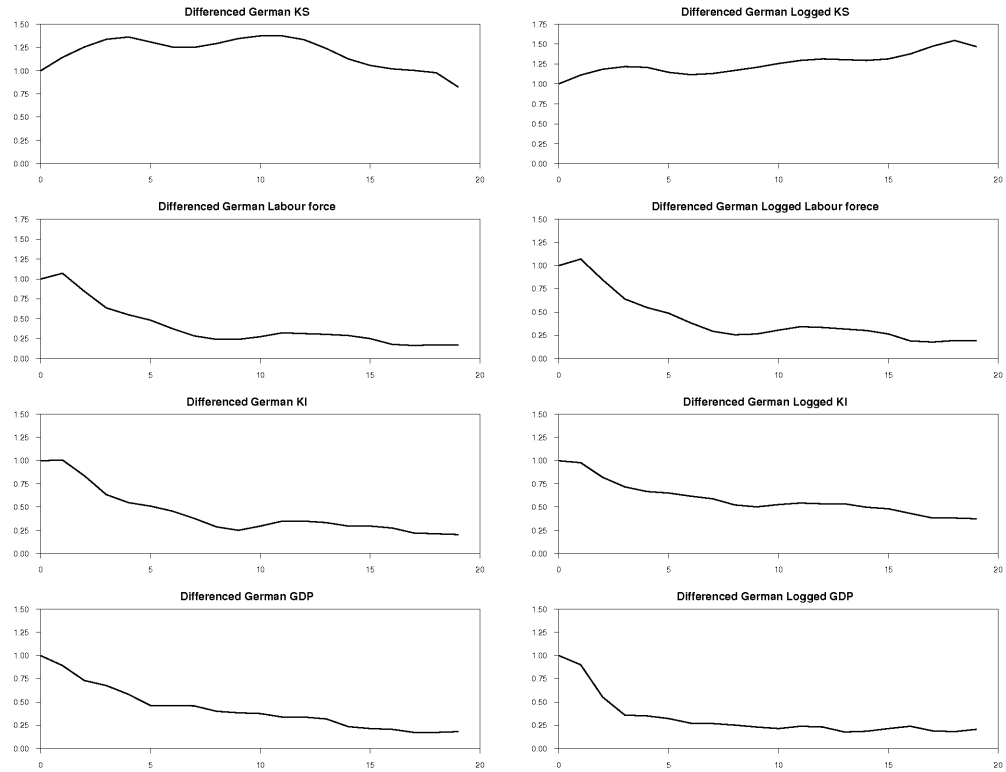

Figure 3: Differenced and differenced logged time series. Variance ratio of German macroeconomic time series

way of thinking, the present investigations show that the annual *differenced German capital stock* and the annual *differenced logged German capital stock* contain a large random walk component, but the differenced German GDP and the differenced logged German GDP contain a large temporary component, see Figure 3[4].

[4]The same statement is true in analogy for the differenced German labour, the differenced logged German labour, the differenced German capital intensity, the differenced logged German capital intensity.

p	KS	L	r	GDP	$log(KS)$	$log(L)$	$log(r)$	$log(GDP)$
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	1.14	1.07	1.01	0.89	1.11	1.06	0.98	0.90
2	1.25	0.85	0.84	0.73	1.18	0.85	0.82	0.55
3	1.33	0.64	0.64	0.68	1.21	0.64	0.72	0.36
4	1.36	0.55	0.55	0.58	1.20	0.55	0.67	0.32
5	1.30	0.48	0.51	0.46	1.14	0.49	0.65	0.27
6	1.25	0.37	0.46	0.46	1.12	0.38	0.62	0.27
7	1.25	0.28	0.38	0.46	1.13	0.29	0.59	0.29
8	1.29	0.24	0.29	0.40	1.17	0.26	0.52	0.25
9	1.34	0.24	0.25	0.38	1.21	0.27	0.50	0.23
10	1.37	0.28	0.29	0.38	1.26	0.30	0.53	0.22

Table 2: Variance Ratio of the four *differenced* German and four *differenced* logged German macro economic time series

3.3. The Presence of Jumps in the Macroeconomic German Time Series

In this subsection the results of empirical investigations concerning the presence of a random walk component within the four German macro-economic time series are described.

