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THE GENERALIZED SOLUTIONS OF THE FUZZY DIFFERENTIAL INCLUSIONS

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Abstract: In this article we introduce the definition of the generalized solution of the fuzzy differential inclusion and find the conditions when this solution is equivalent to the ordinary one.

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1. Introduction

When a real world problem is transferred into a deterministic initial value problem of ordinary differential equations (ODE), namely

$$x' = f(t,x),\ x(t_0) = x_0,$$

we cannot usually be sure that the model is perfect. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model, is to utilize the aspect of fuzziness, which leads to the consideration of fuzzy differential equations (FDE). The intricacies involved in incorporating fuzziness into the theory of ODE pose a certain disadvantage and other possibilities are being explored to address this problem. One of the approaches is to connect FDE to multivalued differential equations and examine the intercon-

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nection between them [10], [11], [12], [19], [20], [24], [26]. The other approach is to transform FDE into differential inclusion with the fuzzy right-hand sides so as to employ the existing theory of differential inclusions [2], [4], [5], [8], [16], [17].

In this article we consider fuzzy differential inclusions.

2. Preliminaries

Let $conv(R^n)$ be the family of all nonempty compact convex subsets of R^n with the Hausdorff metric

$$h(A,B) = \max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n. It is clear that $(conv(R^n), h)$ becomes a metric space.

Let E^n be the family of mappings $x : R^n \to [0,1]$ satisfying the following conditions:

(i) x is normal, i.e. there exists an $y_0 \in R^n$ such that $x(y_0) = 1$;

(ii) x is fuzzy convex, i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in R^n$ and $\lambda \in [0,1]$;

(iii) x is upper semicontinuous, i.e. for any $y_0 \in R^n$ and $\varepsilon > 0$ exists $\delta(y_0,\varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $||y - y_0|| < \delta,\ y \in R^n$;

(iv) $[x]^0 = cl\{y \in R^n : x(y) > 0\}$ is compact.

Denote $[x]^\alpha = \{y \in R^n : x(y) \geq \alpha\}$ for $\alpha \in (0,1]$.

Then from (i)–(iv) follows that the α-level set $[x]^\alpha \in conv(R^n)$ for all $\alpha \in [0,1]$.

Let $\widehat{0}$ be the fuzzy mapping defined by $\widehat{0}(y) = 0$ if $y \neq 0$ and $\widehat{0}(0) = 1$.

Define $D : E^n \times E^n \to R_+$ by the equation $D(x,y) = \sup\limits_{\alpha\in[0,1]} h([x]^\alpha, [y]^\alpha)$. It is easy to show that D is a metric in E^n.

Definition 1. (see [19]) A mapping $f : I \to E^n$ is called measurable if for any $\alpha \in [0,1]$ the multivalued mapping $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable.

Definition 2. (see [19]) A mapping $f : I \to E^n$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta,\ t \in I$.

Definition 3. (see [9]) A mapping $f : I \to E^n$ is called differentiable at point $t \in I$ if for any $\alpha \in [0,1]$ the multivalued mapping $f_\alpha(t)$ is Hukuhara differentiable at point t (see [7]) and the family $\{D_H f_\alpha(t) \,:\, \alpha \in [0,1]\}$ defines a fuzzy number $f'(t) \in E^n$ (which is called the fuzzy derivative of $f(t)$ at point t).

Definition 4. (see [9]) The element $g \in E^n$ is called the integral of $f : I \to E^n$ over I if $[g]^\alpha = (A)\int\limits_I f_\alpha(t)dt$ for any $\alpha \in (0,1]$, where $(A)\int\limits_I f_\alpha(t)dt$ is the Aumann integral, see [3].

