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MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR THIRD ORDER GENERAL TWO-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract: We consider third order nonlinear general two-point boundary value problem on time scales,

$$y^{\Delta\Delta\Delta}(t) + f(t, y(t)) = 0, \qquad t \in [a, b], \quad a < b,$$

subject to the general boundary conditions

$$\begin{aligned} &\alpha_{11}y(a) - \alpha_{12}y(b) = 0, \\ &\alpha_{21}y^{\Delta}(a) - \alpha_{22}y^{\Delta}(b) = 0, \\ &\alpha_{31}y^{\Delta\Delta}(a) - \alpha_{32}y^{\Delta\Delta}(\sigma(b)) = 0, \end{aligned}$$

where the coefficients $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}$, $i = 1, 2$ are real positive constants. We establish the existence of at least one positive solution, and existence of at least three positive solutions for the two-point boundary value problem using Krasnosel'skii and Leggett-Williams fixed point theorems on a cone.

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1. Introduction

The theory of time scales, which has received a lot of attention, was introduced by Stefan Hilger [14] in order to unify the theory on continuous and discrete analysis. Their theory unifies the theories of differential equations and difference equations, and also it is able to extend these classical cases to cases in between, and can be applied on different types of time scales. A time scale $\mathbb{T}$ is an arbitrary closed subset of the real, and the cases when this time scale is equal to the real or to the integers that represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications in the study of population dynamic models. The book on the subject of time scales by Bohner and Peterson [6, 7], summarizes and organizes much of the time scale calculus. By an interval we mean that the intersection of the real interval with a given time scale. The jump operators introduced on a time scale $\mathbb{T}$ may be connected or disconnected. To overcome this topological difficulty, the concept of jump operators is introduced in the following way. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by $\sigma(t)=\inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t)=\sup\{s \in \mathbb{T} : s < t\}$ are called jump operators. If σ is bounded above and ρ is bounded below, then we define $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ and $\rho(\min \mathbb{T}) = \min \mathbb{T}$. These operators allow us to classify the points of time scale $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, left-dense if $\rho(t) = t$, right-scattered if $\sigma(t) > t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$ and dense if $\rho(t) = t = \sigma(t)$.

We consider the existence of positive solutions and multiple positive solutions to third order nonlinear differential equation on time scales,

$$y^{\Delta\Delta\Delta}(t) + f(t, y(t)) = 0, \qquad t \in [a, b], \quad a < b, \tag{1}$$

subject to the two-point boundary conditions

$$\begin{aligned} &\alpha_{11} y(a) - \alpha_{12} y(b) = 0, \\ &\alpha_{21} y^{\Delta}(a) - \alpha_{22} y^{\Delta}(b) = 0, \\ &\alpha_{31} y^{\Delta\Delta}(a) - \alpha_{32} y^{\Delta\Delta}(\sigma(b)) = 0. \end{aligned} \tag{2}$$

The study of the existence of a positive solution or multiple positive solutions of the third order boundary value problems (BVPs) arises in a variety of different areas of applied mathematics and physics. In the modeling of nonlinear diffusion via nonlinear sources, thermal ignition of gases, and in chemical concentrates in biological problem [11]. In these applied settings, only positive solutions are meaningful. The existence of positive solutions to the BVPs is studied by many authors, first for ordinary differential equations, then finite difference equations, and recently, unifying results for dynamic equations. We list some papers, Eloe and Henderson [8, 9, 10], Erbe and Wang [11] for ordinary differential equations, and Anderson and Avery [2], Avery and Peterson [5] for finite difference equations, and Anderson and Avery [3], Erbe and Peterson [12], Sun [18] for time scales. Recently Sun and Wen [17] considered the existence of multiple positive solutions to third order equation,

$$y'''(t) = a(t)f(y(t)), \quad 0 < t < 1,$$

under the boundary conditions

$$\alpha y'(0) - \beta y''(0) = 0, \quad y(1) = y'(1) = 0.$$

We extend these results to general two point boundary value problems on time scales (1) that satisfies (2). We use following notation for simplicity, $\gamma_i = \alpha_{i1} - \alpha_{i2}$, $i = 1, 2, 3$, and $\beta_i = a\alpha_{i1} - b\alpha_{i2}, i = 1, 2$.

We make the following assumptions throughout:

$(A1) f : [a, \sigma^3(b)] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, where $\mathbb{R}^+$ is the set of non negative real numbers.