The persistence of the random walk component is explored with the variance ratio statistic (6). The variance ratio values have been calculated for each of the eight series of Figure 1 and Figure 2. The results are presented with the eight plots in Figure 3. Obviously none of them show convergence to 0. This means, none of them show pure time stationarity (TS). The computational results are presented in Table 1[5]. Remember that the presence of a unit root has been tested with an ADF-test.

The main present empirical results permit to support the statement through the variance ratio test of the presence of a *random walk component* in each of the *differenced* German series $L(t)$, $r(t)$, $GDP(t)$ and the *differenced* $log(L(t))$, $log(r(t))$, $log(GDP)$, because the variance ratio does not decline lower than about $\frac{1}{4}$ of 1. For the *differenced* capital stocks $K(t)$, $log(K(t))$ the variance ratio remains on a value of about 1 or is even higher. Due to Cochrane ([8], p. 894), that the "series will continue to diverge from its previously forecast value following a shock".

[5]Cochrane ([8], p. 894-895) exemplifies that the variance ratio test values $VR_p > 1$ reveal "series that will continue to diverge from its previously forecast value following a shock". Campbell and Mankiw [19] originated and emphasise this interpretation.

This means that there are sufficient reasons to consider these four German time series, differenced logged or differenced in levels, as containing random walk parts. Therefore at least at some of the annual periods of measurement, the series $r(t)$ and $K(t)$ exhibit jumps or threshold effects. This is the deep motivation to revisit the impulsive Solow equation (2) with delays!

4. The Existence of Solutions of a Solow Equation with Jumps or Threshold Effects Including a Delay Parameter

The main purpose of our paper is to investigate the stability of its equilibrium states. The traditional approach has been to compute an equilibrium of interest, and then introduce a change of variables that translates this equilibrium to the origin.

It is important to note, however, that fixed equilibrium assumptions are not realistic in such a general model. In fact, it is often the case that variations in the system parameters (such as the consumption per capita) result in a moving equilibrium, whose stability properties can vary substantially. In some situations, the equilibrium could even disappear all altogether, as in the case of heavily stressed economic systems. Therefore, for our purpose, it is suitable to use the concept of *parametric stability*, which simultaneously captures the existence and stability of a moving equilibria. Siljak [25] has formulated this concept. By using the Razumikhin technique [17, 23, 26, 31, 32] we establish several criteria for parametric stability. The conditions are independent of the form of specific delay and have important significance in both theory and applications.

4.1. Statement of the Problem: Preliminary Notes and Definitions

Let $R_+ = [0, \infty)$, $t_0 \in R_+$, and $\tau \in R_+$.

We consider the following more general impulsive Solow growth equation of capital with dynamical thresholds

$$\begin{cases} \dot{K}(t) = f(t, K(t - \tau(t)), p),\ t \geq t_0,\ t \neq t_k, \\ \Delta K(t_k) = K(t_k + 0) - K(t_k) = I_k(K(t_k), p),\ t_k > t_0, \\ k = 1, 2, ..., \end{cases} \tag{10}$$

where $f : [t_0, \infty) \times R \times R \to R$; $I_k : R \times R \to R$; p is a real parameter[6]; $\tau(t)$

[6]The parameter p may represent the depreciation rate of capital, the intrinsic growth rate

corresponds to the transmission delay and satisfies $0 \leq \tau(t) \leq \tau$, ($\tau = \text{const.}$); $t_0 < t_1 < t_2 < ...;\ \lim\limits_{k\to\infty} t_k = \infty$.

Let $J \subset R$ be an interval. Define the following classes of functions:

$PC[J,R] = \{\sigma : J \to R :\ \sigma(t)$ is continuous everywhere except at some points $t_k \in J$ at which $\sigma(t_k - 0)$ and $\sigma(t_k + 0)$ exist and $\sigma(t_k - 0) = \sigma(t_k)\}$;

$PCB[J,R] = \{\sigma \in PC[J,R] : \sigma(t)$ is bounded on J $\}$.

Let $K_0 \in PCB[[-\tau,\ 0], R]$, i.e. K_0 corresponds to the initial capital for $t \in [-\tau,\ 0]$. Denote by $K(t) = K(t; t_0, K_0, p)$, $K \in R$ the solution of equation (10) satisfying the initial conditions:

$$\begin{cases} K(t; t_0, K_0, p) = K_0(t - t_0),\ t_0 - \tau \ \leq\ t \leq\ t_0, \\ K(t_0 + 0; t_0, K_0, p) = K_0(0), \end{cases} \tag{11}$$

and by $J^+(t_0, K_0, p)$ - the maximal interval of type $[t_0, \beta)$ in which the solution $K(t; t_0, K_0, p)$ is defined.

