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A HYBRID APPROACH TO TERMINAL-BOUNDARY VALUE PROBLEMS FOR THE BLACK-SCHOLES EQUATION

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Abstract: A hybrid algorithm is developed to deal with terminal-boundary value problems for the Black-Scholes equation. The algorithm is semi-analytical in nature. It is based on a combination of the explicit finite difference scheme, which is applied to the time variable, with the Green's function method used to solve the resulting boundary value problems for ordinary deferential equations. Computational potential of this approach is illustrated by computing profiles of solutions for a number of specific problems.

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1. Direct Green's Function Approach

The objective herein is to develop an efficient semi-analytical algorithm for solution of the terminal-boundary value problem

$$v(S,T)=\varphi(S), \tag{1}$$

$$v(S_1,t)=A(t) \quad \text{and} \quad v(S_2,t)=B(t), \tag{2}$$

posed for the nonhomogeneous Black-Scholes equation

$$\frac{\partial v(S,t)}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2}+rS\frac{\partial v(S,t)}{\partial S}-rv(S,t)=\Phi(S,t) \tag{3}$$

in the rectangular region $\Omega=(S_1<S<S_2)\times(T<t<0)$ of the S,t-plane.

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In the contemporary financial engineering Black and Scholes [2], Merton [6] and Wilmott et al [8], the function $v = v(S,t)$ is referred to as the price of the derivative product, $\varphi(S)$ in (1) is called the pay-off function of a given derivative problem at the expiration time T, with the independent variables S and t interpreted as the share price of the underlying asset and time, respectively. The parameters σ and r, in the Black-Scholes equation, represent the volatility of the underlying asset and the risk-free interest rate. The functions $A(t)$ and $B(t)$ specify in (2) the conditions imposed on the boundary fragments $S = S_1$ and $S = S_2$ of Ω.

By the evident substitution

$$v(S,t) = V(S,t) + \frac{(S_2 - S)A(t) + (S - S_1)B(t)}{S_2 - S_1} \tag{4}$$

the terminal-boundary value problem in (1)-(3) reduces to the form

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = F(S,t), \tag{5}$$

$$V(S,T) = f(S), \tag{6}$$

$$V(S_1,t) = 0 \quad \text{and} \quad V(S_2,t) = 0, \tag{7}$$

with the homogeneous boundary conditions imposed at $S = S_1$ and $S = S_2$. The right-hand side function $F(S,t)$ of (5) is determined by the substitution in (4) as

$$F(S,t) = \Phi(S,t) + \frac{rS[A(t) - B(t)]}{S_2 - S_1} - \left(\frac{d}{dt} - r\right)\left(\frac{S_2 - S}{S_2 - S_1}A(t) + \frac{S - S_1}{S_2 - S_1}B(t)\right), \tag{8}$$

while the right-hand side function $f(S)$ in (6) is defined in terms of the pay-off function $\varphi(S)$ as

$$f(S) = \varphi(S) - \frac{(S_2 - S)A(T) + (S - S_1)B(T)}{S_2 - S_1}. \tag{9}$$

Upon implementing the standard substitutions

$$x = \ln S \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t) \tag{10}$$

and setting $u(x,\tau) = V(S,t)$, the terminal-boundary value problem in (5)-(7) converts to the following initial-boundary value one

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + (c-1)\frac{\partial u(x,\tau)}{\partial x} - cu(x,\tau) - \widetilde{F}(x,\tau), \tag{11}$$

$$u(x,0) = \widetilde{f}(x), \tag{12}$$

$$u(a,\tau)=0 \quad \text{and} \quad u(b,\tau)=0 \tag{13}$$

stated on the rectangle $\widetilde{\Omega}=(a<x<b)\times(0<\tau<\sigma^2T/2)$, on which the region maps by the change of variables in (10). The end-values a and b of the new variable x are evidently determined as

$$a=\ln S_1 \quad \text{and} \quad b=\ln S_2$$

while the constant parameter c in (11) is defined in terms of the volatility σ and the interest rate r as $c=2r/\sigma^2$. The right-hand side functions in (11) and (12) are introduced as

$$\widetilde{F}(x,\tau)=F(\exp(x),T-2\tau/\sigma^2) \quad \text{and} \quad \widetilde{f}(x)=f(\exp(x))$$

