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AN ESTIMATE FOR THE COMMUTATOR $[A^{\alpha}, B]$

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Abstract: Let A and B be operators in $\mathbf{B}(\mathfrak{H})$, with A positive and invertible. We produce an estimate for the commutator $[A^{\alpha}, B]$ in terms of $\|[A, B]\|$ when $0 < \alpha < 1$. This estimate, when combined with a result for $\alpha \in \mathbb{N}$, establishes an estimate for any $\alpha \in \mathbb{R}^{+}$.

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1. Introduction

Let $\mathfrak{H}$ be a Hilbert space with $A, B \in \mathbf{B}(\mathfrak{H})$ bounded linear operators on $\mathfrak{H}$. Let A be positive: it is normal and its spectrum lies on the nonnegative real axis. Estimates for the commutator $[A^{\alpha}, B]$ have been made when α is not an integer. In particular, Pedersen showed in [5] that when $0 < \alpha < 1$, we have

$$\|[A^{\alpha}, B]\| \leq \frac{5}{4}\|B\|^{1-\alpha}\|[A, B]\|^{\alpha}.$$

We produce another estimate for $[A^{\alpha}, B]$ when $0 < \alpha < 1$, assuming that A is invertible as well as positive. Our result uses a technique known as quantization

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that has been employed for similar purposes, as in [3]. Furthermore, we extend this estimate to one valid for any $\alpha \in \mathbb{R}^+$.

2. Quantization

We first describe a device useful for establishing operator identities. Using this technique, algebraic manipulations of commuting objects correspond to algebraic operations on noncommuting linear operators via a "quantization map." We may thus work in a commutative setting, then quantize to return to noncommutative operators, in the spirit of the operational calculus.

Let $A, B \in \mathbf{B}(\mathfrak{H})$, with A normal. Introduce the formal objects A_L and A_R, to be thought of as "A on the left" and "A on the right", respectively; we also introduce the formal adjoints A_L^* and A_R^*. Set $G = \{I, A_L, A_L^*, A_R, A_R^*, B\}$, where I is the identity operator. Form a commutative algebra $\mathfrak{A}$ from the closure under finite formal sums and products of G and symbols $f(T)$, where each f is a continuous function on $\mathbf{B}(\mathfrak{H})$ and $T \in G \setminus \{I, B\}$.

Let $w \in \mathfrak{A}$ be a nonempty word of objects in G and symbols $f_j(T)$. To define the quantization map, we must keep track of the symbols in w. To this end, let N_S be the number of occurrences of each $S \in G$, and for each $T \in G \setminus \{I, B\}$ let Π_T be the product of all operators $f(A)$ for which $f(T)$ occurs in w. We then define the *quantization* map φ on words w by

$$\varphi(w) = A^{N_{A_L}} A^{*N_{A_L^*}} \Pi_{A_L} \Pi_{A_L^*} \cdot B^{N_B} \cdot A^{N_{A_R}} A^{*N_{A_R^*}} \Pi_{A_R} \Pi_{A_R^*},$$

stipulating that it be $\mathbb{C}$-linear on finite sums of words. Since A is normal and each f_j is approximately a polynomial, the quantities $A, A^*, f_j(A), f_j(A^*)$ commute, and φ is well-defined. Note that φ is not defined on all of $\mathfrak{A}$, nor does it surject onto $\mathbf{B}(\mathfrak{H})$. We shall refer to the application of φ as *quantization.*

Additions and scalar multiplications performed in $\mathfrak{A}$ are preserved in $\mathbf{B}(\mathfrak{H})$ due to the linearity of φ. Because only the number, not the order, of multiplications of objects in the argument of φ affects the output, the commutation of elements in $\mathfrak{A}$ does not conflict with the noncommutativity of $\mathbf{B}(\mathfrak{H})$.

3. Trivial Case

We estimate $\|[A^\alpha, B]\|$ when $\alpha = n \in \mathbb{N}$ and A is normal, first using a commutator identity, then with the quantization technique.

Repeatedly applying the identity $[xy, z] = x[y, z] + [x, z]y$ to $[A^n, B]$ yields

$$[A^n, B] = \sum_{j=0}^{n-1} \binom{n-1}{j} A^j [A, B] A^{n-j-1},$$

$$\begin{aligned}\|[A^n, B]\| &\leq \sum_{j=0}^{n-1} \binom{n-1}{j} \|A^j\| \|[A, B]\| \|A^{n-j-1}\| \\ &\leq \sum_{j=0}^{n-1} \binom{n-1}{j} \|A\|^{n-1} \|[A, B]\|. \qquad (1)\end{aligned}$$

This bound can be made sharper, however. Let x and y be real variables. For reasons soon to be clarified, we seek a decomposition for $(x^n - y^n)(x - y)^{-1}$ of the form $\sum_{j=1}^{k} p_j(x) q_j(y)$, for $k \in \mathbb{N}$, and p, q complex-valued rational functions. Using polynomial division, we obtain

$$\frac{x^n - y^n}{x - y} = \sum_{j=1}^{n} x^{n-j} y^{j-1}.$$

Substitute $A_L = x$, $A_R = y$, multiply by $B(A_L - A_R)$, and quantize

$$BA_L^n - BA_R^n = \sum_{j=1}^{n} \left(BA_L^{n-j+1} A_R^{j-1} - BA_R A_L^{n-j} A_R^{j-1} \right),$$

$$A_L^n B - BA_R^n = \sum_{j=1}^{n} \left(A_L^{n-j+1} B A_R^{j-1} - A_L^{n-j} B A_R^{j} \right),$$

$$[A^n, B] = \sum_{j=1}^{n} A^{n-j} \, [A, B] \, A^{j-1}.$$

The estimate is then

$$\begin{aligned}\|[A^n, B]\| &\leq \sum_{j=1}^{n} \|A^{n-j}\| \|[A, B]\| \|A^{j-1}\| \\ &\leq n \|A\|^{n-1} \|[A, B]\|. \qquad (2)\end{aligned}$$

This bound is equivalent to (1) for $n = 2$, and strictly smaller for $n \geq 3$; the quantization device may be advantageous. We note that Section 3 of [2] has a somewhat related approach, used there to study Fréchet differentials, and that this estimate follows from a more general result found in [4, Proposition 2.11].