The solution $K(t) = K(t; t_0, K_0, p)$ of the initial value problem (10), (11) is characterized by the following:

a/ For $t_0 - \tau \ \leq\ t \leq\ t_0$ the solution $K(t)$ satisfies the initial conditions (11).

b/ For $t_0 < t \leq\ t_1$, $K(t)$ coincides with the solution of the problem

$$\begin{cases} \dot{K}(t) = f(t, K(t - \tau(t)), p),\ t > t_0, \\ K_{t_0} = K_0(s),\ -\tau\ \leq\ s \leq\ 0. \end{cases} \tag{12}$$

At the time $t = t_1$ the mapping point $(t, K(t; t_0, K_0, p))$ of the extended phase space jumps momentarily from the position $(t_1, K(t_1; t_0, K_0, p))$ to the position $(t_1, K(t_1; t_0, K_0, p)) + I_1(K(t_1; t_0, K_0, p),\ p))$.

For $t_1 < t \leq\ t_2$ the solution $K(t)$ coincides with the solution of

$$\begin{cases} \dot{y}(t) = f(t, y(t - \tau(t)), p),\ t > t_1, \\ y_{t_1} = \varphi_1,\ \varphi_1 \in\ PCB[[-\tau,\ 0], R], \end{cases}$$

where

$$\varphi_1(t - t_1) = \begin{cases} K_0(t - t_1),\ t \in\ [t_0 - \tau, t_0] \cap\ [t_1 - \tau, t_1], \\ K(t; t_0, K_0, p),\ t \in\ (t_0, t_1) \cap\ [t_1 - \tau, t_1], \\ K(t; t_0, K_0, p) + I_1(K(t; t_0, K_0, p), p),\ t = t_1. \end{cases}$$

At the time $t = t_2$ the mapping point $(t, K(t))$ jumps momentarily, etc.

The solution $K(t; t_0, K_0, p)$ of problem (10), (11) is a piecewise continuous function for $t \in J^+(t_0, K_0, p)$ with points of discontinuity of the first kind

of the population or the carrying capacity of the habitat (in biology).

$t = t_k, k = 1, 2, ...$ at which it is continuous from the left.

Introduce the following notations:

$$G_k = \{(t, x) \in [t_0, \infty) \times R : \ t_{k-1} < t < t_k\}, \ k = 1, 2, ...; \quad G = \bigcup_{k=1}^{\infty} G_k;$$

$\|\phi\| = \sup_{t \in [t_0 - \tau, t_0]} |\phi(t - t_0)|$ is the norm of the function $\phi \in PCB[[-\tau, 0], R]$. We also assume that for some nominal value p^* of the parameter vector p, there is an equilibrium state K^*, that is,

$$\begin{cases} f(t, K^*, p^*) = 0, \ t \geq t_0, \ t \neq t_k, \\ \Delta K^*(t_k) = K^*(t_k + 0) - K^*(t_k) = 0, \ t_k > t_0, k = 1, 2, ..., \end{cases} \tag{13}$$

and K^* is stable. Suppose that the parameter vector p is changed from p^* to another value. The question arises: Does there exist a new equilibrium K^ε of (10)? If K^ε exists, is it stable as K^* was, or is its stability destroyed by the change of p? Consider the equilibrium $K^\varepsilon : R \to R$ as a function $K^\varepsilon(p)$ and introduce the following definitions of parametric stability.

Definition 4.1. The system (10) is said to be *parametrically stable* at $p^* \in R$ if there exists a neighbourhood $N(p^*)$ such that for any $p \in N(p^*)$:

(i) there exists an equilibrium $K^\varepsilon(p) \in R$;

(ii) for any $t_0 \in R_+$ and for any number $\varepsilon > 0$, there exists a number $\delta = \delta(t_0, \varepsilon, p) > 0$ such that $\|K_0 - K^\varepsilon(p)\| < \delta$ implies $|K(t; t_0, K_0, p) - K^\varepsilon(p)| < \varepsilon$ for all $t \geq t_0$.