Definition 5. A mapping $f : I \to E^n$ is called absolutely continuous on I if there exists an integrable map $g : I \to E^n$ such that

$$f(t) = f(t_0) + \int\limits_{t_0}^{t} g(s)ds,\ t_0 \in I \text{ for every } t \in I.$$

Let $comp(E^n)$ $(conv(E^n))$ be the family of all subsets F of the space E^n such that for any $\alpha \in [0,1]$ the family of all $\alpha-$ level sets of the elements from F is the nonempty compact (and convex) element in $comp(R^n)$ (that is the element of $cc(R^n)$ $(cocc(R^n))$ (see [15])) with metric

$$d(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} D(a,b), \sup_{b\in B}\inf_{a\in A} D(a,b)\}.$$

Define also the distance from the element $x \in E^n$ to the set $A \in comp(E^n)$:

$$\rho(x,A) = \min_{a\in A} D(x,a).$$

We introduce in $comp(E^n)$ the usual algebraic operations:

– addition : $X + Y = \{x + y \,:\, x \in X,\, y \in Y\}$;

– multiplication by scalars λ: $\lambda X = \{\lambda x \,:\, x \in X\}$.

The following properties hold (see [25]):

1) $A + B \in comp(E^n)$;	4) $\alpha A \in comp(E^n)$;
2) $A + B = B + A$;	5) $\alpha(\beta A) = (\alpha\beta)A$;
3) $(A + B) + C = A + (B + C)$;	6) $\alpha(A + B) = \alpha A + \alpha B$;
4) $A + \{0\} = A$;	7) $1 \cdot A = A$.

Definition 6. A mapping $F : I \to comp(E^n)$ is called measurable on I if the set $\{t \in I :\ F(t) \bigcap G \neq \emptyset\}$ is measurable for every $G \in comp(E^n)$.

Definition 7. A mapping $F : I \to conv(E^n)$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(F(t), F(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta,\ t \in I$.

Define an integral of $F : I \to comp(E^n)$ over I :

$$\int_I F(t)dt = \left\{ \int_I f(t)dt \; : \; f(t) \in F(t) \text{ almost everywhere on } I \right\}.$$

3. Main Results

Consider the fuzzy differential inclusion

$$x' \in F(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $F : I \times E^n \to comp(E^n),\ t_0 \in I,\ x_0 \in E^n$.

Definition 8. The absolutely continuous mapping $x(\cdot)$ is called the ordinary solution of the differential inclusion (1) if $x'(t) \in F(t, x(t))$ almost everywhere on I.

Let $O(F)$ be the set of all ordinary solutions of the differential inclusion (1).

Definition 9. The continuous mapping $x(\cdot)$ is called the generalized solution of the differential inclusion (1) if the integral inclusion

$$x(t'') \in x(t') + \int_{t'}^{t''} F(t, x(t))dt \tag{2}$$

fulfills for all $t', t'' \in I$ such that $t' < t''$.

Let $G(F)$ be the set of all generalized solutions of the differential inclusion (1).

Definition 10. The multivalued mapping $F : I \times D \to comp(E^n), D \subset E^n$ satisfies the Caratheodory conditions if:

1) for every fixed $x \in D$ the multivalued mapping $F(\cdot, x)$ is measurable on the interval I;

2) for almost every fixed $t \in I$ the multivalued mapping $F(t, \cdot)$ is continuous on D;

3) there exists a Lebesgue summable function $m : I \to R$ such that $d(F(t,x), \{\widehat{0}\}) \le m(t)$ for almost every $(t, x) \in I \times D$.

Using the ideas of (see [13]) we can proof the following lemma.

Lemma 1. *Let the multivalued mapping $F : I \times D \to comp(E^n)$ satisfy the Caratheodory conditions, $\varphi : I \to D$ is measurable. Then the multivalued* *mapping $F(t,\varphi(t))$ is summable on I.*

Theorem 11. *Let $F : I \times D \to conv(E^n)$ satisfy the Caratheodory conditions. Then $O(F) = G(F)$.*

Proof. Let $x : I \to D$ be the ordinary solution of the inclusion (1) and $\Phi(t) = F(t, x(t))$. The multivalued mapping $\Phi : I \to conv(E^n)$ is summable on I by Lemma 1.