Earlier in Melnikov [4], we have implemented a Green's function-based algorithm to terminal-boundary value problems of the type in (5)-(7). Explicit analytical exdpressions of Green's functions $G(S,t;\widetilde{S})$ are required for the algorithm wth the solution of (5)-(7) coming to the integral form

$$V(S,t) \;=\; \int_{S_1}^{S_2} G(S,t;\widetilde{S})f(\widetilde{S})d\widetilde{S} \;+\; \int_t^T\int_{S_1}^{S_2} G(S,t-\widetilde{t},\widetilde{S})F(\widetilde{S},\widetilde{t})d\widetilde{S}d\widetilde{t}. \tag{14}$$

Computer-friendly representations of Green's functions were constructed in Melnikov et al [5] for a number of terminal-boundary value problems. For the Dirichlet problem of (5)-(7), for example, it was obtained in the form

$$G(S,t;\widetilde{S})=\frac{1}{\sigma\widetilde{S}\sqrt{2\pi(T-t)}}\exp\left(-\frac{r-\sigma^2/2}{\sigma^2}\ln\frac{S}{\widetilde{S}}-\frac{(r+\sigma^2/2)^2}{2\sigma^2}(T-t)\right)$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[\ln(SS_1^{2m}/\widetilde{S}S_2^{2m})]^2}{2\sigma^2(T-t)}\right)-\exp\left(-\frac{[\ln(S_2^{2(m+1)}/S\widetilde{S}S_1^{2m})]^2}{2\sigma^2(T-t)}\right)\right\},$$

whose series component converges at a high rate.

For the mixed problem

$$|V(0,t)|<\infty \quad \text{and} \quad \frac{\partial V(S_2,t)}{\partial S}+\rho V(S_2,t)=0, \quad \rho\geq 0$$

stated on the region $\Omega=(0<S<S_2)\times(T<t<0)$, its Green's function was found as

$$G(S,t;\widetilde{S})=\frac{1}{\widetilde{S}}\left(\frac{S}{\widetilde{S}}\right)^{\omega}\exp\left(-\frac{(\sigma^2+2r)^2}{8\sigma^2}(T-t)\right)$$
$$\times\left\{\frac{1}{\sigma\sqrt{2\pi(T-t)}}\left[\exp\left(-\frac{[\ln(S/\widetilde{S})]^2}{2\sigma^2(T-t)}\right)+\exp\left(-\frac{[\ln(S\widetilde{S}/S_2^2)]^2}{2\sigma^2(T-t)}\right)\right]\right.$$

$$-\varphi\left(\frac{S\widetilde{S}}{S_2^2}\right)^{\varphi}\exp\left(\frac{\varphi^2\sigma^2(T-t)}{2}\right)\operatorname{erfc}\left(\frac{\varphi\sigma}{2}\sqrt{2(T-t)}-\frac{\ln(S\widetilde{S}/S_2^2)}{\sigma\sqrt{2(T-t)}}\right)\Bigg\},$$

where

$$\omega=\frac{\sigma^2-2r}{2\sigma^2} \qquad \text{and} \qquad \varphi=\rho S_2+\omega$$

and 'erfc' stays for the complementary Gauss error function.

Hence solution to a terminal-boundary value problem in (5)-(7) reduces to the integral-only form of (14). This implies that the algorithm is supposed to be of a high efficiency. This assertion is convincingly supported in Melnikov [4] by actual computations.

The objective herein is to show that another Green's function-based approach could be suggested to treat the problem in (5)-(7). The approach, whose use was recommended in Buttler et al [3], combines finite difference and the Green's function ideas.

2. Solution Algorithm

The solution algorithm developed in the present study is hybrid in nature and consists of two different phases. The first phase is numerical whereas the second analytical. An explicit finite difference scheme (Samarskii [7]) is first applied and then a Green's function method goes to the picture to analytically solve boundary value problems for resulting ordinary differential equations.