Remark 4.1. Definition 4.1 says that the variation of parameter p is limited to a neighbourhood $N(p^*)$. If the system (10) is not stable in the above sense, we say it is *parametrically unstable* at p^*. This means that if for any neighbourhood $N(p^*)$, there exists a $p \in N(p^*)$ for which either there is no equilibrium $K^\varepsilon(p)$ of (10), or there is an equilibrium $K^\varepsilon(p)$ which is unstable in the sense of Lyapunov.

Definition 4.2. The system (10) is said to be *parametrically uniformly stable* at $p^* \in R$ if the number δ from Definition 4.1 is independent of $t_0 \in R_+$.

Definition 4.3. The system (10) is said to be *parametrically uniformly asymptotically stable* at $p^* \in R$ if there exists a neighbourhood $N(p^*)$ such that for any $p \in N(p^*)$:

(i) it is parametrically uniformly stable at p^*;

(ii) for all $p \in N(p^*)$, there exists a number $\mu = \mu(p) > 0$ such that $\|K_0 - K^\varepsilon(p)\| < \mu$ implies

$$\lim_{t \to \infty} |K(t; t_0, K_0, p) - K^\varepsilon(p)| = 0.$$

We introduce the following conditions:

H1. $f \in C[[t_0, \infty) \times R \times R, R]$.

H2. The function $f(t, K, p)$ is Lipchitzian with respect to K in R and $p \in R$ uniformly on $t \in [t_0, \infty)$.

H3. There exists a constant $M > 0$ such that for all $(t, K, p) \in [t_0, \infty) \times R \times R$

$$|f(t, K, p)| \leq M < \infty.$$

H4. $I_k \in C[R \times R, R]$, $k = 1, 2, ...$

H5. $t_0 < t_1 < t_2 < ...$ and $\lim_{k \to \infty} t_k = \infty$.

Define the following classes of functions:

$\mathcal{K} = \{a \in C[R_+, R_+] : \ a(u) \text{ is strictly increasing and } a(0) = 0\}$;

$V_0 = \{V : [t_0, \infty) \times R \to R_+ : V \in C[G, R_+], \ V(t, K^{\varepsilon}(p)) = 0, \ t \in [t_0, \infty),$ V is locally Lipschitzian in $K \in R$ on each of the sets G_k, $V(t_k - 0, K) = V(t_k, K)$ and $V(t_k + 0, K) = \lim_{\substack{t \to t_k \\ t > t_k}} V(t, K)$ exists$\}$.

For a function $V \in V_0$ and for some $t \geq t_0$, we define the set

$$\Omega_1 = \{K \in PC[(t_0, \infty), R] : \ V(s, K(s)) \leq V(t, K(t)), \ t - \tau \leq s \leq t\}.$$

Let $V \in V_0$. For $K \in PC[[t_0, \infty), R]$ and $t \neq t_k$, $k = 1, 2, ...$ we define the upper right-hand Dini derivative of $V \in V_0$ (with respect to system (10))

$$D^+V(t, K(t)) = \limsup_{h \to 0^+} h^{-1}[V(t + h, K(t + h)) - V(t, K(t))].$$

In the proof of the main results we shall use the following lemmas:

Lemma 4.1. *If the conditions H1, H3-H5 hold, then* $J^+(t_0, K_0, p) = [t_0, \infty)$.

Proof. Since the conditions H1 and H3 hold, from the existence theorem for the functional differential equation without impulses $\dot{x} = f(t, x_t, p)$ ([13], Theorem 2.2.1) it follows that the solution $K(t) = K(t; t_0, K_0, p)$ of the problem (10), (11) is defined on each of the intervals $(t_{k-1}, t_k]$, $k = 1, 2, ...$ From conditions H4 and H5 we conclude that it is continuable for $t \geq t_0$. □

Let us note that the problems of existence, uniqueness, and continuability of the solutions of functional differential equations without impulses has been investigated in the monographs [13], [14] and [15].

