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CURVES OVER $\mathbb{F}_q$ THROUGH POINTS IN A HYPERBOLIC QUADRIC SURFACE AND IN HIRZEBRUCH SURFACES

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Abstract: Let X be either a hyperbolic quadric surface over $\mathbb{F}_q$ or a Hirzebruch surface. Here we give conditions for the existence of a curve $C \subset X$ defined over $\mathbb{F}_q$ and containing $X(\mathbb{F}_q)$ or a prescribed subset of it.

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Fix a prime power q. For any algebraic set $A \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ let $A(q)$ be the set of its $\mathbb{F}_q$-points. With this convention $\mathbb{P}^n(q) = PG(n, q)$. Now assume that A is a curve, perhaps singular. We stress that in $A(q)$ we also count the singular points of A defined over $\mathbb{F}_q$, but we do not count the point over $\mathbb{F}_q$ of the normalization of A. This is the convention used in [2] and [3]. We want to find curves C in a Hirzebruch surface or in a hyperbolic quadric surface X containing many $\mathbb{F}_q$-points (e.g. containing $X(q)$) and with "small invariants". For more refined results in the case of plane curves, see [2] and [3]. Here there is one of the results of this note.

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Theorem 1. *Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric surface. Fix integers a, b such that $a \ge q+1$, $b \ge q+1$ and $(a,b) \ne (q+2, q+2)$. There is $A \in |\mathcal{I}_{X(q)}(a,b)|(q)$ which is smooth at each point of $X(q)$.*

Since we work over a finite field we need to distinguish the case $e > 0$ and the case $e = 0$ of the Hirzebruch surfaces F_e. In Example 1 we first state both cases on an algebraically closed base field and at the end study the case $e > 0$ over $\mathbb{F}_q$

Remark 1. Let F_e, $e \ge 0$, the Hirzebruch surface with invariant e, i.e. such that $-e$ is the minimal self-intersection of a section of the ruling of F_e. We have $\mathrm{Pic}(F_e) \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_e)$ a section h of f with $h^2 = -e$ and a class, f, of the ruling u. Thus $h \cdot f = 1$ and $f^2 = 0$. We will use both the additive and the multiplicative notation for line bundles and divisors on F_e. We have $\omega_{F_e} \cong -2h - (2+e)f$. we have $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a < 0$. By the projection formula we have $f_*(\mathcal{O}_{F_e}(ah+bf)) \cong \oplus_{i=0}^{a} \mathcal{O}_{\mathbf{P}^1}(b - ie)$ for every $a \ge 0$. Thus $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \ge 0$ and $b < ea$, $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = \sum_{i=1}^{a}(b - ie + 1) = (2b + 2 - ae)(a+1)/2$ if $a \ge 0$ and $b \ge 0$ and $h^1(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \ge 0$ and $b \ge ae - 1$. Set $H_{b,a} := ah + bf$. The line bundle $H_{b,a}$ is ample if and only if $a > 0$ and $b > ae$. The ruling of F_e is unique and hence defined over any field on which our surface is defined if and only if $e > 0$. Now assume that we work over $\mathbb{F}_q$ and that we have a surface W defined over $\mathbb{F}_q$ and which is isomorphic to F_e over $\overline{\mathbb{F}}_q$. If $e > 0$, then $W \cong F_e$ over $\mathbb{F}_q$ and hence we may apply the previous discussion of $\mathrm{Pic}(F_e)$ to W, i.e. each $H_{b,a}$ is defined over $\mathbb{F}_q$. If $e > 0$, then $\sharp(F_e(q)) = (q+1)^2$.

In Examples 1 we fix the notation for the hyperbolic quadric surface. We have no result on the elliptic quadric surface. We remark here that in both cases the embedding $W \hookrightarrow F_e$ is defined over $\mathbb{F}_q$ because it is the anticanonical embedding.

