A DELAYED RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH STAGE-STRUCTURED AND IMPULSIVE STOCKING ON PREDATOR

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Abstract: In this paper, a delayed ratio-dependent Holling-III predator-prey system with stage-structured and impulsive stocking on predator and continuous harvesting on prey is considered. We obtain sufficient conditions of the global attractivity of prey-extinction periodic solution and the permanence of the system. These results show that the behavior of impulsive stocking on predator plays an important role for the permanence of the system. We also prove that all solutions of the system are uniformly ultimately bounded. Our results show that the biological resource management is effective and reliable.

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1. Introduction

Traditional Lotka-Volterra type predator-prey model has received great attention from both theoretical and mathematical biologists, and has been well studied [1], [17], [9], [7]. The basic predator-prey model is

\[
\begin{cases}
    x'(t) = x(t)(r - ax(t) - by(t)), \\
    y'(t) = y(t)(-d + cx(t)),
\end{cases}
\] (1.1)

where \(x(t)\) and \(y(t)\) represent the densities of the prey and the predator at time.
Recently, the traditional predator-prey model has been challenged by several biologists (see, for example [3], [4]). There is growing explicit biological and physiological evidence that in some situations, especially when predator have search for food, a more suitable general predator-prey model should be based on the ratio-dependent theory. This roughly states that the per capita predator growth rate should be a function of the ratio of prey to predator abundance. This is strongly supported by numerous field and laboratory experiments and observations [2], [3], [4], [10].

Based on the Michaelis-Menten or Holling type-II function, Arditi and Ginzburg [2] proposed a ratio-dependent function of the form:

\[ P\left(\frac{x}{y}\right) = \frac{c(x/y)}{m + (x/y)} = \frac{cx}{my + x}, \]

and the following ratio-dependent predator-prey model:

\[
\begin{aligned}
\dot{x} &= x(a - bx) - \frac{cxy}{my + x}, \\
\dot{y} &= y(-d + \frac{fx}{my + x}).
\end{aligned}
\]  

(1.2)

Here \(x(t)\) and \(y(t)\) represent the densities of the prey and the predator at time \(t\), respectively. \(a/b\) is the carrying capacity, \(d > 0\) is the death rate of the predator, and \(a, c, m\) and \(f/c\) are positive constants that stand for the intrinsic growth rate of the prey, capturing rate, half saturation constant and conversion rate of the predator, respectively. Based on the form, we obtain a ratio-dependent function with the Holling type III functional response of the form:

\[ P\left(\frac{x}{y}\right) = \frac{c(x/y)^2}{m + (x/y)^2} = \frac{cx^2}{my^2 + x^2}. \]

Stage-structured models have received much attention in recent years (see [1], [15], [18], [20]). In [1], a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model was derived and investigated. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immature and a reduced survival of immature to their maturity. The model takes the form

\[
\begin{aligned}
\dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma \tau} x_m(t - \tau), \\
\dot{x}_m(t) &= \alpha e^{-\gamma \tau} x_m(t - \tau) - \beta x_m^2(t), \quad t > \tau.
\end{aligned}
\]  

(1.3)
where \( x_i(t) \) represents the immature population density, \( x_m(t) \) denotes the mature population density, \( \alpha > 0 \) represents the birth rate, \( \gamma > 0 \) is the immature death rate, \( \beta > 0 \) is the mature death and overcrowding rate, \( \tau \) is the time to maturity. The term \( \alpha e^{-\gamma \tau} x_m(t-\tau) \) represents the immature who were born at time \( t-\tau \) and survive at time \( t \) (with the immature death rate \( \gamma \)), and therefore represents the transformation of immature to mature.

However, if \( rc - ad < 0 \), the basic predator-prey model (1.1) do not have any positive equilibrium point, and the only boundary equilibrium point \( (\frac{r}{a}, 0) \) is globally asymptotically stable, which implies that the predator population will go extinction. If the prey is stocked at constant rate, then system (1.1) can be the following differential equation,

\[
\begin{align*}
\frac{dx}{dt} &= x(t)(r - ax(t) - by(t)) + \mu, \\
\frac{dy}{dt} &= y(t)(-d + cx(t)).
\end{align*}
\]

It can be easily derived that if \( \mu > \frac{d(ad-rc)}{c} \), the above system has a unique globally asymptotically stable positive equilibrium \((\frac{r}{c}, \frac{rdc-ad^2+\mu c}{bcd})\). This implies that the behavior of stocking prey keeps the above system permanent.