Lemma 4.2. (see [31, 32]) *Let the following conditions hold:*

1. Conditions H1, H3-H5 are met.

2. The solution $K = K(t; t_0, K_0, p)$ *of the problem (10), (11) is such that* $K \in PC[(t_0 - \tau, \infty), R]$.

3. The function $V \in V_0$ *is such that*

$$V(t_k + 0, K(t_k) + I_k(K(t_k), p)) \le V(t_k, K(t_k)),\ p \in R,\ k = 1, 2, ...$$

and the inequality

$$D^+V(t, K(t)) \le 0,\ t \ne t_k,\ k = 1, 2, ...,$$

is valid for $t \in [t_0, \infty)$ *and* $K \in \Omega_1$.

Then

$$V(t, K(t; t_0, K_0, p)) \le V(t_0 + 0, K_0(0)),\ t \in [t_0, \infty).$$

5. Main Results

The following theorem provides sufficient conditions for requirement (i) of Definition 4.1.

Theorem 5.1. *Let the following conditions hold:*

1. Conditions H1 - H5 hold.

2. For some nominal value p^* *of the parameter vector* p*, there is an equilibrium state* K^* *which satisfies (13).*

3. $detD_x f(t, K^*, p^*) \ne 0,\ t \ne t_k,\ \ k = 1, 2, ...$

Then there exists a neighbourhood $N(p^*)$ *of* p^* *such that for any* $p \in N(p^*)$*, the equation*

$$\begin{cases} f(t, K(t - \tau(t)), p) = 0,\ t > t_0,\ t \ne t_k, \\ \Delta K(t_k) = K(t_k + 0) - K(t_k) = 0,\ t_k > t_0,\ k = 1, 2, ..., \end{cases} \tag{14}$$

has a solution $K^\varepsilon(p) \in R$.

Proof. Conditions H1-H3 and condition 3 of Theorem 5.1 as well as the existence theorem applied to problem (12) [6, 7] imply that if for the value p^* of the parameter vector p, there is an equilibrium state K^* which satisfies $f(t, K^*, p^*) = 0$, then there exists a neighbourhood $N(p^*)$ of p^* such that for any $p \in N(p^*)$, the equation

$$f(t, K(t - \tau(t)), p) = 0 \tag{15}$$

has a solution $\Phi_1^\varepsilon(t, p) \in R$ for $t \ge t_0$. Let t_1 be the first moment of impulsive perturbation. Setting $K^\varepsilon(p) = \Phi_1^\varepsilon(t, p)$ as $t \in [t_0, t_1]$, we have $\Phi_1^\varepsilon(t_1 + 0, p) = I_1(\Phi_1^\varepsilon(t_1, p), p) + \Phi_1^\varepsilon(t_1, p) = \Phi_1^+ = \Phi_1^\varepsilon(t_1, p)$.

Now the above mentioned existence theorem applied to the equation (15) in the interval (t_1, t_2) ensures that there exists a solution $\Phi_2^\varepsilon(t,p)$ such that $\Phi_2^\varepsilon(t,p) = \Phi_1^\varepsilon(t,p)$ for $t_1 - \tau \le t \le t_1$ and $\Phi_2(t_1) = \Phi_1^+$. The solution $K^\varepsilon(p)$ of problem (14) can be extended to the moment $t = t_2$ by setting $K^\varepsilon(p) = \Phi_2^\varepsilon(t,p)$ for $t_1 < t \le t_2$.

In the same way, let us denote by $\Phi_k^\varepsilon(t,p)$ the solutions of the equation (15) in the intervals $(t_{k-1}, t_k]$, $k = 1, 2, ...$ respectively. Then for $t = t_k$ we have

$$\Phi_k^\varepsilon(t_k + 0, p) = I_k(\Phi_k^\varepsilon(t_k, p), p) + \Phi_k^\varepsilon(t_k, p) = \Phi_k^+ = \Phi_k^\varepsilon(t_k, p).$$

It follows from the existence theorem for problem (15) on the interval $(t_k, t_{k+1}]$ that there exists a solution $\Phi_{k+1}^\varepsilon(t,p)$ such that $\Phi_{k+1}^\varepsilon(t,p) = \Phi_k^\varepsilon(t,p)$ for $t_k - \tau \le t \le t_k$ and $\Phi_{k+1}(t_1) = \Phi_k^+$. Thus the solution $K^\varepsilon(p)$ of problem (14) can be extended to the moment t_{k+1}, $k = 2, 3, ...$, by setting $K^\varepsilon(p) = \Phi_{k+1}^\varepsilon(t,p)$ for $t_k < t \le t_{k+1}$.