$(A2)$ $\gamma_i < 0, \quad i = 1, 2, 3.$

$(A3)$The point $t \in [a, \sigma^3(b)]$ is not left-dense and right-scattered at the same time.

We define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 = \lim_{y \to 0^+} \min_{t \in [a, \sigma^3(b)]} \frac{f(t, y)}{y}, \quad f^0 = \lim_{y \to 0^+} \max_{t \in [a, \sigma^3(b)]} \frac{f(t, y)}{y},$$

$$f_\infty = \lim_{y \to \infty} \min_{t \in [a, \sigma^3(b)]} \frac{f(t, y)}{y}, \text{ and } f^\infty = \lim_{y \to \infty} \max_{t \in [a, \sigma^3(b)]} \frac{f(t, y)}{y},$$

and assume that they will exist. We note that $f^0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f^\infty = 0$ correspond to the sublinear case. By the positive solution of (1)-(2) we mean that $y(t)$ is positive on $[a, \sigma^3(b)]$ and satisfies the differential equation (1) together with the boundary conditions (2).

This paper is organized as follows. In Section 2, we estimate bounds for the Green's function. Later we prove a Lemma 2.2 which is needed in the main result. In Section 3, we establish a criteria for the existence of at least one positive solution of the BVP (1)-(2) by using Krasnosel'skii fixed point theorem. In Section 4, some existence criterions for at least three positive solutions to the BVP (1)-(2) are established by using the Leggett-Williams fixed point theorem. Finally as an application, we give some examples to demonstrate our results.

2. Green's Function and Bounds

In this section, we estimate the bounds of the Green's function for the homogeneous two-point boundary value problem corresponding to (1)-(2). Later we prove a Lemma 2.2 which is needed in the later sections.

The Green's function for the homogeneous problem $-y^{\Delta\Delta\Delta} = 0$, satisfying the boundary conditions (2), can be constructed after computation and is given by

$$G(t,s) = \begin{cases} \frac{1}{2\gamma_1\gamma_2\gamma_3}[-\alpha_{12}\gamma_2\gamma_3(\sigma(b)-\sigma(s))(\sigma(b)-\sigma^2(s)) \\ -\alpha_{22}\gamma_3(-\beta_1+t\gamma_1)(\sigma(b)+\sigma^2(b)-\sigma(s)-\sigma^2(s)) \\ -\alpha_{32}(p-t\gamma_1(\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b))+t^2\gamma_1\gamma_2)], \\ \qquad\qquad a \le t \le s \le \sigma^3(b), \\ \frac{1}{2\gamma_1\gamma_2\gamma_3}[-\alpha_{11}\gamma_2\gamma_3(\sigma(s)-a)(\sigma^2(s)-a) \\ +\alpha_{21}\gamma_3(-\beta_1+t\gamma_1)(\sigma(s)+\sigma^2(s)-a-\sigma(a)) \\ -\alpha_{31}(p-t\gamma_1(\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b))+t^2\gamma_1\gamma_2)], \\ \qquad\qquad a \le \sigma(s) \le t \le \sigma^3(b), \end{cases} \tag{3}$$

where $p = \beta_1(\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)) - \gamma_2(a^2\alpha_{11}-b^2\alpha_{12})$.

Theorem 2.1. *Let $G(t,s)$ be the Green's function for the homogeneous problem $-y^{\Delta\Delta\Delta}(t) = 0$ satisfying the boundary conditions* (2). *Then the inequality*

$$\gamma G(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s) \tag{4}$$

holds for all $(t,s) \in [a,\sigma^3(b)] \times [a,b]$, *where* $0 < \gamma = \min\{m_1, m_2\} \le 1$.

Proof. The Green's function $G(t,s)$ for the BVP (1)-(2) is given in equation(3). Clearly,

$$G(t,s) > 0, \quad \text{on} \quad (a,\sigma^3(b)) \times (a,b). \tag{5}$$

For $a \le t \le s \le \sigma^3(b)$, $a \le \sigma(s) \le t \le \sigma^3(b)$ and from $(A2)$ we have

$$G(t,s) \le G(\sigma(s),s).$$

Therefore, we have

$$G(t,s) \le G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [a,\sigma^3(b)] \times [a,b].$$