Example 1. Let $X \subset \mathbb{P}^3$ be a smooth and hyperbolic quadric surface defined over $\mathbb{F}_q$. Hence $\sharp(X(q)) = (q+1)^2$ and X has 2 ruling defined over $\mathbb{F}_q$ (see [1]). Every effective divisor $A \subset X$ defined over $\mathbb{F}_q$ has a bidegree (a,b) and the lines of X are the curves of bidegree $(1,0)$ or $(0,1)$. Conversely, for all $(a,b) \in \mathbb{N}^{\oplus 2} \backslash \{(0,0)\}$ there are curves $A \subset X$ defined over $\mathbb{F}_q$ and with bidegree (a,b). There are even curves with no singular point over $\overline{\mathbb{F}}_q$. If $ab \ne 0$ there are even curves A as above which are integral and with no singular point over $\overline{\mathbb{F}}_q$, i.e. curves which are smooth and geometrically integral. Any curve of type $(a,0)$, $a > 0$, is the union of a lines of type $(1,0)$ (although some line may appear with multiplicity > 1). Hence the computation of $\sharp(A(q))$ is trivial if

$ab = 0$. Hence from now on we assume $a \geq b > 0$. Let π_1 and π_2 be the two rulings of X, with the convention that the fibers of π_1 (resp. π_2) are the lines of type $(0,1)$ (resp. $(1,0)$). For all $(a,b) \in \mathbb{N}^{\oplus 2}$ let $M(a,b,q)_1$ denote the maximal number of elements of $X(q)$ contained in one $A \in |\mathcal{O}_X(a,b)|(q)$.

Proposition 1. *Fix integers $e \geq 0$, $a > 0$ and $b \geq \max\{ae, 1\}$. If $e = 0$ take as F_0 the hyperbolic quadic surface. Fix $A \in |H_{b,a}|(q)$ such that A contains no ruling of u. Then $\sharp(A(q)) \leq (q+1) \cdot \min\{a, q+1\}$.*

Proof. Since both A and the ruling u are defined over $\mathbb{F}_q$, we have $u(A(q)) \subseteq \mathbb{P}^1(q)$. Since no fiber of u over some point of $\mathbb{P}^1(q)$ is contained in A, $\sharp(A(q) \cap u^{-1}(Q)) \leq \min\{a, q+1\}$ for all $Q \in \mathbb{P}^1(q)$. □

Notice that the upper bound for $\sharp(A(q))$ in Proposition 1 does not depend from the integer b, although for a fixed $a \geq 2$ we have $\lim_{b \to +\infty} p_a(A) = +\infty$: smooth curves in $|H_{b,a}|$ are very far from having many points among the curve with their genus if $a \geq 2$ and $b \gg ae$.

Proposition 2. *Take the assumptions of Proposition 1. Assume that A has at least x singular points with mutiplicity $\mu_1, \ldots, \mu_x$ defined over $\mathbb{F}_q$. Then $\sharp(A(q)) \leq (q+1) \cdot \min\{a, q+1\} - \sum_{i=1}^{x}(\mu_i - 1)$.*

Proof. Fix $Q \in \mathbb{P}^1(q)$ and assume that A has $y \geq 1$ singular points $Q_1, \ldots, Q-y$ with multiplicity $m_1, \ldots, m_y$ such that $Q_i \in A(q) \cap u^{-1}(Q)$ for all i. Then $\sharp(A(q) \cap u^{-1}(Q)) \leq a - \sum_{i=1}^{y}(m_i - 1)$. Copy the proof of Proposition 1. □

Corollary 1. *Fix an integer $e > 0$ and integers $b \geq ae > 0$. Let $A \in |H_{b,a}|(q)$ such that $A(q) = F_e(q)$. Then $a \geq q+1$. If $a = q+1$, then no singular point of A is defined over $\mathbb{F}_q$.*

Proof. Since $\sharp(F_e(q)) = (q+1)^2$, the first statement follows from Proposition 1. For the second statement use Proposition 2. □

Reversing if necessary the two rulings the case $e = 0$ of Propositions 1 and 2 gives the following result.