According to the definition 8 $x(\cdot)$ is absolutely continuous on I and $x'(t) \in \Phi(t)$ for almost every $t \in I$. In addition

$$D(x'(t), \widehat{0}) \le d(\Phi(t), \{\widehat{0}\}) \le m(t).$$

So $x'(\cdot)$ is a summable single-valued selection of the multivalued mapping $\Phi(\cdot)$ and according to the definition of the integral we have

$$\int_{t'}^{t''} x'(t)dt \in \int_{t'}^{t''} \Phi(t)dt$$

for all $t', t'' \in I$ such that $t' < t''$. It means that the inclusion

$$x(t'') \in x(t') + \int_{t'}^{t''} \Phi(t)dt$$

is true for all $t', t'' \in I$ such that $t' < t''$.

Therefore $x(\cdot)$ is the generalized solution of the differential inclusion (1), that is $O(F) \subset G(F)$.

Let us prove the inverse inclusion. Let $x(\cdot)$ be the generalized solution of the inclusion (1). Then $x(\cdot)$ is continuous and $x(t'') \in x(t') + \int_{t'}^{t''} F(t, x(t))dt$ for all $t', t'' \in I$ such that $t' < t''$.

Using the properties of the integral of the multivalued mappings we have

$$D(x(t''), x(t')) \le d\left(\int_{t'}^{t''} F(s, x(s))ds, \{\widehat{0}\}\right) \le \int_{t'}^{t''} d(F(s, x(s)), \{\widehat{0}\})ds \le \int_{t'}^{t''} m(s)ds, \quad (3)$$

for all $t', t'' \in I$ such that $t' < t''$.

Consider the function $\varphi(t)=\int\limits_{t_0}^{t} m(s)ds$. As $m(\cdot)$ is summable on I, then $\varphi(\cdot)$ is absolutely continuous on I as the integral with a variable top limit. Therefore for any $\varepsilon>0$ exists $\delta(\varepsilon)>0$ such that for every $m\in N$ and every $t_i'<t_i''$, $i=\overline{1,m}$ such that $\sum\limits_{i=1}^{m}(t_i''-t_i')<\delta$ fulfills the inequation

$$\sum_{i=1}^{m}(\varphi(t_i'')-\varphi(t_i'))<\varepsilon.$$

Thereby using (3) we have

$$\sum_{i=1}^{m}D(x(t_i''),x(t_i'))<\varepsilon.$$

Moreover according to (2) for all t',t'' exists $r(t',t'')\in E^n$ such that $x(t'')=x(t')+r(t',t'')$. Then similarly to [1] we have that the mapping $x(t)$ is absolutely continuous and there exists $x(t'')-x(t')$ for all t'',t' such that $t''>t'$.

Using the properties of the metric we get

$$0\le\rho(x'(t),F(t,x(t)))\le D\left(x'(t),\frac{1}{\eta}(x(t+\eta)-x(t))\right)$$
$$+\,d\left(\frac{1}{\eta}\int\limits_{t}^{t+\eta}F(s,x(s))ds,F(t,x(t))\right)=A_1(t,\eta)+A_2(t,\eta),\quad(4)$$

for any $\eta>0$ and $t\in I$.

The absolutely continuous mapping $x(\cdot)$ is differentiable almost everywhere, so $A_1(t,\eta)\to 0$ when $\eta\downarrow 0$ for almost every $t\in I$. By [14] for almost every t we have

$$A_2(t,\eta)=d\left(\frac{1}{\eta}\int\limits_{t}^{t+\eta}F(s,x(s))ds,F(t,x(t))\right)$$
$$=d\left(\frac{1}{\eta}\int\limits_{t}^{t+\eta}F(s,x(s))ds,\frac{1}{\eta}\int\limits_{t}^{t+\eta}F(t,x(t))ds\right)$$
$$\le\frac{1}{\eta}\int\limits_{t}^{t+\eta}d(F(s,x(s)),F(t,x(t))ds\to 0\quad\text{when }\eta\downarrow 0.$$

Let η in (4) go to 0, then we have $\rho(x'(t),F(t,x(t))=0$ almost everywhere on I. So $x'(t) \in F(t, x(t))$ almost everywhere on I. It means that $x(\cdot)$ is the ordinary solution of the differential inclusion (1), that is $G(F) \subset O(F)$. $\square$

Remark. This result generalizes the results of M. Dawidowski [6] for the ordinary differential inclusions and A.V. Plotnikov [15], [21] for the differential inclusion with the Hukuhara derivative.

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