To establish, the other inequality, for $a \le t \le s \le \sigma^3(b)$ we have from $(A2)$

$$G(t,s) \ge m_1 G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [a,\sigma^3(b)] \times [a,b],$$

where $m_1 = \frac{a_1}{b_1+b_2+b_3}$, and

$$a_1 = \frac{\alpha_{32}}{2\gamma_3}\left[\left(\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right)^2 - \frac{p}{\gamma_1\gamma_2}\right], \quad b_1 = -\frac{\alpha_{12}}{2\gamma_1}(\sigma(b)-a)^2,$$

$$b_2 = -\frac{\alpha_{22}}{\gamma_2}\left[(-\frac{\beta_1}{\gamma_1}+\sigma^3(b))(\sigma^2(b)-a)\right],$$

$$b_3 = -\frac{\alpha_{32}}{2\gamma_3}\left[\left[b-\left(\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right)\right]^2 - \left[\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right]^2 + \frac{p}{\gamma_1\gamma_2}\right]$$

and for $a \le \sigma(s) \le t \le \sigma^3(b)$, we have from $(A2)$

$$G(t,s) \ge m_2 G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [a,\sigma^3(b)] \times [a,b],$$

where

$$m_2 = \frac{c_1}{d_1+d_2+d_3}, \quad c_1 = \frac{\alpha_{31}}{2\gamma_3}\left[\left(\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right)^2 - \frac{p}{\gamma_1\gamma_2}\right],$$

$$d_1 = -\frac{\alpha_{11}}{2\gamma_1}(a-b)^2, \quad d_2 = \frac{\alpha_{21}}{\gamma_2}\left[(-\frac{\beta_1}{\gamma_1}+a)(\sigma^2(b)-a)\right]$$

$$d_3 = -\frac{\alpha_{31}}{2\gamma_3}\left[\left[b-\left(\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right)\right]^2 - \left[\frac{\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)}{2\gamma_2}\right]^2 + \frac{p}{\gamma_1\gamma_2}\right].$$

Therefore we have

$$G(t,s) \ge \gamma G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [a,\sigma^3(b)] \times [a,b],$$

where $0 < \gamma = \min\{m_1, m_2\} \le 1$. □

Let $y(t)$ be the solution of the BVP (1)-(2), and it is given by

$$y(t) = \int_a^{\sigma(b)} G(t,s) f(s,y(s))\Delta s, \quad \text{for all } t \in [a,\sigma^3(b)]. \tag{6}$$

Define

$$X = \left\{ \ y \mid y \in C[a,\sigma^3(b)] \ \right\},$$

with norm

$$\| y \| = \max_{t\in[a,\sigma^3(b)]} | y(t) | .$$

Then $(X, \| \,.\, \|)$ is a Banach space. Define a set κ by

$$\kappa = \left\{ u \in X : u(t) \geq 0 \;\; \text{on } [a, \sigma^3(b)] \;\; \text{and} \;\; \min_{t \in [a, \sigma^3(b)]} u(t) \geq \gamma \| u \| \right\}, \tag{7}$$

then it is easy to see that κ is a positive cone in X.

Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s) f(s, y(s)) \Delta s, \quad \text{for all } t \in [a, \sigma^3(b)]. \tag{8}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (6) and hence y is a positive solution of the BVP (1)-(2). We seek the fixed points of the operator T in the cone κ.

Lemma 2.2. *The operator T defined by* (8) *is a self map on κ.*

Proof. If $y \in \kappa$, then by (4)

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s) f(s, y(s)) \Delta s \leq \int_a^{\sigma(b)} G(\sigma(s), s) f(s, y(s)) \Delta s,$$

then

$$\| Ty \| \leq \int_a^{\sigma(b)} G(\sigma(s), s) f(s, y(s)) \Delta s, \quad t \in [a, \sigma^3(b)].$$

Moreover, if $y \in \kappa$,

$$\begin{aligned} (Ty)(t) &= \int_a^{\sigma(b)} G(t, s) f(s, y(s)) \Delta s \geq \int_a^{\sigma(b)} \gamma G(\sigma(s), s) f(s, y(s)) \Delta s \\ &\geq \gamma \int_a^{\sigma(b)} \max_{t \in [a, \sigma^3(b)]} G(t, s) f(s, y(s)) \Delta s \geq \gamma \max_{t \in [a, \sigma^3(b)]} \int_a^{\sigma(b)} G(t, s) f(s, y(s)) \Delta s \\ &= \gamma \| Ty \| . \end{aligned}$$

Therefore,

$$\min_{t \in [a, \sigma^3(b)]} (Ty)(t) \geq \gamma \| Ty \| .$$

Also, from the positivity of $G(t, s)$, it clear that for $y \in \kappa$, that $(Ty)(t) \geq 0, \; a \leq t \leq \sigma^3(b)$, and so $Ty \in \kappa$; thus $T : \kappa \to \kappa$. Further arguments yield that T is completely continuous. □

3. Existence of Positive Solutions

In the section we establish the existence of at least one positive solution of the BVP (1)-(2) is based on an application of the Krasnosel'skii fixed point theorem

[15].