Corollary 2. *Fix integers $a > 0$ and $b > 0$. Let A be a curve of bidegree (a,b) on a hyperbolic quadric surface and defined over $\mathbb{F}_q$. Assume that A contains no line defined over $\mathbb{F}_q$. Then $\sharp(A(q)) \leq (q+1) \cdot \min\{a, b, q+1\}$. If A has $x \geq 1$ singular points with multiplicities $\mu_1, \ldots, \mu_x$ defined over $\mathbb{F}_q$, then $\sharp(A(q)) \leq (q+1) \cdot \min\{a, b\} - \sum_{i=1}^{x}(\mu_i - 1)$.*

Corollary 2 is sharp (i.e. $\sharp(A(q)) = q+1$) if $\min\{a, b\} = 1$, because A is a smooth rational curve and (calling u the ruling associated to the integer

$\min\{a,b\})$, $u|A(q) : A(q) \to \mathbb{P}^1(q)$ is a bijection in this case.

Lemma 1. *Fix integers a, b such that $a \le q$ and $b \le q$. Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric. Then $h^0(X, \mathcal{I}_{X(q)}(a,b)) = 0$.*

Proof. For any $D \in |\mathcal{O}_X(0,1)|(q)$ we have $\sharp(D \cap X(q)) = q+1$. Since $a \le q$, any such line is contained in the base locus of $|\mathcal{I}_{X(q)}(a,b)|$. Since $b \le q$, we get $|\mathcal{I}_{X(q)}(a,b)| = \emptyset$. □

Lemma 2. *Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric. Then $h^0(X, \mathcal{I}_{X(q)}(q,q)) = h^1(X, \mathcal{I}_{X(q)}(q,q)) = 0$.*

Proof. Since $\sharp(X(q)) = (q+1)^2 = h^0(X, \mathcal{O}_X(q,q))$, we have

$$h^0(X, \mathcal{I}_{X(q)}(q,q)) = h^1(X, \mathcal{I}_{X(q)}(q,q)).$$

Lemma 1 gives $h^0(X, \mathcal{I}_{X(q)}(q,q)) = 0$. □

Corollary 3. *Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric. Fix $P \in X(q)$ and any $S \subseteq X(q)$. Then $h^1(X, \mathcal{I}_S(q,q)) = 0$, $h^0(X, \mathcal{I}_{X(q)\setminus\{P\}}(q,q)) = 1$, and the unique $A \in |\mathcal{I}_{X(q)\setminus\{P\}}(q,q)|$ containing $X(q)\setminus\{P\}$ is the union of the q lines of bidegree $(0,1)$ and the q lines of bidegree $(1,0)$ defined over $\mathbb{F}_q$ and not containing P.*

Proof. Since $S \subseteq X(q)$, Lemma 2 gives $h^1(X, \mathcal{I}_S(q,q)) = 0$. Taking $S := X(q)\setminus\{P\}$ we get $h^0(X, \mathcal{I}_{X(q)\setminus\{P\}}(q,q)) = 1$. Hence the element of $|\mathcal{I}_{X(q)\setminus\{P\}}(q,q)|$ described in the statement of Corollary 3 is the unique element of $|\mathcal{I}_{X(q)\setminus\{P\}}(q,q)|$. □

Proposition 3. *For all integers $e > 0$, $a \ge 0$ and $b \ge q+1$ there is $A \in |H_{b,a}|(q)$ such that $F_e \subseteq A$.*

Proof. Set $D := u^{-1}(\mathbb{P}^1(q))$. If $b = a+1$, then take $A := D + ah$. If $b > 1$, then fix $P \in \mathbb{P}^1(q)$ and set $A := D + (b-q-1)u^{-1}(P) + ah$. □

Proposition 4. *Fix integers $a \ge 0$ and $b \ge 0$. Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric surface.*

(a) If $\max\{a,b\} \ge q+1$, then $M(a,b,q)_1 = (q+1)^2$.