However, in the real life, it is impossible to release prey for the constant release, or release a lot of prey, which will cause great waste, and may undermine the balance of the ecosystem. Therefore, we should put reasonable prey by the pulse periodically release prey, which is in line with reality.

The impulsive equation has become a widely concerned subject in recent years, particularly applied to biological impulsive equation model more appropriate for the actual reality. There are a few papers devoted to impulsive delay equations (for example [12], [11], [16], [21]), while equations with Holling-III functional response and ratio-dependent delayed impulsive equations have not been studied until now.

In the paper, we consider the following delayed ratio-dependent predator-prey system with stage structure and impulsive effect:
show the effect of continuous harvesting prey, the stocking amount of the predator at the period of the impulsive stocking of the predator.

\[ \begin{aligned}
\frac{dx_1(t)}{dt} &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma \tau} x_2(t - \tau), \\
\frac{dx_2(t)}{dt} &= \alpha e^{-\gamma \tau} x_2(t - \tau) - \beta x_2^3(t) - \frac{a_1 x_2^2(t) x_3(t)}{x_3^2(t) + cx_2^2(t)} - Ex_2(t), \\
\frac{dx_3(t)}{dt} &= x_3(t)(-r_1 + \frac{a_2 x_2^2(t)}{x_3^2(t) + cx_2^2(t)} - bx_3(t)). \\
\end{aligned} \tag{1.4}
\]

\[ \{ \begin{aligned}
\Delta x_1(t) &= 0, \\
\Delta x_2(t) &= 0, \\
\Delta x_3(t) &= \mu. \\
\end{aligned} \quad t = nT, \]

\[ (\varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta)) \in \mathbf{C}_+ = \mathbf{C}([-\tau_1, 0], \mathbf{R}^3_+), \varphi_i(0) > 0, \quad i = 1, 2, 3, \]

where \( x_1(t), x_2(t) \) represent the immature and mature prey (pest) densities, respectively. \( x_3(t) \) represents the density of predator (nature enemy). \( \tau \) denotes a constant time from immaturity to maturity. The model is derived under the following assumptions: The birth rate of the immature population is proportional to the existing mature population with a proportionality \( \alpha > 0 \); the death rate of the immature population is proportional to the existing immature population with a proportionality \( \gamma > 0 \); The death rate of the mature population is a logistic nature, i.e., it is proportional to square of the population with a proportionality \( \beta > 0 \); it also is assumed in (1.4) that immature individual predators do not feed on prey and do not have the ability to reproduce. It seems reasonable that a number of mammals, whose immature predators are raised by their parents; the rate of the prey being attacked and reproductive rate can be ignored. The coefficients \( a_1 > 0, r_1 > 0, a_2 > 0, b > 0, c > 0 \) and \( 0 < E < 1 \) show the effect of continuous harvesting prey, \( \Delta x_i(t) = x_i(t^+) - x_i(t), \mu \geq 0 \) is the stocking amount of the predator at \( t = nT, n \in \mathbf{Z}_+ \) and \( \mathbf{Z}_+ = 1, 2, \cdots, T \) is the period of the impulsive stocking of the predator.

The organization of this paper is as follows: Firstly, we give some notation and lemmas; Secondly we show that the system (1.4) has a pest-extinction periodic solution and it is globally attractive; Finally, we will proof the system (1.4) is permanent.