4. Discussion of Main Problem

The problem of estimating $\|[A^\alpha, B]\|$ for any $\alpha \in \mathbb{R}^+$ may be reduced to considering $0 < \alpha < 1$ as follows. Let n and β be the integral and fractional parts of α, respectively. Then

$$\begin{aligned} [A^\alpha, B] &= A^\beta[A^n, B] + [A^\beta, B]A^n, \\ \|[A^\alpha, B]\| &\leq \|A^\beta\|\|[A^n, B]\| + \|[A^\beta, B]\|\|A^n\| \\ &\leq n\|A\|^{n+\beta-1}\|[A, B]\| + \|[A^\beta, B]\|\|A^n\|; \end{aligned} \tag{3}$$

we have used (2), and $\|A^\beta\| \leq \|A\|^\beta$. The only unknown quantity is $\|[A^\beta, B]\|$.

So let $A, B \in \mathbf{B}(\mathfrak{H})$ with A is positive and invertible, and $0 < \beta < 1$. Let x, y be positive real variables. We proceed as in the $\alpha \in \mathbb{N}$ case, but whereas a finite sum decomposition was employed there, we now use the integral identity

$$\frac{x^\beta - y^\beta}{x - y} = -\frac{1}{\pi}\sin((\beta+1)\pi) \cdot \int_0^\infty \frac{\lambda^\beta}{(x+\lambda)(y+\lambda)} \mathrm{d}\lambda,$$

valid for $x, y \in \mathbb{R}^+$ and $0 < \beta < 1$, see [1]. Let $k = -\pi^{-1}\sin((\beta+1)\pi)$ and substitute $A_L = x$, $A_R = y$. Multiply by $B(A_L - A_R)$, quantize, then take norms

$$\begin{aligned} A_L^\beta B - BA_R^\beta &= k\int_0^\infty \lambda^\beta \left(\frac{A_L}{A_L+\lambda}B\frac{1}{A_R+\lambda} - \frac{1}{A_L+\lambda}B\frac{A_R}{A_R+\lambda}\right)\mathrm{d}\lambda, \\ [A^\beta, B] &= k\int_0^\infty \lambda^\beta \frac{1}{A+\lambda}[A, B]\frac{1}{A+\lambda}\mathrm{d}\lambda, \\ \|[A^\beta, B]\| &\leq |k|\int_0^\infty \left\|\lambda^\beta \frac{1}{A+\lambda}[A, B]\frac{1}{A+\lambda}\right\|\mathrm{d}\lambda \\ &\leq |k|\|[A, B]\|\int_0^\infty |\lambda^\beta| \left\|\frac{1}{A+\lambda}\right\|^2\mathrm{d}\lambda. \end{aligned}$$

Since $\lambda \geq 0$, we have $\|(A+\lambda)^{-1}\| = (\mu+\lambda)^{-1}$, where $\mu = \min \mathrm{sp}A$. Then

$$\begin{aligned} \|[A^\beta, B]\| &\leq |k|\|[A, B]\|\int_0^\infty \lambda^\beta \left(\frac{1}{\mu+\lambda}\right)^2 \mathrm{d}\lambda \\ &\leq |k|\|[A, B]\| \left(\pi\beta\mu^{\beta-1}\csc(\beta\pi)\right). \end{aligned}$$

Because $|k| = |-\pi^{-1}\sin((\beta+1)\pi)| = \pi^{-1}\sin(\beta\pi)$ when $0 < \beta < 1$, we see

$$\|[A^\beta, B]\| \leq \beta\mu^{\beta-1}\|[A, B]\|. \tag{4}$$

The form of this bound is quite similar to that of (2). This relates to the result in [4], but here the estimate involves μ, rather than $\|A\|$.

Noting that $\mu^{-1} = \|A^{-1}\|$, we summarize results (2), (3), and (4) as follows

Theorem 1. *Let $A, B \in \mathbf{B}(\mathfrak{H})$, with A positive and invertible. If $\alpha \in \mathbb{R}^+$ has integral part n and fractional part β, then we may estimate $\|[A^\alpha, B]\|$ by*

$$\|[A^\alpha, B]\| \leq \|A\|^n \left(n\|A\|^{\beta-1} + \beta\|A^{-1}\|^{1-\beta} \right) \|[A, B]\|.$$

If $0 < \alpha < 1$ so that $\alpha = \beta$, this reduces to

$$\|[A^\alpha, B]\| \leq \alpha\|A^{-1}\|^{1-\alpha}\|[A, B]\|;$$

if $\alpha \in \mathbb{N}$ so that $\alpha = n$, it then becomes

$$\|[A^\alpha, B]\| \leq \alpha\|A\|^{\alpha-1}\|[A, B]\|.$$

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