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BLOCKING SETS OF HIRZEBRUCH SURFACES OVER A FINITE FIELD

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study the blocking sets of Hirzebruch surfaces over $\mathbb{F}_q$, showing the huge differences with respect to the classical case of $PG(n, q)$.

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1. Introduction

Fix a prime power q and a set $S \subseteq PG(n, q)$. Contrary to the established rule (see [2], p. 366) we say that S is a *blocking set* if each hyperplane of $PG(n, q)$ intersects S; thus for us a set containing a line is a bloking sets. S is called a *minimal bloking set* if it is a blocking set and no proper subset of S is a bloking set. It is obvious that lines are minimal bloking sets and that every bloking set has at least cardinality $q+1$ (use a pencil of hyperplanes). With our conventions any subset of $PG(n, q)$ containing a blocking set is a blocking set.

For any algebraic set $A \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ let $A(q)$ be the set of its $\mathbb{F}_q$-points. With this convention $\mathbb{P}^n(q) = PG(n, q)$.

Now we introduce our players (the Hirzebruch surfaces). Since we work over a finite field we need to distinguish the case $e > 0$ and the case $e = 0$. In Example 1 we first state both cases over an algebraically closed base field and at the end study the case $e > 0$ over $\mathbb{F}_q$.

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Remark 1. Let F_e, $e \ge 0$, the Hirzebruch surface with invariant e, i.e. such that $-e$ is the minimal self-intersection of a section of the ruling of F_e. We have $\mathrm{Pic}(F_e) \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_e)$ a section h of f with $h^2 = -e$ and a class, f, of the ruling u. Thus $h \cdot f = 1$ and $f^2 = 0$. We will use both the additive and the multiplicative notation for line bundles and divisors on F_e. We have $\omega_{F_e} \cong -2h - (2+e)f$. We have $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a < 0$. By the projection formula we have $f_*(\mathcal{O}_{F_e}(ah+bf)) \cong \oplus_{i=0}^{a} \mathcal{O}_{\mathbf{P}^1}(b - ie)$ for every $a \ge 0$. Thus $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \ge 0$ and $b < ea$, $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = \sum_{i=1}^{a}(b - ie + 1) = (2b + 2 - ae)(a+1)/2$ if $a \ge 0$ and $b \ge 0$ and $h^1(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \ge 0$ and $b \ge ae - 1$. We write $H_{b,a} := ah + bf$ and $H_b := H_{b,1}$. The line bundle $H_{b,a}$ is ample if and only if it is very ample if and only if $a > 0$ and $b > ae$. The ruling of F_e is unique and hence defined over any field on which our surface is defined if and only if $e > 0$. Now assume that we work over $\mathbb{F}_q$ and that we have a surface W defined over $\mathbb{F}_q$ and which is isomorphic to F_e over $\overline{\mathbb{F}}_q$. If $e > 0$, then $W \cong F_e$ over $\mathbb{F}_q$ and hence we may apply the previous discussion of $\mathrm{Pic}(F_e)$ to W, i.e. each $H_{b,a}$ is defined over $\mathbb{F}_q$. In Examples 1 and 2 we fix the notation for the two cases (hyperbolic quadric surface and elliptic quadric surface) which may arise when $e = 0$. We remark here that in both cases the embedding $W \hookrightarrow F_e$ is defined over $\mathbb{F}_q$ because it is the anticanonical embedding. For arbitrary e the very ample line bundle H_b, $b > e$, embedds F_e as a minimal degree smooth surface $X_{e,b} \subset \mathbb{P}^{2b-e+1}$ (i.e. $\deg(X_{e,b}) = 2b - e$) in which the fibers of the ruling of F_e (or of the ruling of type $(1,0)$ are embedded as lines. If $b = e+1$ and $e > 0$, then $X_{e,b}$ contains exactly another line (the image of the minimal section h of the ruling). If $b \ge e+2$, then $X_{e,b}$ contains no line. For any $a \ge 1$ and any $b > ae$ let $X_{e,b,a} \subset \mathbb{P}^N$, $N := (a+1)(b+1) - ea(a+1)/2 - 1$, denote the embedding induced by the complete linear system $|H_{b,a}|$. If $a \ge 2$, then $X_{e,b,a}$ contains a line if and only if $e = 0$ and $b = 1$ (notice that $X_{0,1,1}$ is a hyperbolic quadric) or $e > 0$ and $b = ae + 1$ (in this case the line is unique and it is the image of the section h of the ruling. If $e > 0$ we will often identify h with its image in $X_{e,b,a}$.