Because the second and third equations of (1.4) do not contain \( x_1(t) \), we can simplify system (1.4) and pay attention to the following subsystem:

\[ \begin{aligned}
\frac{dx_2(t)}{dt} &= \alpha e^{-\gamma \tau} x_2(t - \tau) - \beta x_2^3(t) - \frac{a_1 x_2^2(t) x_3(t)}{x_3^2(t) + cx_2^2(t)} - Ex_2(t), \\
\frac{dx_3(t)}{dt} &= x_3(t)(-r_1 + \frac{a_2 x_2^2(t)}{x_3^2(t) + cx_2^2(t)} - bx_3(t)), \\
\Delta x_2(t) &= 0, \\
\Delta x_3(t) &= \mu. \\
\end{aligned} \tag{1.5}
\]

\[ t = nT, \]
The initial conditions for (1.5) are
\[(\varphi_2(\zeta), \varphi_3(\zeta)) \in \mathbb{C}_+ = \mathbb{C}([−\tau, 0], \mathbb{R}_+^2), \quad \varphi_i(0) > 0, \quad i = 2, 3. \quad (1.6)\]

2. Notation and Lemmas

Let \(x(t) = (x_1(t), x_2(t), x_3(t))^T\) denote the solution of (1.4). It is a piecewise continuous function \(x : \mathbb{R}_+ \rightarrow \mathbb{R}_+^3\). More precisely: \(x(t)\) is continuous on \((nT, (n + 1)T], n \in \mathbb{Z}_+\) and the limits \(x(nT^+) = \lim_{t \to nT^+} x(t)\) exists. Obviously the global existence and uniqueness of the solutions of (1.4) is guaranteed by the smoothness properties of \(f\), which denotes the mapping defined by right-hand side of system (1.4) (see [5], [14]). For continuity of the initial conditions, we require
\[\varphi_1(0) = \int_{-\tau}^{0} \alpha e^{\gamma s} \varphi_2(s) ds. \quad (2.1)\]

Before we have main results, we need to give some lemmas which will be used in the next.

**Lemma 2.1.** Let \((\varphi_1(t), \varphi_2(t), \varphi_3(t)) > 0\) for \(-\tau < t < 0\). Then any solution of the system (1.4) is strictly positive.

**Proof.** Firstly, we show that \(x_2(t) > 0\) for all \(t > 0\). Notice \(x_2(t) > 0\), hence if there exists \(t_0\) such that \(x_2(t_0) = 0\), then \(t_0 > 0\). Assume that \(t_0\) is the first time such that \(x_2(t) = 0\), that is \(t_0 = \inf\{t > 0 | x_2(t) = 0\}\), then \(x_2'(t_0) = \alpha e^{-\gamma \tau} x_2(t_0 - \tau) > 0\). Hence for sufficiently small \(\epsilon > 0\), \(x_2'(t_0 - \epsilon) > 0\).

By the definition of \(t_0\), \(x_2'(t_0 - \epsilon) \leq 0\), the contradiction shows that \(x_2(t) > 0\) for all \(t > 0\).

By uniqueness of the solutions of the system (1.4) and \(x_3'(t) = 0\) whenever \(x_3(t) = 0, t \neq nT\), and \(x_3(nT^+) = x_3(nT) + \mu, \mu \geq 0\). It is easy to see that \(x_3(t) > 0\) for all \(t > 0\).

We now consider the following equation
\[y'(t) = -\alpha e^{-\gamma \tau} x_2(t - \tau) - \gamma y(t), \quad (2.2)\]
and comparing with (1.4), we note that if \(y(t)\) is the solution of (2.2) and if \(x_1(t)\) can solve (1.4), then \(x_1(t) > y(t)\) on \(0 < t < \tau\). Solving (2.2) gives
\[y(t) = e^{-\gamma t}(x_1(0) - \int_{0}^{t} \alpha e^{\alpha(u-\tau)} x_2(u - \tau) du).\]
From (2.1) we can obtain
\[ y(\tau) = e^{-\gamma \tau} \left( \int_{-\tau}^{0} \alpha e^{\alpha \tau} \varphi_2(s) \, ds - \int_{0}^{\tau} \alpha e^{\alpha(u-\tau)} x_2(u-\tau) \, du \right). \]

By transformation \( x_2(t) \to \varphi_2(t) \) for \( t \in [-\tau, 0] \), we know that \( \int_{-\tau}^{0} \alpha e^{\alpha u} \varphi_2(s) \, ds = \int_{0}^{\tau} \alpha e^{\alpha(u-\tau)} x_2(u-\tau) \, du \). So we obtain \( y(\tau) = 0 \). Hence \( x_1(t) > 0 \), since \( y(t) \) is strictly decreasing, then \( x_1(t) > y(t) > 0 \) for \( t \in (0, \tau) \). So \( x_1(t) > 0 \) on \( 0 \leq t \leq \tau \). By induction and similar method to the proof of Theorem 1 of [6], we can show that \( x_1(t) > 0 \) for all \( t \geq 0 \). This completes the proof. 