The explicit finite difference scheme is applied to the problem in (11)-(13). In doing so, we uniformly divide the domain $[0,\sigma^2T/2]$ for the variable τ into N intervals, with the partition step $h=\sigma^2T/(2N)$, and come up with the regular discretization

$$\tau_k=kh, \qquad k=0,1,2,\ \dots\ ,N.$$

With this partition, we approximate the τ derivative in (11) at $\tau=\tau_{k+1}$ with the forward difference

$$\frac{\partial u(x,\tau_{k+1})}{\partial\tau}\approx\frac{u(x,\tau_{k+1})-u(x,\tau_k)}{h}$$

and obtain the boundary value problem

$$\frac{d^2U(x,\tau_{k+1})}{dx^2}+(c-1)\frac{dU(x,\tau_{k+1})}{dx}-\left(c+\frac{1}{h}\right)U(x,\tau_{k+1})$$

$$=\widetilde{F}(x,\tau_k)-\frac{1}{h}U(x,\tau_k), \qquad k=0,1,2,\ \dots\ ,N, \tag{15}$$

$$U(a,\tau_{k+1}) = 0, \qquad U(b,\tau_{k+1}) = 0 \tag{16}$$

in an approximation $U(x, \tau_{k+1})$ of $u(x, \tau_{k+1})$. Note that to start the process we assume

$$U(x, \tau_0) = \widetilde{f}(x).$$

Before going any further with the solution procedure for the boundary value problem in (15) and (16), we convert the governing differential equation in (15) to a self-adjoint form. This can be done by introducing the integrating factor of $\exp((c-1)x)$

$$\exp((c-1)x)\frac{d^2U(x,\tau_{k+1})}{dx^2} + (c-1)\exp((c-1)x)\frac{dU(x,\tau_{k+1})}{dx} - \left(c+\frac{1}{h}\right)\exp((c-1)x)U(x,\tau_{k+1})$$
$$= \exp((c-1)x)\left[\widetilde{F}(x,\tau_k) - \frac{1}{h}U(x,\tau_k)\right], \qquad k = 0,1,2,\ \dots\ ,N. \tag{17}$$

The idea behind the above conversion is to guarantee for the Green's function, which will be found later, a reciprocal (symmetrical) form making it more convenient in numerical implementations.

Let $g(x,s)$ represent the Green's function to the homogeneous boundary value problem associated with (16)-(17). If so then the solution of the problem in (16)-(17) itself is given in Arsenin [1] as

$$U(x,\tau_{k+1}) = \int_a^b g(x,s)\exp((c-1)s)\left[\widetilde{F}(s,\tau_k) - \frac{1}{h}U(s,\tau_k)\right] ds. \tag{18}$$

Thus, at each τ-step of the procedure, one is required to simply compute the above integral, which represents a function of the variable x, at a required point set. Our experience brings a strong evidence of the efficiency of standard quadratures for such a computation. Marching towards the right-end point of the segment $[0, \sigma^2 T/2]$ resembles a recurrence procedure and can easily be conducted. As soon as approximations $U(x,\tau_{k+1})$ of $u(x,\tau_{k+1})$ are found, the backward substitution, in compliance with the relations in (10) and (4), allows to find approximate values of $V(S,t)$, and then of the original function $v(S,t)$. In Section 4, an evidence is presented of the potential of the proposed approach.

3. Construction of Green's Function

Construction of the Green's function $g(x, s)$, which represents the kernel in the integral of (18), will be conducted in an indirect way. That is, the method of variation of parameters will be applied to actually solve the problem in (16) and (17). By this, the solution to the latter appears in the form of (18) bringing an explicit expression for the Green's function we are looking for.

Since the auxiliary equation

$$m^2 + (c-1)m - \left(c + \frac{1}{h}\right) = 0$$

for (17) has two distinct real roots

$$m_{1,2} = \frac{1}{2}\left[(1-c) \pm \sqrt{(1+c)^2 + \frac{4}{h}}\right] \tag{19}$$

the fundamental set of solutions to the homogeneous equation corresponding to (17) can be constituted as

$$\exp(m_1 x) \qquad \text{and} \qquad \exp(m_2 x).$$

With this, the general solution to (17) appears in the form

$$U(x,\tau_{k+1}) = C(x)\exp(m_1 x) + D(x)\exp(m_2 x). \tag{20}$$

Tracing out the variation of parameters procedure, one obtains the following well-posed system of linear algebraic equations