Theorem 3.1. (Krasnosel'skii) *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either:*

(i) $\| Tu \| \leq \| u \|,\ u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \geq \| u \|,\ u \in K \cap \partial\Omega_2$, *or*

(ii) $\| Tu \| \geq \| u \|,\ u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \leq \| u \|,\ u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *Assume that conditions $(A1)-(A3)$ are satisfied. If $f^0 = 0$ and $f_\infty = \infty$, then the BVP (1)-(2) has at least one positive solution that lies in κ.*

Proof. Let T be the cone preserving, completely continuous operator defined as in (8). Since $f^0 = 0$, we may choose $H_1 > 0$ so that

$$\max_{t\in[a,\sigma^3(b)]} \frac{f(t,y)}{y} \leq \eta_1, \text{ for } 0 < y \leq H_1,$$

where $\eta_1 > 0$ satisfies

$$\eta_1 \int_a^{\sigma(b)} G(\sigma(s), s)\Delta s \leq 1.$$

Thus, if $y \in \kappa$ and $\| y \| = H_1$, then we have

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s) f(s, y(s))\Delta s \leq \int_a^{\sigma(b)} G(\sigma(s), s) f(s, y(s))\Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma(s), s)\eta_1 y(s)\Delta s \leq \eta_1 \int_a^{\sigma(b)} G(\sigma(s), s) \| y \| \Delta s \leq \| y \| .\end{aligned}$$

Therefore,

$$\| Ty \| \leq \| y \| .$$

Now if we let

$$\Omega_1 = \{y \in X : \quad \| y \| < H_1\},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{9}$$

Further, since $f_\infty = \infty$, there exists $\overline{H}_2 > 0$ such that

$$\min_{t\in[a,\sigma^3(b)]} \frac{f(t,y)}{y} \geq \eta_2, \quad \text{for} \quad y \geq \overline{H}_2,$$

where $\eta_2 > 0$ is chosen so that

$$\eta_2\gamma^2 \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s \geq 1.$$

Let

$$H_2 = \max\left\{2H_1, \frac{1}{\gamma}\overline{H}_2\right\},$$

and

$$\Omega_2 = \{y \in X : \quad \| y \| < H_2\},$$

then $y \in \kappa$ and $\| y \| = H_2$ implies

$$\min_{t\in[a,\sigma^3(b)]} y(t) \geq \gamma \| y \| \geq \overline{H}_2,$$

and so

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s \geq \int_a^{\sigma(b)} \gamma G(\sigma(s),s)f(s,y(s))\Delta s \\ &\geq \gamma \int_a^{\sigma(b)} G(\sigma(s),s)\eta_2 y(s)\Delta s \geq \gamma^2\eta_2 \int_a^{\sigma(b)} G(\sigma(s),s) \| y \| \Delta s \geq \| y \| .\end{aligned}$$

Hence,

$$\| Ty \| \geq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{10}$$

Therefore, by part (i) of Theorem 3.1 applied to (9) and (10), T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2\backslash\Omega_1)$ such that $H_1 \leq \| y \| \leq H_2$. This fixed point is the positive solution of the BVP (1)-(2). □

Theorem 3.3. *Assume that conditions* $(A1)-(A3)$ *are satisfied. If* $f_0 = \infty$ *and* $f^\infty = 0$*, then the BVP* (1)-(2) *has at least one positive solution that lies in* κ*.*

Proof. Let T be the cone preserving, completely continuous operator defined as (8). Since $f_0 = \infty$, we choose $J_1 > 0$ such that

$$\min_{t\in[a,\sigma^3(b)]} \frac{f(t,y)}{y} \geq \overline{\eta}_1, \text{ for } 0 < y \leq J_1,$$

where $\overline{\eta}_1\gamma^2 \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s \geq 1$. Then for $y \in \kappa$ and $\| y \| = J_1$, we have