**Lemma 2.2.** (see [14]) Let the function \( m(t) \in PC'[R_+, R] \) satisfies the inequalities
\[
\begin{aligned}
\{ \quad & m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k, \quad k = 1, 2, \ldots, \\
& m(t_k) \leq d_km(t_k) + b_k, \quad t = t_k, \quad t \geq t_0,
\end{aligned}
\]
where \( p, q \in PC[R_+, R] \) and \( d_k \geq 0, b_k \) are constants, then
\[
m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left( \int_{t_0}^{t} p(s) \, ds \right) + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} d_j \exp\left( \int_{t_0}^{t_j} p(s) \, ds \right) b_k \\
+ \int_{t_0}^{t} \prod_{t < t_k < t} d_k \exp(p(\sigma) q(s)) \, ds, \quad t \geq t_0.
\]

Next, we show that all solutions of (1.4) are uniformly ultimately bounded.

**Lemma 2.3.** There exists a constant \( M > 0 \) such that \( x_1(t) \leq M, x_2(t) \leq M, x_3(t) \leq \frac{a_2M}{\alpha_1} \) for each solution \( (x_1(t), x_2(t), x_3(t)) \) of (1.4) with all \( t \) large enough.

**Proof.** Define \( V(t) = x_1(t) + x_2(t) + \frac{a_1}{a_2} x_3(t) \), and then \( t \neq nT \) we have
\[
D^+V(t) + \gamma V(t)
= (\alpha + \gamma - E)x_2(t) - \beta x_2^2(t) + \frac{a_1}{a_2}(\gamma - r_1)x_3(t) - \frac{a_1}{a_2} bx_3^2(t) \leq M_0,
\]
where \( M_0 = \left( \frac{(a+\gamma-E)^2}{4\beta} + \frac{a_1(a_1-\gamma)}{4a_2 \beta} \right) \). When \( t = nT \), \( V(nT^+) = V(nT) + \mu \), by Lemma 2.2, for \( t \in (nT, (n+1)T] \) we have
\[
V(t) \leq V(0) \exp(-\gamma t) + \int_{0}^{t} M_0 \exp(-\gamma (t-s)) \, ds + \sum_{0<nT<t} \mu \exp(-\gamma (t-nT))
= V(0) \exp(-\gamma t) + \frac{M_0}{\gamma} (1 - \exp(-\gamma t))
+ \frac{\exp(-\gamma (t-T)) - \exp(-\gamma (t-(n+1)T))}{1 - \exp(\gamma T)}
\]
\begin{align*}
< V(0) \exp(-\gamma t) + \frac{M_0}{\gamma} (1 - \exp(-\gamma t)) + \frac{\mu \exp(-\gamma (t - T))}{1 - \exp(\gamma T)} + \frac{\mu \exp(\gamma T)}{\exp(\gamma T) - 1} \\
\rightarrow \frac{M_0}{\gamma} + \frac{\mu \exp(\gamma T)}{\exp(\gamma T) - 1}, \quad \text{as } t \to \infty.
\end{align*}

So \( V(t) \) is uniformly ultimately bounded. Hence, by he definition of \( V(t) \), we have there exists a constant \( M > 0 \) such that \( x_1(t) < M, x_2(t) < M, x_3(t) < \frac{a_2 M}{a_1} \) for \( t \) large enough. The proof is complete.