Since the solution $K^\varepsilon(p)$ is defined on each of the intervals $(t_k, t_{k+1}]$, $k =$ 1, 2, ... , then from conditions H4 and H5 we conclude that it is continuable for $t \ge t_0$. □

Theorem 5.2. *Let the following conditions hold:*

1. Conditions H1-H5 hold.

2. The functions $V \in V_0$ and $a,\ b \in \mathcal{K}$ are such that for $p \in N(p^*)$

$$a(|K - K^\varepsilon(p)|) \le V(t, K) \le b(|K - K^\varepsilon(p)|),\ (t, K) \in [t_0, \infty) \times R, \tag{16}$$

$$V(t + 0, K(t) + I_k(K(t), p)) \le V(t, K(t)),\ t = t_k,\ k = 1, 2, ... \tag{17}$$

and the inequality

$$D^+V(t, K(t)) \le 0,\ t \ne t_k,\ k = 1, 2, ...$$

is valid for $t \in [t_0, \infty),\ K \in \Omega_1$.

Then the system (10) is parametrically uniformly stable at p^*.

Proof. Let $\varepsilon > 0$ be chosen. Chose $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

Let $K_0 \in PCB[[-\tau,\ 0], R] :\ ||K_0 - K^\varepsilon(p)|| < \delta$. Let $K(t; t_0, K_0, p)$ be the solution of the problem (10), (11). By Lemma 4.1 we have $J^+(t_0, K_0, p) = [t_0, \infty)$.

Since the conditions of Lemma 4.2 are met,

$$V(t, K(t; t_0, K_0, p)) \le V(t_0 + 0, K_0(0)),\ t \in [t_0, \infty). \tag{18}$$

From the condition 2 of Theorem 5.2 and from (18) there follow the inequalities

$$a(|K(t; t_0, K_0, p) - K^\varepsilon(p)|) \le V(t, K(t; t_0, K_0, p)) \le V(t_0 + 0, K_0(0))$$

$$\leq b(|K_0(0) - K^{\varepsilon}(p)|) \leq b(||K_0 - K^{\varepsilon}(p)||) < b(\delta) < a(\varepsilon),$$

which imply that $|K(t; t_0, K_0, p) - K^{\varepsilon}(p)| < \varepsilon$ for $t \geq t_0$. This implies that the system (10) is parametrically uniformly stable at p^*. □

Theorem 5.3. *Let the following conditions hold:*

1. Conditions H1-H5 hold.

2. The functions $V \in V_0$, $a, b \in \mathcal{K}$ are such that (16), (17) are satisfied, and the inequality

$$D^+V(t, K(t)) \leq -c(|K(t) - K^{\varepsilon}(p)|),\ t \neq t_k,\ k = 1, 2, ... \tag{19}$$

is valid for $c \in \mathcal{K}$, $p \in N(p^*)$, and $K \in \Omega_1$.

Then the system (10) is parametrically uniformly asymptotically stable at p^*.

Proof. 1. Let $\alpha = const > 0$ be given, then $\{K \in R :\ |K - K^{\varepsilon}(p)| \leq \alpha\} \subset R$. For any $t \geq t_0$ denote

$$V^{-1}_{t,\alpha} = \{K \in R :\ V(t + 0, K) \leq a(\alpha)\}.$$

From (16) we deduce

$$V^{-1}_{t,\alpha} \subset \{K \in R : |K - K^{\varepsilon}(p)| \leq \alpha\}.$$

From condition 2 of Theorem 5.3 it follows that for any function $K_0 \in PCB[[-\tau,\ 0], R] :\ K_0(0) \in V^{-1}_{t_0,\alpha}$ we have $x(t; t_0, K_0, p) \in V^{-1}_{t,\alpha}$, $t > t_0$.

Let $\varepsilon > 0$ be chosen. Chose $\eta = \eta(\varepsilon)$ so that $b(\eta) < a(\varepsilon)$ and let $T > \frac{b(\alpha)}{c(\eta)}$.