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s \geq \int_a^{\sigma(b)} \gamma G(\sigma(s),s)f(s,y(s))\Delta s \\ &\geq \gamma \int_a^{\sigma(b)} G(\sigma(s),s)\overline{\eta}_1 y(s)\Delta s \geq \gamma^2\overline{\eta}_1 \int_a^{\sigma(b)} G(\sigma(s),s) \| y \| \Delta s \geq \| y \| .\end{aligned}$$

Thus, we may let

$$\Omega_1 = \{y \in X : \| y \| < J_1\},$$

so that

$$\| Ty \| \geq \| y \|, \text{ for } y \in \kappa \cap \partial\Omega_1. \tag{11}$$

Now, since $f^\infty = 0$, there exists $\overline{J}_2 > 0$ so that

$$\max_{t\in[a,\sigma^3(b)]} \frac{f(t,y)}{y} \leq \overline{\eta}_2, \text{ for } y \geq \overline{J}_2,$$

where $\overline{\eta}_2 > 0$ satisfies

$$\overline{\eta}_2 \int_a^{\sigma(b)} G(\sigma(s), s)\Delta s \leq 1.$$

It follows that

$$f(t,y) \leq \overline{\eta}_2 y, \quad \text{for} \quad y \geq \overline{J}_2.$$

We consider two cases:

Case (i). f is bounded. Suppose $L > 0$ is such that $f(t,y) \leq L$, for all $0 < y < \infty$. In this case, we may choose

$$J_2 = \max\left\{2J_1, L\int_a^{\sigma(b)} G(\sigma(s), s)\Delta s\right\},$$

so that $y \in \kappa$ with $\| y \| = J_2$, we have

$$\begin{aligned}(Ty)(t) = \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s &\leq \int_a^{\sigma(b)} G(\sigma(s),s)f(s,y(s))\Delta s \\ &\leq L\int_a^{\sigma(b)} G(\sigma(s),s)\Delta s \leq J_2 = \| y \|,\end{aligned}$$

and therefore

$$\| Ty \| \leq \| y \| .$$

Case (ii). f is unbounded. Choose $J_2 > \max\{2J_1, \overline{J}_2\}$ be such that $f(t,y) \leq f(t,J_2)$, for $0 < y \leq J_2$. Then for $y \in \kappa$ and $\| y \| = J_2$, we have

$$\begin{aligned}(Ty)(t) &= \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s \leq \int_a^{\sigma(b)} G(\sigma(s),s)f(s,y(s))\Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma(s),s)f(s,J_2)\Delta s \leq \int_a^{\sigma(b)} G(\sigma(s),s)\overline{\eta}_2 J_2\Delta s \\ &\leq \overline{\eta}_2 \int_a^{\sigma(b)} G(\sigma(s),s)J_2\Delta s \leq J_2 = \| y \| .\end{aligned}$$

Therefore, in either case we put

$$\Omega_2 = \{y \in X : \| y \| < J_2\},$$

we have

$$\| Ty \| \leq \| y \|, \text{ for } y \in \kappa \cap \partial\Omega_2. \tag{12}$$

Therefore, by the part (ii) of Theorem 3.1 applied to (11) and (12), T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $J_1 \leq \| y \| \leq J_2$. This fixed point is the positive solution of the BVP (1)-(2). □

4. Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the BVP (1)-(2).

Let E be a Banach space with cone P. A map $S : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1 - \lambda)y) \geq \lambda S(x) + (1 - \lambda)S(y),$$

for all $x, y \in P$ and $\lambda \in [0, 1]$. Let α and β be two real numbers such that $0 < \alpha < \beta$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_\alpha = \{y \in P : \| y \| < \alpha\},$$

and

$$P(S, \alpha, \beta) = \{y \in P : \alpha \leq S(y), \| y \| \leq \beta\}.$$

We now state the famous Leggett-Williams fixed point theorem.

Theorem 4.1. *Let $T : \overline{P_{a_3}} \to \overline{P_{a_3}}$ be completely continuous and S be a nonnegative continuous concave functional on P such that $S(y) \leq \| y \|$ for all $y \in \overline{P_{a_3}}$. Suppose that there exist $0 < d < a_1 < a_2 \leq a_3$ such that the following conditions hold.*

(i) $\{y \in P(S, a_1, a_2) : S(y) > a_1\} \neq \emptyset$ and $S(Ty) > a_1$ for all $y \in P(S, a_1, a_2)$;

(ii) $\| Ty \| < d$ for all $y \in \overline{P_d}$;

(iii) $S(Ty) > a_1$ for $y \in P(S, a_1, a_3)$ with $\| Ty \| > a_2$.