Consider the following delay equation

\[ x'(t) = a_1(t - \tau) - a_2 x(t). \quad (2.4) \]

We assume that \( a_1, a_2, \tau > 0; x(t) > 0 \) for \( -\tau \leq t \leq 0 \). The following result for system (2.5) can be easily obtained from Lemma 2.4.

**Lemma 2.4.** (see [13]) For system (2.4), assume that \( a_1 < a_2 \). Then

\[ \lim_{t \to \infty} x(t) = 0. \]

**Lemma 2.5.** Consider the following impulsive system

\[ \begin{cases}
    v'(t) = v(t)(-r_1 - bv(t)), & t \neq nT, \\
    v(nT^+) = v(nT) + \mu, & t = nT, \quad n = 1, 2, \ldots,
\end{cases} \quad (2.5) \]

where \( r_1 > 0, b > 0, \mu > 0 \). Then there exists a unique positive periodic solution of system (2.5)

\[ \hat{v}(t) = \frac{r_1 v^*}{-bv^* + (r_1 + bv^*) \exp(r_1(t - nT))}, \quad t \in (nT, (n + 1)T], \quad n \in \mathbb{Z}_+, \quad (2.6) \]

which is globally asymptotically stable, where

\[ v^* = \frac{(b\mu - r_1) + \sqrt{(b\mu - r_1)^2 + 4r_1b\mu \exp(r_1T)/(e^{r_1T} - 1)}}{2b}. \]

**Proof.** Integrating and solving the first equation of system (2.5) between pulses, we obtain \( v(t) = \frac{r_1 v_{nT}^*}{(r_1 + bv_{nT}) \exp(r_1(t - nT) - bv_{nT})}, \quad nT < t \leq (n + 1)T, \) where \( v_{nT} \) is the number of population immediately after the \( n \)-th pulse at time \( t = nT \). Using the second equation of system (2.5), we deduce the stroboscopic map such that \( v_{n+1} = \mu + \frac{r_1 v_n^*}{(r_1 + bv_n) \exp(r_1T) - bv_nT} \). The map has the unique positive fixed point: \( v^* = \frac{(b\mu - r_1) + \sqrt{(b\mu - r_1)^2 + 4r_1b\mu \exp(r_1T)/(e^{r_1T} - 1)}}{2b}. \) Set \( f(v) = \mu + \frac{r_1 v_n^*}{(r_1 + bv_n) \exp(r_1T) - bv_nT} \), then \( v^* \) satisfies \( v < f(v) < v^* \) if \( 0 < v < v^* \) and \( v^* < f(v) < v \) if \( v > v^* \). From Cull [7], we derive that \( v^* \) is globally asymptotically stable. It implies the corresponding periodic solution of system (2.5)

\[ \hat{v}(t) = \frac{r_1 v^*}{-bv^* + (r_1 + bv^*) \exp(r_1(t - nT))}, \quad t \in (nT, (n + 1)T], \quad n \in \mathbb{Z}_+, \]

is globally asymptotically stable. This completes proof of the lemma. \( \square \)
Lemma 2.6. (see [19]) Consider the following equation:
\[ \dot{x}(t) = ax(t - \tau) - bx(t) - cx^2(t), \]
where \(a, b, c\) and \(\tau\) are positive constants, \(x(t) > 0\) for \(t \in [-\tau, 0]\). We have
(i) if \(a > b\), then \(\lim_{t \to +\infty} x(t) = \frac{a - b}{c}\);  
(ii) if \(a < b\), then \(\lim_{t \to +\infty} x(t) = 0\).