(b) If $\max\{a,b\} \le q$, then $M(a,b,q)_1 = (a+b)(q+1) - ab$ and $A \in |\mathcal{O}_X(a,b)|(q)$ computes $M(a,b,q)_1$ if and only if it is the union of a distinct lines of bidegree $(1,0)$ and b distinct lines of bidegree $(0,1)$, all of them defined over $\mathbb{F}_q$.

Proof. Part (a) follows from the proof of Proposition 3, exchanging the two rulings if $b \le q$. The inequality $M(a,b,q)_1 \ge (a+b)(q+1) - ab$ is obvious, taking a union of a distinct lines of bidegree $(1,0)$ and b distinct lines of bidegree

$(0,1)$, all of them defined over $\mathbb{F}_q$. To get the reverse inequality in part (b) we fix $A \in |\mathcal{O}_X(a,b)|(q)$. If A contains no line of bidegree $(0,1)$ defined over $\mathbb{F}_q$, then $\sharp(A(q)) \le a(q+1)$ (Corollary 2). Since $b \le q$, we get $\sharp(A(q)) < (a+b)(q+1)-ab$ in this case. Now assume that A contains exactly c lines of bidegree $(0,1)$ defined over $\mathbb{F}_q$. Call A'' their union and write $A = A'+A''$ with $A'' \in |\mathcal{O}_X(a,b-c)|(q)$. We have $\sharp(A''(q)) = c(q+1)$. Since $\sharp(u(A'(q)\backslash(A' \cap A'')(q))) \le q+1-c$, the proof of Proposition 1 gives $\sharp(A'(q)\backslash A''(q)) \le a(q+1-c)$. Hence $\sharp(A(q)) \le a(q+1-c)+c(q+1) = a(q+1)+c(q+1-a)$. The function $\psi_{q,a} : \mathbb{Z} \to \mathbb{Z}$ defined by $\psi_{q,a}(c) = a(q+1)_+c(q+1-a)$ is strictly increasing in the interval $[0,q]$. Hence $\psi_{q,a}(b) > \psi_{q,a}(c)$ for all $c < b$. Hence $\sharp(A(q)) < (a+b)(q+1)-ab$, unless $c = b$. In this case A'' has bidegree $(a,0)$ and we easily get that the maximal $\sharp(A(q))$ is achieved if and only if A is reduced and all its lines are defined over $\mathbb{F}_q$. □

Proposition 5. *Fix integer a, a' such that $1 \le a \le q+1$, $1 \le a' \le q+1$ and $a+a' \le 2q$. Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric surface. Let $S \subset X(q)$ be the complete intersection of the union a distinct lines of bidegree $(1,0)$ and of the union of a' distinct lines of bidegree $(0,1)$, all of them defined over $\mathbb{F}_q$. Then $h^0(X, \mathcal{I}_S(a,a')) = a+a'+1$, $h^1(X, \mathcal{I}_S(a,a')) = 0$ and there is $A \in |\mathcal{I}_S(a,a')|(q)$ which is smooth at each point of S.*

Proof. The proof of Lemma 2 gives $h^1(X, \mathcal{I}_S(a,a')) = 0$. Thus

$$h^0(X, \mathcal{I}_S(a,a')) = a+a'+1.$$

Hence $\sharp(|\mathcal{I}_S(a,a')|(q)) = (q^{a+a'+1}-1)/(q-1)$. Fix $A \in |\mathcal{I}_S(a,a)|(q)$ and $P \in X(q)$. If A is singular at P, then A has as a component the union D_P of the two lines of X passing through P. Let A_P be the other component. Since $P \in S$, D_P is defined over $\mathbb{F}_q$. Hence A_P is defined over $\mathbb{F}_q$. Set $S_P := S\backslash(D_P(q)$. We have $\sharp(S_P) = (a-1)(a'-1)$ and S_P is the complete intersection of the union of $a-1$ lines of bidegree $(0,1)$ and the union of $a'-1$ lines of bidegree $(1,0)$. Hence $h^1(X, \mathcal{I}_{S_P}(a-1,a'-1)) = 0$ and $h^0(X, \mathcal{I}_{S_P}(a-1,a'-1)) = aa'-(a-1)(a'-1) = a+a'-1$. Hence $\sharp(|\mathcal{I}_{S_P}(a-1,a'-1)|(q)) = (q^{a+a'-1}-1)/(q-1)$. Since $a+a' \le 2q$, we have $aa' \le q^2$. Thus $aa' \cdot (q^{a+a'-1}-1)/(q-1) \le q^2 \cdot (q^{a+a'-1}-1)/(q-1) < (q^{a+a'+1}-1)/(q-1)$. Since $\sharp(S) = aa'$, we conclude. □