$$\begin{pmatrix} \exp(m_1 x) & \exp(m_2 x) \\ m_1 \exp(m_1 x) & m_2 \exp(m_2 x) \end{pmatrix} \begin{pmatrix} C'(x) \\ D'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\Psi(x) \end{pmatrix} \tag{21}$$

in the derivatives of $C(x)$ and $D(x)$. The entry $\Psi(x)$ in the above system represents a short-handed for

$$\widetilde{F}(s,\tau_k) - \frac{1}{h}U(s,\tau_k).$$

Upon solving the system in (21), one obtains

$$C'(x) = \frac{\exp(m_2 x)\Psi(x)}{\Delta(x)} \qquad \text{and} \qquad D'(x) = -\frac{\exp(m_1 x)\Psi(x)}{\Delta(x)},$$

where

$$\Delta(x) = (m_2 - m_1)\exp((m_1 + m_2)x).$$

Integrating the above expressions for $C'(x)$ and $D'(x)$, we have

$$C(x) = \int_a^x \frac{\exp(m_2 s)\Psi(s)ds}{\Delta(s)} + M$$

and

$$D(x) = -\int_a^x \frac{\exp(m_1 s)\Psi(s)ds}{\Delta(s)} + N.$$

Implementing the boundary conditions from equation (16), we specify the constants of integration M and N. The condition $U(a,\tau_{k+1})=0$, in particular, yields

$$M\exp(m_1 a) + N\exp(m_2 a) = 0 \tag{22}$$

while the condition $U(b,\tau_{k+1})=0$ results in

$$M\exp(m_1 b) + N\exp(m_2 b) = \int_a^b \frac{\Psi(s)}{\Delta(s)}[\exp(m_2 b+m_1 s) - \exp(m_1 b+m_2 s)]ds. \tag{23}$$

The relations in (22) and (23) constitute a well-posed system of linear algebraic equations in the constants M and N. Solution to the system is found as

$$M=\int_a^b \frac{[\exp(m_2 b+m_1 s) - \exp(m_1 b+m_2 s)]\Psi(s)}{\Delta(s)\exp(m_2 b)[\exp((m_1-m_2)b)-\exp((m_1-m_2)a)]}ds$$

and

$$N=-\int_a^b \frac{[\exp(m_2 b+m_1 s) - \exp(m_1 b+m_2 s)]\Psi(s)}{\Delta(s)\exp(m_1 b)[\exp((m_2-m_1)a)-\exp((m_2-m_1)b)]}ds.$$

When these expressions for M and N are substituted in the above forms for $C(x)$ and $D(x)$, while the latter are then plugged in (20), the solution of the boundary value problem as of (16) and (17) appears in the form

$$U(x,\tau_{k+1})=\int_a^x \frac{\Psi(s)}{\Delta(s)}[\exp(m_1 x+m_2 s) - \exp(m_2 x+m_1 s)]ds + \int_a^b \frac{\exp(m_2 b+m_1 s) - \exp(m_1 b+m_2 s)}{\exp((m_1-m_2)b)-\exp((m_1-m_2)a)} \times \frac{\exp((m_1-m_2)x)-\exp((m_1-m_2)a)}{\Delta(s)\exp(m_2(b-x))}\Psi(s)ds \tag{24}$$

It is evident that the two integrals in (24) can be reduced to a single integral representation whose integrand is defined in two pieces. Thus, in light of (18), an expression for the Green's function $g(x,s)$, which is valid for $x\le s$, is arrived at as

$$g(x,s) = \frac{\exp(m_2 b+m_1 s) - \exp(m_1 b+m_2 s)}{\exp((m_1-m_2)b)-\exp((m_1-m_2)a)} \times \frac{\exp((m_1-m_2)x)-\exp((m_1-m_2)a)}{(m_1-m_2)\exp(m_2(b-x))}, \qquad x\le s,$$

whereas the expression

$$g(x,s) = \frac{\exp(m_2 b + m_1 x) - \exp(m_1 b + m_2 x)}{\exp((m_1 - m_2)b) - \exp((m_1 - m_2)a)} \times \frac{\exp((m_1 - m_2)s) - \exp((m_1 - m_2)a)}{(m_1 - m_2)\exp(m_2(b-s))}, \qquad x \geq s,$$

which is valid for $x \geq s$, can be obtained from that valid for $x \geq s$ by interchanging x with s.