According to the system (1.4), we easily know that there exists \(t_1 \in Z_+, t > t_1\) such that \(x_2(t - \tau) = 0\) and \(x_2(t) = 0\). Then by the following system
\[
\begin{cases}
    x_3'(t) = x_3(t)(-r_1 - bx_1(t)), & t \neq nT, \\
    x_3(nT^+) = x_3(nT) + \mu, & t = nT, \ n = 1, 2, \cdots,
\end{cases}
\]
and Lemma 2.5, we know that (1.4) has a pest-extinction periodic solution
\[
(0, 0, \tilde{x}_3(t)) = (0, 0, \frac{r_1x_3^*}{-bx_3^* + (r_1 + bx_3^*) \exp(r_1(t - nT))}),
\]
\(t \in (nT,(n+1)T], \ n \in Z_+\), (2.8)
or (1.5) has a pest-extinction periodic solution
\[
(0, \tilde{x}_3(t)) = (0, \frac{r_1x_3^*}{-bx_3^* + (r_1 + bx_3^*) \exp(r_1(t - nT))}),
\]
\(t \in (nT,(n+1)T], \ n \in Z_+\), (2.9)
which is globally asymptotically stable, where
\[
x_3^* = \frac{(b\mu - r_1) + \sqrt{(b\mu - r_1)^2 + 4r_1b\mu \exp(r_1T)/(e^{r_1T} - 1)}}{2b}.
\]

3. Globally Asymptotically Stable

In this section, we will study the stability of pest-extinction periodic solution of system (1.4).

Theorem 3.1. Let \((x_1(t), x_2(t), x_3(t))\) be any solution of system (1.4). If
\[
E > \alpha e^{-\gamma \tau}
\]
holds, the pest-extinction periodic solution \((0, 0, \tilde{x}_3(t))\) of system (1.4) is globally attractive.

Proof. It is clear that the global attraction of pest-extinction periodic solution \((0, 0, \tilde{x}_3(t))\) of system (1.4) is equivalent to the global attraction of pest-extinction periodic solution \((0, x_3(t))\) of system (1.5). So we only
devote to system (1.5).

By system (1.5) and Lemma 2.1, we know that
\[
\frac{dx_2(t)}{dt} \leq \alpha e^{-\gamma t} x_2(t) - E x_2(t) - \beta x_2^2(t). \tag{3.2}
\]

Consider the following comparison differential system
\[
\frac{dy(t)}{dt} = \alpha e^{-\gamma t} y(t) - E y(t) - \beta y^2(t), \quad t \in (nT, (n + 1)T]. \tag{3.3}
\]

Form (2.1), we have \(\alpha e^{-\gamma t} < E\). According to Lemma 2.6, we have \(\lim_{t \to \infty} y(t) = 0\).

Let \((x_2(t), x_3(t))\) be the solution of system (1.5) with initial conditions (1.6) and \(x_2(\zeta) = \varphi_2(\zeta)(\zeta \in [-\tau, 0])\), \(y(t)\) is the solution of system (2.3) with initial conditions \(y(\zeta) = \varphi_2(\zeta)(\zeta \in [-\tau, 0])\). By the comparison theorem, we have \(\lim_{t \to \infty} x_2(t) < \lim_{t \to \infty} y(t) = 0\). Incorporating into the positivity of \(x_2(t)\), we know that
\[
\lim_{t \to \infty} x_2(t) = 0. \tag{3.4}
\]

We now proof that \(x_3(t) \to \tilde{x}_3(t)\) as \(t \to \infty\). By Lemma 2.1 and Lemma 2.5, we may choose a constant \(m > 0\) such that \(x_3(t) > m\) as \(t\) large enough. For simplification we may assume \(x_3(t) > m\) holds for all \(t \geq 0\). By \(\lim_{t \to \infty} x_2(t) = 0\), there exist \(T_0 > 0\) and a sufficiently small \(\epsilon > 0\) such that \(0 < x_2(t) < \epsilon\) for \(t \geq T_0\). Without loss of generality, we assume \(0 < x_2(t) < \epsilon\) for all \(t \geq 0\), then from system (1.5) we have
\[
x_3(t)(-r_1 - bx_3(t)) \leq \frac{dx_3(t)}{dt} \leq x_3(t)(-r_1 - bx_3(t)) + \frac{e a_2 x_3(t)}{2m \sqrt{c}}, \tag{3.5}
\]