If we assume that for each $t \in [t_0, t_0 + T]$ the inequality $|K(t; t_0, \varphi_0, p) - K^{\varepsilon}(p)| \geq \eta$ is valid, then from (17) and (19) we deduce the inequalities

$$V(t_0 + T, K(t_0 + T; t_0, K_0, p)) \leq V(t_0 + 0, K_0(0)) - \int_{t_0}^{t_0+T} c(|K(s; t_0, K_0, p) - K^{\varepsilon}(p)|)ds \leq b(\alpha) - c(\eta)T < 0,$$

which contradicts (16). The contradiction obtained shows that there exists $t^* \in [t_0, t_0 + T]$ such that $|K(t^*; t_0, K_0, p) - K^{\varepsilon}(p)| < \eta$. Then for $t \geq t^*$ (hence for any $t \geq t_0 + T$) the following inequalities hold:

$$a(|K(t; t_0, K_0, p) - K^{\varepsilon}(p)|) \leq V(t; K(t; t_0, K_0, p)) \leq V(t^*, K(t^*; t_0, K_0, p)) \leq b(|K(t^*; t_0, K_0, p) - K^{\varepsilon}(p)|) < b(\eta) < a(\varepsilon).$$

Therefore $|K(t; t_0, K_0, p) - K^{\varepsilon}(p)| < \varepsilon$ for $t \geq t_0 + T$.

2. Let $\mu = const > 0$ be such that $b(\mu) \leq a(\alpha)$. Then if

$$K_0 \in PCB[[-\tau, 0], R] : ||K_0 - K^{\varepsilon}(p)|| < \mu,$$

(16) implies

$$V(t_0+0, K_0(0)) \leq b(|K_0(0) - K^{\varepsilon}(p)|) \leq b(||K_0 - x^{\varepsilon}(p)||) < b(\mu) \leq a(\alpha),$$

which shows that for $K_0 \in PCB[[-\tau,\ 0], R] :\ K_0(0) \in V^{-1}_{t_0,\alpha}$. From what we proved in item 1 it follows that the system (10) satisfies (ii) from the Definition 4.3, and since Theorem 5.2 implies that it is parametrically uniformly stable, then the system (10) is parametrically uniformly asymptotically stable. □

6. Conclusions

This paper investigates an extension of the initial Solow equation to a general impulsive Solow differential equation with a delay function to be applied in modern growth theory. The solution of the model is parametrically and asymptotically stable.

As an application, an analysis of the presence of random walk parts in the actual four German time series of capital stock, labour force, capital intensity and GDP, in levels or logged, along the lines of the work of Nelson and Plosser (1982) and of Cochrane (1988) is performed. It can be shown that these *differenced* four German macroeconomic time series (in levels or logged) exhibit a *random walk structure*. For this reason the existence of jumps at some moments cannot be excluded, because the time series (in levels or logged) exhibit at least at some critical moments the jumps present in the differences at those moments. Taking into account the developments of the Solow model for instance along the lines of Boucekkine, Licandro and Paul [7], the German capital intensity is identified as an appropriate candidate to be modelled by an *impulsive Solow equation* with *a delay function*, giving the possibility to investigate more accurately its dynamics, even with threshold effects[7].

References

[1] Ph. Aghion, St.N. Durlauf, *Handbook of Economic Growth*, Volume 1A, 1B, Elsevier B.V. (2005).

[2] D.D. Bainov, P.S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood, Chichester (1989).

[7] The magnitude $I_k(K(t_k), p)$ of the impulsive perturbations due to which the capital stock K changes from the positions $K(t_k)$ into the position $K(t_k+0)$ depends on the state $K(t_k)$ and on parameter p. We find efficient conditions under which small variations of the capital and of the parameter p do not alter very much the economic growth process.

[3] D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations: Periodic Solutions and Applications*, Longman, Harlow (1993).

[4] J. Benhabib, A. Rustichini, Vintage capital, investement and growth, *Journal of Economic Theory*, **55** (1991), 323-339.

[5] St. Beveridge, Ch. R. Nelson A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the business cycle, *Journal Monetary Economy*, **7** (March 1981), 151-174.

[6] W.A. Brock, M.S. Taylor, Economic growth and the environment: Review of theory and empirics, In: *Handbook of Economic Growth*, Chapter 28 (Ed-s: Ph. Aghion, St. N. Durlauf), Volumes 1A, 1B, Elsevier B.V. (2005).

[7] R. Boucekkine, O. Licandro, Ch. Paul, Differential-difference equations in economics: On the numerical solution of vintage capital growth models, Preprint submitted to Elsevier Science (April 1996).

[8] J.H. Cochrane, How big is the random walk in GNP? *Journal of Political Economy* (1998), 893-920.