Example 1. Let $X \subset \mathbb{P}^3$ be a smooth and hyperbolic quadric surface defined over $\mathbb{F}_q$. Hence $\sharp(X(q)) = (q+1)^2$ and X has 2 ruling defined over $\mathbb{F}_q$ (see [2]). Every effective divisor $A \subset X$ defined over $\mathbb{F}_q$ has a bidegree (a,b) and the lines of X are the curves of bidegree $(1,0)$ or $(0,1)$. Conversely, for all $(a,b) \in \mathbb{N}^{\oplus 2} \backslash \{(0,0)\}$ there are curves $A \subset X$ defined over $\mathbb{F}_q$ has a bidegree (a,b). There are even curves with no singular point over $\overline{\mathbb{F}}_q$. If $ab \neq 0$ there are even curves A as above which are integral and with no singular point over $\overline{\mathbb{F}}_q$, i.e. curves which are smooth and geometrically integral. Any curve of type

$(a, 0)$, $a > 0$, is the union of a lines of type $(1, 0)$ (although some line may appear with multiplicity > 1. Let π_1 and π_2 be the two rulings of X, with the convention that the fibers of π_1 (resp. π_2) are the lines of type $(0, 1)$ (resp. $(1, 0)$). In this case each $H_{b,a}$ and each H_b is defined over $\mathbb{F}_q$.

Example 2. Let $Y \subset \mathbb{P}^3$ be a smooth and elliptic quadric surface defined over $\mathbb{F}_q$. Hence $\sharp(X(q)) = q^2 + 1$ and Y contains no line (see [2]). Every curve A of Y defined over $\mathbb{F}_q$ is also a divisor of Y defined over $\overline{\mathbb{F}}_q$. Over $\overline{\mathbb{F}}_q$ it has a bidegree (a, b). Since A has degree $a + b$ as a space curve, the ellipticity of Y implies $b = a$. Hence in this case we only consider curve of type (a, a) (i.e. space curves of degree $2a$ contained in Y and defined over $\mathbb{F}_q$). Here only $H_{a,a}$ is defined over $\mathbb{F}_q$.

Remark 2. Assume either $e > 0$ or that we are looking at a surface F_0 as in Example 1, i.e. in which the two rulings are defined over $\mathbb{F}_q$. Fix an integer $b > e$ and use $|H_b|(q)$ to define the blocking sets of $F_e(q)$, i.e. take the blokings sets of $X_{e,b}(q) \subset PG(2b-e+1, q)$ with respect to the hyperplanes (or, equivalently, working inside $X_{e,b}$ instead of looking at $PG(2b - e + 1, q)$) with respect to $|H_b|(q)$). Take any line $D \subset X_{e,b}$. Obiously $\sharp(D) = q + 1$ and D is a bloking set.

Proposition 1. *Fix integers $e > 0$ and $b > ae > 0$. Let $A \subseteq X_{e,b,a}(q)$ be any blocking set with respect to $|H_{b,a}|(q)$. Then $A(q) \cap h(q) \neq \emptyset$.*

Proof. Since $h \cong \mathbb{P}^1$ over $\mathbb{F}_q$, we may find $B \subset h$ defined over $\mathbb{F}_q$, formed by b distinct points, but such that $\mathbb{F}_{q^b}$ is the minimal overfield of $\mathbb{F}_q$ such that each of these points is defined over $\mathbb{F}_{q^b}$. Hence $u^{-1}(B) \in |H_{b,0}|(q)$. Set $D := u^{-1}(B) + ah \in |H_{b,a}|(q)$. Since $D(q) = h(q)$, the result follows. □

Proposition 2. *Fix integers $a \geq 2$ and $b \geq 2$. Let X be a hyperbolic quadric surface. There is no blocking set of $X_{0,b,a}(q)$ with respect to $H_{b,a}$.*

Proof. The statement is equivalent to the assertion that $X_{e,a,b}(q) = X(q)$ is not a blocking set, i.e. it is equivalent to the existence of $D \in |H_{b,a}|(q)$ such that $D(q) = \emptyset$. Fix $A \in \mathbb{P}^1$ such that $\sharp(A) = a$, A is defined over $\mathbb{F}_q$, but $\mathbb{F}_{q^b}$ is the minimal overfield of $\mathbb{F}_q$ such that each of these points is defined over $\mathbb{F}_{q^b}$. Fix $B \subset \mathbb{P}^1$ defined over $\mathbb{F}_q$, formed by b distinct points, but such that $\mathbb{F}_{q^b}$ is the minimal overfield of $\mathbb{F}_q$ such that each of these points is defined over $\mathbb{F}_{q^b}$. Set $D := \pi_1^{-1}(A) + \pi_2^{-1}(B)$. □

Lemma 1. *Fix integer $e > 0$ and $a > 0$. There is $A \in |H_{ea,a}|(q)$ such that $A(\mathbb{F}_q) \cap h(\mathbb{F}_q) = \emptyset$.*