By Lemma 2.5 and comparison theorem of impulsive equation (Theorem 3.1.1 in [14]), we have \(z_1(t) \leq x_1(t) \leq z_2(t)\) and \(z_1(t) \to x_3(t), z_2(t) \to \tilde{x}_2(t)\) as \(t \to \infty\), where \(z_1(t)\) is solution of
\[
\begin{align*}
\frac{dz_1(t)}{dt} & = z_1(t)(-r_1 - bz_1(t)), \quad t \neq nT, \\
\Delta z_1(t) & = \mu, \quad t = nT, \\
z_1(0^+) & = x_3(0^+) \geq 0,
\end{align*} \tag{3.6}
\]

and \(z_2(t)\) is solution of
\[
\begin{align*}
\frac{dz_2(t)}{dt} & = z_2(t)(-r_1 + \frac{e a_2}{2m \sqrt{c}} - bz_2(t)), \quad t \neq nT, \\
\Delta z_2(t) & = \mu, \quad t = nT, \\
z_2(0^+) & = x_3(0^+) \geq 0.
\end{align*} \tag{3.7}
\]

For \(nT < t \leq (n + 1)T\),
\[ z_2(t) = \frac{(r_1 - \frac{ea_2}{2m\sqrt{c}})z_2^* - bz_2^* + ((r_1 - \frac{ea_2}{2m\sqrt{c}}) + bz_2^*)\exp((r_1 - \frac{ea_2}{2m\sqrt{c}})(t-nT) - \epsilon^a_2 \frac{m}{\sqrt{c}})}{t \in (nT, (n+1)T], \ n \in \mathbb{Z}_+} \]

where

\[ z_2^* = \frac{(b\mu - r_1 + \frac{ea_2}{2m\sqrt{c}}) + \sqrt{(b\mu - r_1 + \frac{ea_2}{2m\sqrt{c}})^2 + 4(r_1 - \frac{ea_2}{2m\sqrt{c}})b \mu \exp((r_1 - \frac{ea_2}{2m\sqrt{c}})T)}}{(\exp(r_1 - \frac{ea_2}{2m\sqrt{c}})T - 1) \epsilon^a_2 \frac{m}{\sqrt{c}}} \]

Therefore, for any \( \epsilon_1 > 0 \) there exists a \( T_1 > 0 \) such that

\[ x_3(t) - \epsilon_1 < x_3(t) < z_2(t) + \epsilon_1 \quad \text{for all } t > T_1. \]

Let \( \epsilon \to 0 \), we have

\[ x_3(t) - \epsilon_1 < x_1(t) < x_3(t) + \epsilon_1, \]

for \( t \) large enough, which implies \( x_3(t) \to x_3(t) \) as \( t \to \infty \). This completes the proof.

**4. The Permanence of the System (1.4)**

Now we investigate the permanence of the system (1.4). Before starting our theorem, we give the definition of permanence.

**Definition 4.1.** System (1.4) is said to be permanent if there exist constants \( M \geq m > 0 \) such that for all solutions \((x_1(t), x_2(t), x_3(t))\) with initial values \( x_1(0^+) > 0, x_2(0^+) > 0, x_3(0^+) > 0, m \leq x_1(t) \leq M, m \leq x_2(t) \leq M, m \leq x_3(t) \leq \frac{a_2M}{a_1} \) holds for all \( t \) sufficiently large.

From system (1.5), we have

\[ \frac{dx_3(t)}{dt} \geq x_3(t)(-r_1 - bx_3(t)). \]  

By Lemma 2.5 and comparison theorem of impulsive equation (Theorem 3.1.1 in [14]), we obtain

\[ \lim_{t \to \infty} x_3(t) \geq m_3, \]

where

\[ m_3 = \frac{(b\mu - r_1) + \sqrt{(b\mu - r_1)^2 + 4r_1b\mu \exp(r_1T)}/(e^{r_1T} - 1) - \epsilon}{2b} \]

for a sufficiently small \( \epsilon > 0 \). So we know that there exists a \( T_2 > 0 \) such that
Theorem 4.2. Suppose

\[ E < \alpha^{-\gamma \tau} - \frac{a_1}{2\sqrt{c}} \]  

(4.4)

holds, then there is a positive constant \( m_2 \) such that each positive solution \((x_2(t), x_3(t))\) of (1.5) satisfies \( x_2(t) \geq m_2 \).