4. Numerical Implementations

Before going any further with numerical implementations of the hybrid algorithm, proposed in this study, it is important to accumulate data on the accuracy level that it allows to attain. Such data could ideally be obtained by applying the algorithm to a sample problem whose exact solution is available. With this in mind, we turn back to the problem in (5)-(7) stated on $\Omega = (S_1 < S < S_2) \times (T < t < 0)$ and compose the function

$$V_0(S,t) = (S - S_1)(S_2 - S)\,t, \tag{25}$$

which could be considered as the exact solution to (5)-(7) with specifically chosen function $F(S,t)$ in the governing equation of (5) and function $f(S)$ in the terminal condition of (6). Indeed, it is evident that $V_0(S,t)$ vanishes on the fragments $S = S_1$ and $S = S_2$ of Ω satisfying the boundary conditions of (7). It is also clear that $V_0(S,t)$ satisfies the terminal condition of (6) if the right-hand side function in (6) is fixed as $f(S) = T(S - S_1)(S_2 - S)$. If, in addition, the right-hand side function in (5) is chosen as

$$F(S,t) = -\left[1 + (r + \sigma^2)\,t\right] S^2 + (S_1 + S_2)\,S - (1 - rt)S_1 S_2,$$

then, upon inspection, we conclude that $V_0(S,t)$ does indeed represent the exact solution to the whole problem in (5)-(7).

The solution algorithm described in Section 2 was used to find an approximate solution to the sample terminal-boundary value problem just introduced. Approximate values of $V_0(S,t)$, computed at a few observation points in Ω, are exhibited in Table 1. The constant parameters in the setting were fixed as: $r = 0.06$, $\sigma = 0.8$, $T = 2$, $S_1 = 1$ and $S_2 = 5$. The time variable step in the finite difference method was defined by the partition number $N = 100$, while values of the definite integral in (18) were computed by the standard trapezoidal rule with just 20 partitions.

As it follows from the data presented in Table 1, although a primitive com-

Value	Observation point (S,t) in Ω					
$V_0(S,t)$	$(2.,1.5)$	$(3.,1.5)$	$(4.,1.5)$	$(2.,.5)$	$(3.,.5)$	$(4.,.5)$
Approx.	4.4999	5.9993	4.4998	1.4990	1.9988	1.4989
Exact	4.5000	6.0000	4.5000	1.5000	2.0000	1.5000

Table 1: To the accuracy level attained for the sample problem

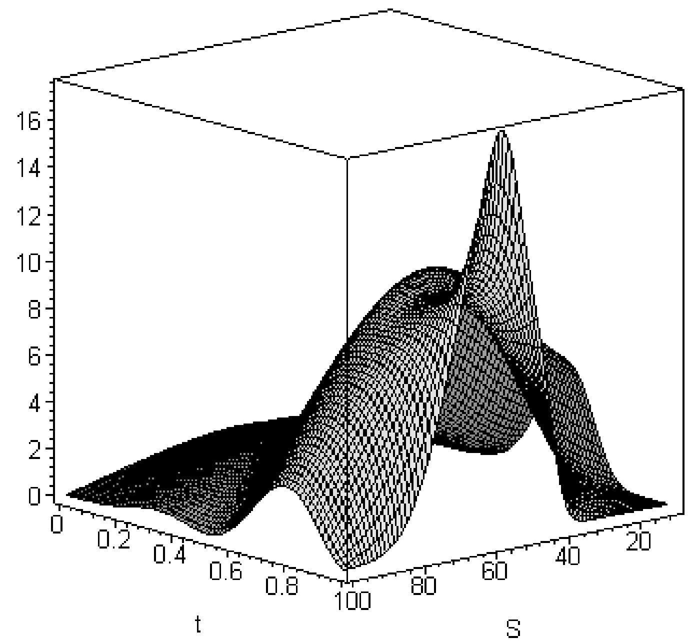

Figure 1:

putational scheme has been used with limited partition numbers, it became possible to attain a relatively high accuracy level in solving the sample problem.