Proof. The line bundle $H_{ea,a}$ is spanned and a general element of $|H_{ea,a}|$ is disjoint from h. The lemma says that we may find such a divisor with the additional condition that it is defived over $\mathbb{F}_q$, if we weaken the thsis and only require $A(\mathbb{F}_q) \cap h(\mathbb{F}_q) = \emptyset$. First assume $a = 1$. Fix any $D \in |h + ef|$. Since $h \cdot (h + ef) = 0$ and for all $0 \le x < e$ every element of $|h + xf|$ is the union of h and x fibers of u (counting multiplicities), $D(\overline{\mathbb{F}}_q) \cap h(\overline{\mathbb{F}}_q) \neq \emptyset$ if and only if $D = h + D'$, with $D' \in |ef|$. D is defined over $\mathbb{F}_q$ if and only if D' is defined over $\mathbb{F}_q$. If $a \ge 2$, then take $A := aB$. The projective space $|h + ef|$ has dimension $e + 1$. Hence $\sharp(|h + ef|) = (q^{e+2} - 1)/(q - 1)$. Fix $P \in h(q)$. We saw that the set V of all $U \in |h + ef|$ containing P is a projective space of dimension $e - 1$. Hence $\sharp(V(q)) = (q^e - 1)/(q - 1)$. Since $\sharp(h(q)) = q + 1$ and $(q + 1) \cdot (q^e - 1)/(q - 1) < (q^{e+2} - 1)/(q - 1)$, we get the lemma. □

Proposition 3. *Fix integers $e > 0$ and $b > ae > 0$. The set $h(q)$ is a blocking set of $X_{e,b,a}$ with respect to $|H_{b,a}|(q)$ (and hence a minimal blocking set) if and only if $b = ae + 1$.*

Proof. Notice that $b = ae + 1$ if and only if $h \subset X_{e,b,a}$ is a line. This observation gives the "if" part of the proposition. Now assume $b \ge ae+2$. Since $b - ae \ge 2$ and $h \cong \mathbb{P}^1$ over $\mathbb{F}_q$, we may find $B \subset h$ defined over $\mathbb{F}_q$, formed by $b-ae$ distinct points, but such that $\mathbb{F}_{q^{b-ae}}$ is the minimal overfield of $\mathbb{F}_q$ such that each of these points is defined over $\mathbb{F}_{q^{b-ae}}$. Set $B' := u^{-1}(B)$. Take fix integer $e > 0$ and $a > 0$. There is $A \in |H_{ea,a}|(q)$ such that $A(\overline{\mathbb{F}}_q) \cap h(\overline{\mathbb{F}}_q) = \emptyset$ (Lemma 1). Set $D := B' + A$. Notice that $D \in |H_{b,a}|(q)$ and that $D(q) \cap h(q) = \emptyset$. Hence $h(q)$ is not a bloking set with respect to $|H_{b,a}|(q)$. Proposition 2 shows the truth of the sentence "(and hence a minimal blocking set)". □

Proposition 4. *Let $Y \subset \mathbb{P}^3$ be the elliptic quadric described. Then $Y(q)$ is the only bloking set of $Y(q)$ with respect to $|\mathcal{O}_Y(1)|(q)$.*

Proof. We first show that no proper subset of $Y(q)$ is a bloking set. Fix $P \in Y(q)$. Since Y becomes a hyperbolic quadric if we extend the base field to the field $\mathbb{F}_{q^2}$, the two lines, D_1, D_2, of $Y(\overline{\mathbb{F}}_q)$ containg P are defined over $\mathbb{F}_{q^2}$ and exchaged by the generator of the field extension $[\mathbb{F}_{q^2}, \mathbb{F}_q]$. Hence $D_1 \cup D_2 \in |\mathcal{O}_Y(1)|(q)$ and $\{P\} = (D_1 \cup D_2)(q)$. Thus $X \backslash \{P\}$ is not a bloking set. Now we prove that $Y(q)$ is a bloking set, i.e. that $A(q) \neq \emptyset$ for all $A \in |\mathcal{O}_Y(1)|(q)$. If the plane $\langle A \rangle$ is not tangent to Y, then A is a smooth conic defined over $\mathbb{F}_q$. Hence $\sharp(A(q)) = q + 1$ in this case. If $\langle A \rangle$ is tangent to Y, then the tangent point is the only element of $A(q)$. □

Proposition 5. *Fix an integer $a \ge 2$. Let $Y \subset \mathbb{P}^3$ be the elliptic quadric described in Example 2. There is no blocking set of $Y(q)$ with respect to*

$|\mathcal{O}_Y(a)|(q)$, *i.e. there is* $A \in |\mathcal{O}_Y(a)|(q)$ *such that* $A(q) = \emptyset$.

Proof. Since Y becomes a hyperbolic quadric if we extend the base field to the field $\mathbb{F}_{q^2}$, there is a an effective divisor D_1 of bidegree $(a, 0)$ on $Y(\overline{\mathbb{F}}_q)$ defined over $\mathbb{F}_{q^2}$, but such that none of its lines is defined over $\mathbb{F}_{q^2}$. Let D_2 be the effective divisors of $Y(\overline{\mathbb{F}}_q)$ obtained from D_1 by the action of the generator of the Galois extension $[\mathbb{F}_{q^2}, \mathbb{F}_q]$. Hence $D_1 \cup D_2$ is defined over $\mathbb{F}_q$. Since Y is an elliptic quadric, D_2 has bidegree $(0, a)$, i.e. $D_1 \cup D_2$ has bidegree (a, a), i.e. $D_1 \cup D_2 \in |\mathcal{O}_Y(a)|(q)$. Since $D_1(q^2) = \emptyset$, $(D_1 \cup D_2)(q) = \emptyset$. □

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