Proof. From the second equation of system (1.5), we have

\[ \frac{dx_2(t)}{dt} \geq \alpha e^{-\gamma \tau} x_2(t - \tau) - Ex_2(t) - \frac{a_1 x_2(t)}{2\sqrt{c}} - \beta x_2^2(t). \]  

(4.5)

We consider the following auxiliary equation:

\[ \frac{du(t)}{dt} = \alpha e^{-\gamma \tau} u(t - \tau) - Eu(t) - \frac{a_1 u(t)}{2\sqrt{c}} - \beta u^2(t). \]

By Lemma 2.6 and (3.5) we derive

\[ \lim_{t \to \infty} u(t) = \alpha^{-\gamma \tau} - \frac{a_1}{2\sqrt{c}} - E, \]

where \( \alpha^{-\gamma \tau} - \frac{a_1}{2\sqrt{c}} - E < \frac{\alpha \beta}{2} \).

By comparison, there exist a \( T_3 > 0 \) and a positive constant \( m_2 < m_2^* \) such that \( x_2(t) > m_2 \) for \( t > T_3 + \tau \). This completes the proof of the theorem. \( \Box \)

Theorem 4.3. Suppose

\[ E < \alpha^{-\gamma \tau} - \frac{a_1}{2\sqrt{c}}, \]

holds, then the system (1.4) is permanent.

Proof. Denote \((x_1(t), x_2(t), x_3(t))\) be any solution of system (1.4). By (4.3), Theorem 4.2 and Lemma 2.3, we know \( x_1(t) \leq M, m_2 < x_2(t) \leq M \) and \( m_3 \leq x_3(t) \leq \frac{a_3 M}{a_1} \). Next, we only proof that there exist a positive constant \( m_1 \) and \( T_4 > 0 \) such that \( x_1(t) \geq m_1 \) for \( t > T_4 + \tau \).

From the first equation of system (1.4) and the above argument, we have

\[ \frac{dx_1(t)}{dt} \geq \alpha (m_2 - e^{-\gamma \tau} M) - \gamma x_1(t), \]  

(4.6)

where \( m_2 \) is defined in proof of Theorem 4.2, \( M \) is defined by Lemma 2.3. It is easy to obtain

\[ \lim_{t \to \infty} x_1(t) \geq m_1, \]  

(4.7)

where \( m_1 = \frac{\alpha (m_2 - e^{-\gamma \tau} M)}{\gamma} - \epsilon \). Let \( m = \min\{m_1, m_2, m_3\} \), by the above discussion, we have \( m \leq x_1(t) \leq M, m \leq x_2(t) \leq M \) and \( m \leq x_3(t) \leq \frac{a_3 M}{a_1} \). So system (1.4) is permanent. The proof of Theorem 4.3 is complete.
5. Discussion

According to the fact of biological resource management, in this paper, we consider a delayed ratio-dependent predator-prey system with stage-structured and impulsive effect. We obtain that the pest-extinction periodic solution of system (1.4) is globally attractive, and the sufficient condition for the permanent of system (1.4). By (2.3) and (4.4), we easily know that $\alpha e^{-\gamma \tau} > \alpha e^{-\gamma \tau} - \frac{\sigma}{2\sqrt{c}}$. So we can guess that there must exists a threshold $E^*$. If $E < E^*$, system (1.4) is permanent. If $E < E^*$, the pest-extinction periodic solution $(0,0,\tilde{x}_1(t))$ is globally attractive. The results show that the behavior of impulsive stocking on predator plays an important role for the permanent of system (1.4), that is, it can prevent the pest from dying out. We also can find that harvesting on pest affects the permanent of system (1.4). If harvesting on pest catches more than a certain number, i.e., $E > E^*$, the predator will go to extinct. Conversely, $E < E^*$, the system (1.4) is permanent. This meet in biological balance protection. However, there are many interesting questions we need further study, for example: Can we get more appropriate parameters than Theorem 3.1 and Theorem 4.2, i.e., more optimal harvesting policy? Under what circumstances the system (1.4) will have bifurcation and chaotic solutions? Whether periodic impulsive stocking or investing by the ratio on predator of system (1.4) does a better job than constant stocking on predator?

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References