To provide some illustrations of the computational potential for the hybrid approach, which is advocated in this study, we deliver two particular examples.

For the first of these illustrations, the homogeneous Black-Scholes equation ($\Phi(S,t)=0$) is considered. The constant parameters specifying the setting in (1)-(3) are fixed as: $r=0.06$, $\sigma=0.8$, $T=1$, $S_1=10$ and $S_2=100$; the functions defining the boundary conditions are chosen as: $A(t)=4\cdot 10^4(t-t^4)^{12}$ and $B(t)=5t^2\sin^2(2\pi t/T)$, while continuous piecewise-linear pay-off function $\varphi(S)$

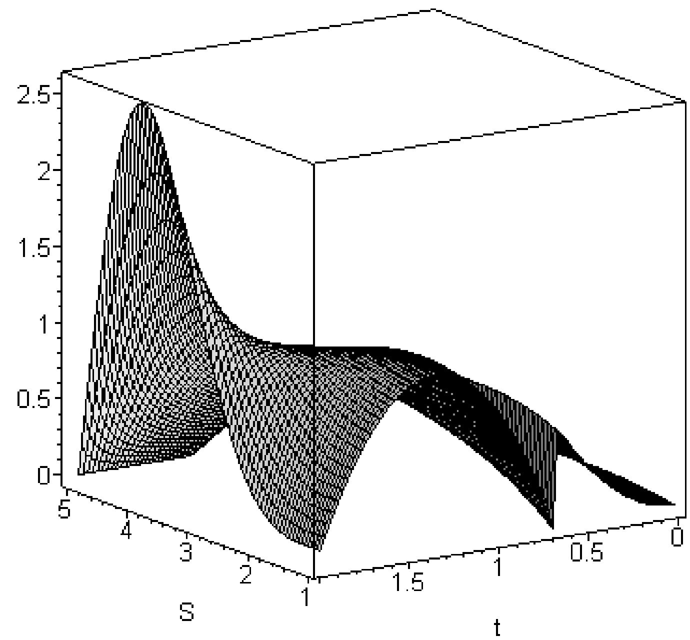

Figure 2:

is defined on the segment $[S_1, S_2]$ as

$$\varphi(S) = \begin{cases} 0, & S \leq 40 \ , \\ 4(S-40)/3, & 40 < S \leq 55 \ , \\ 4(80-S)/5, & 55 < S \leq 80 \ , \\ 0, & S > 80. \end{cases}$$

Solution profile for the above setting is depicted in Figure 1. An evident conclusion following from this illustration is that our algorithm appears efficient in the case where the pay-off function $\varphi(S)$ is not smooth. It is not actually differentiable on $[S_1, S_2]$.

For another illustrative example, we consider the problem with boundary conditions imposed in terms of data which are not smooth and even represent discontinuous functions. The parameters defining the setting are in this case fixed as: $r = 0.06$, $\sigma = 0.8$, $T = 2$, $S_1 = 1$ and $S_2 = 5$; $\Phi(S, t) = 1$. The pay-off function $\varphi(S) = 0.04(S - S_1)^4(S_2 - S)$ is smooth on $[S_1, S_2]$, while one of the functions defining the boundary conditions is discontinuous. They are chosen as

$$A(t) = \begin{cases} 1.5t^2, & 0 \le t < T/3\ , \\ 2.5(t - T/3)(T - t), & T/3 \le t \le T, \end{cases}$$

and

$$B(t) = \begin{cases} 1.25t(2T/3 - t), & 0 \le t < 2T/3\ , \\ 0, & 2T/3 \le t \le T. \end{cases}$$

Solution profile for the problem under consideration is depicted in Figure 2 illustrating the potential of our algorithm in cases with discontinuous boundary data.

5. Conclusion

A new algorithm is proposed for terminal-boundary value problems stated for the Black-Scholes equation. The algorithm is based on a combination of the methods of finite difference and Green's function. High accuracy level is reported as attained for statements with boundary and terminal conditions imposed in terms of data which are not smooth.

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