Remark 2. Fix integers a, b such that $1 \le a \le q+1$ and $1 \le b \le q+1$. Let $X \subset \mathbb{P}^3$ be a hyperbolic quadric surface. Let $S \subseteq X(q)$ be the complete intersection of the union a distinct lines of bidegree $(1,0)$ and of the union of b distinct lines of bidegree $(0,1)$, all of them defined over $\mathbb{F}_q$. Let $A(q,a,b)_S$ be the set of all $A \in |\mathcal{I}_S(a,b)|(q)$ which are smooth at each point of S. Lemma 5 gives $A(q,a,b)_S \ne \emptyset$ if $a+b \le 2q$. The integer $\alpha(q,a,b) := \sharp(A(q,a,b)_S)$

only depends from q, a and b, not the choice of the complete intersection S. Proposition 5 gives $\alpha(q,a,b) > 0$ if $a+b \le 2q$. Since $\dim(|\mathcal{I}_P(1,1)|) = 2$ for all $P \in X$, we have $\alpha(q,1,1) = (q^3-1)/(q-1) - 1 = q^2+q$. The proof of Proposition 5 gives

$$\alpha(q,a,b) = (q^{a+b+1}-1)/(q-1) - \sum_{i=1}^{a-1}\sum_{j=1}^{b-1}\binom{a}{i}\binom{b}{j}\cdot\alpha(q,i,j) \quad (1)$$

for all integers a,b such that $1 \le a \le q+1$, $1 \le b \le q+1$ and $(a,b) \ne (1,1)$. One could recursively compute all integers $\alpha(q,a,b)$ using (1).

Proof of Theorem 1. Fix $M \subset \mathbb{P}^1(\overline{\mathbb{F}}_q))$ such that $\sharp(M) = a$, M is defined over $\mathbb{F}_q$, but each $P \in M$ has $\mathbb{F}_{q^a}$ as the minimal extension of $\mathbb{F}_q$ on which it is defined. Set $D := \pi_2^{-1}(M)$. Thus $D \subset X$ be a curve of bidegree $(a,0)$ defined over $\mathbb{F}_q$. Since $M(q) = \emptyset$, $D(q) = \emptyset$. Set $B := \pi_1^{-1}(\mathbb{P}^1(q))$

(a) Assume $b = q+1$. Set $A := B+D$. Since $X(q) \subset B$, $A \in |\mathcal{I}_{X(q)}(a,b)|(q)$. Since B is smooth and $D(q) = \emptyset$, A is smooth at each point of $X(q)$.

(b) Assume $b \ge q+3$. Fix $N \subset \mathbb{P}^1(\overline{\mathbb{F}}_q))$ such that $\sharp(N) = b-q-1$, M is defined over $\mathbb{F}_q$, but each $P \in N$ has $\mathbb{F}_{q^{b-q-1}}$ as the minimal extension of $\mathbb{F}_q$ on which it is defined. Set $D' := \pi_1^{-1}(N)$. Since $D'(q) = \emptyset$, we may take $A := B + D + D'$.

(c) Assume $b = q+2$. If $a \ne q+2$, then we may apply steps (a) or (b) exchanging the two rulings of X. Hence we cover all cases with $(a,b) \ne (q+2,q+2)$. □

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