[9] F.X. Diebold, Random walks versus fractional integration: Power comparision of scalar and joint tests of the variance-time function, *Advances in Econometrics and Modeling* (Ed. Baldev Raj), Needham, Mass., Kluwer Academic Publishers (1989), 29-45.

[10] T.A. Doan, *RATS* software package, user's manual, Version 6.3, Estima, Evanston, Illinois (2007).

[11] J.-F. Emmenegger, I.M. Stamova, Shocks to capital intensity make the Solow equation an impulsive differential equation, *International Journal of Differential Equations and Applications*, **6** (2002), 93-110.

[12] K. Gopalsamy, B.G. Zhang, On delay differential equation with impulses, *J. Math. Anal. Appl.*, **139** (1989), 110-122.

[13] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York-Heidelberg-Berlin (1977).

[14] J.K. Hale, V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag (1993).

[15] V.B. Kolmanovskii, V.R. Nosov, *Stability of Functional Differential Equations*, Academic Press (1986).

[16] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore-New Jersey-London (1989).

[17] Z. Luo, J. Shen, Stability results for impulsive functional differential equations with infinite delays, *Journal of Computational and Applied Mathematics*, **131** (2001), 55-64.

[18] G.S. Maddala, In-Moo Kim, *Unit Roots, Cointegration, and Structural Change*, Cambridge University Press, Series: Themes in Modern Econometrics (1998).

[19] J.Y. Campbell, N.G. Mankiw, Are output fluctuations transitory? *Quarterly Journal of Economics*, **102** (November 1987), 151-174.

[20] Ch.R. Nelson, Ch.I. Plosser, Trends and random walks in macroeconomic time series: Some evidence and implications, *Journal of Monetary Economics*, **10** (1982), 139-162.

[21] B. Snowdon, V. Howard, P. Wynarczyk, *A Modern Guide to Macroeconomics: An Introduction to Competing Schools in Thought*, Edward Elgar Publishing Company (1994).

[22] W. Poole, Speculative prices as random walks: An analysis of ten time series of flexible exchange rates, *Southern Economic Journal*, **33** (April 1967), 468-478.

[23] B.S. Razumikhin, *Stability of Hereditary Systems*, Nauka, Mosow (1988), In Russian.

[24] D. Romer, *Advanced Macroeconomics*, McGraw-Hill, New York (1996).

[25] D.D. Siljak, M. Ikeda, Y. Ohta, Parametric stability, In: *Proceedings of the Universita di Genova-Ohio State University Joint Conference*, Boston, MA, Birkhäuser (1991), 1-20.

[26] J. Shen, J. Yan, Razumikhin type stability theorems for impulsive functional differential equations, *Nonlinear Anal.*, **33** (1998), 519-531.

[27] D.J. de Solla Price, J. Derek, *Networks of Scientific Papers Science*, New Series, Volume 149, No. 3683 (July 30, 1965), 510-515.

[28] R.M. Solow, A contribution to the theory of economic growth, *Quarterly Journal of Economics*, **70**, 1 (February 1956), 65-94.

[29] R.M. Solow, Technical change and the aggregate production function, *Review of Economics and Statistics*, **39** (August 1957), 312-320.

[30] R. Solow, T. Tobin, C. von Weizacker, M. Yaari, Neoclassical growth with fixed factor propostions, *Review of Economic Studies*, **33** (1966), 79-115.

[31] I.M. Stamova, Vector Lyapunov functions for practical stability of nonlinear impulsive functional differential equations, *Journal of Mathematical Analysis and Applications*, **325** (2007), 612-623.

[32] I.M. Stamova, G.T. Stamov, Lyapunov-Razumikhin method for impulsive functional differential equations and applications to the population dynamics, *Journal of Computational and Applied Mathematics*, **130** (2001), 163-171.

[33] M. Szydlowski, A. Krawiec, Scientific cycle model with delay, Jointly Published by Akademiai Kiado, Budapest and Kluwer Academic Publishers, Dordrecht, *Scientometrics*, **52**, No. 1 (2001), 83-95.

[34] H. Working, *The Investigation of Economic Expectations*, **39** (May 1949), 150-166.

[35] W.E. Young, Random walk of stock prices: A test of the variance-time function, *Econometrica*, **39**, No. 5 (September 1971).