Then, T has at least three fixed points y_1, y_2, y_3 in $\overline{P_{a_3}}$ satisfying

$$\| y_1 \| < d, \quad a_1 < S(y_2), \quad \| y_3 \| > d, \quad S(y_3) < a_1.$$

For convenience, we let

$$D = \max_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)\Delta s; \qquad C = \min_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)\Delta s.$$

Theorem 4.2. *Assume that the conditions (A1)-(A3) are satisfied and also there exist real numbers d_0, d_1 and c with $0 < d_0 < d_1 < \frac{d_1}{\gamma} < c$ such that*

$$f(t,y(t)) < \frac{d_0}{D}, \quad \text{for } y \in [0,d_0], \tag{13}$$

$$f(t,y(t)) > \frac{d_1}{C}, \quad \text{for } y \in [d_1,\frac{d_1}{\gamma}], \tag{14}$$

$$f(t,y(t)) < \frac{c}{D}, \quad \text{for } y \in [0,c]. \tag{15}$$

Then the BVP (1)-(2) *has at least three positive solutions.*

Proof. Let the Banach space $E = C[a,\sigma^3(b)]$ be equipped with the norm

$$\| y \| = \max_{t\in[a,\sigma^3(b)]} | y(t) | .$$

We denote

$$P = \{y \in E : y(t) \geq 0,\ t \in [a,\sigma^3(b)]\}.$$

Then, it is obvious that P is a cone in E. For $y \in P$, we define

$$S(y) = \min_{t\in[a,\sigma^3(b)]} | y(t) |,$$

and

$$(Ty)(t) = \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s, \quad t \in [a,\sigma^3(b)].$$

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y) \leq \| y \|$ for $y \in P$ and that $T : P \to P$ is a completely continuous and the fixed points of T are the solutions of the BVP (1)-(2).

First, we prove that if there exists a positive number r such that $f(t,y(t)) < \frac{r}{D}$ for $y \in [0,r]$, then $T : \overline{P_r} \to P_r$. Indeed, if $y \in \overline{P_r}$, then for $t \in [a,\sigma^3(b)]$,

$$\begin{aligned}(Ty)(t) = \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s &< \frac{r}{D}\int_a^{\sigma(b)} G(t,s)\Delta s \\ &\leq \frac{r}{D}\max_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)\Delta s = r.\end{aligned}$$

Thus, $\| Ty \| < r$, that is, $Ty \in P_r$. Hence, we have shown that if (13) and (15) hold, then T maps $\overline{P_{d_0}}$ into P_{d_0} and $\overline{P_c}$ into P_c.

Next, we show that $\{y \in P(S,d_1,\frac{d_1}{\gamma}) : S(y) > d_1\} \neq \emptyset$ and $S(Ty) > d_1$ for all $y \in P(S, d_1, \frac{d_1}{\gamma})$. In fact, the constant function is

$$\frac{d_1 + \frac{d_1}{\gamma}}{2} \in \{y \in P(S, d_1, \frac{d_1}{\gamma}) : S(y) > d_1\}.$$

Hence it is nonempty. Moreover, for $y \in P(S, d_1, \frac{d_1}{\gamma})$, we have

$$\frac{d_1}{\gamma} \geq \| y \| \geq y(t) \geq \min_{t\in[a,\sigma^3(b)]} y(t) = S(y) \geq d_1,$$

for all $t \in [a, \sigma^3(b)]$. Thus, in view of (14) we see that

$$S(Ty) = \min_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s$$
$$> \frac{d_1}{C} \min_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)\Delta s = d_1,$$

as required.

Finally, we show that if $y \in P(S, d_1, c)$ with $\| Ty \| > \frac{d_1}{\gamma}$, then $S(Ty) > d_1$. To see this, we suppose that $y \in P(S, d_1, c)$ and $\| Ty \| > \frac{d_1}{\gamma}$, then, by Theorem 2.1, we have

$$S(Ty) = \min_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s \geq \gamma \int_a^{\sigma(b)} G(\sigma(s),s)f(s,y(s))\Delta s$$
$$\geq \gamma \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s,$$

for all $t \in [a, \sigma^3(b)]$. Thus

$$S(Ty) \geq \gamma \max_{t\in[a,\sigma^3(b)]} \int_a^{\sigma(b)} G(t,s)f(s,y(s))\Delta s = \gamma \| Ty \| > \gamma \frac{d_1}{\gamma} = d_1.$$

Hence the hypotheses of the Leggett Williams Theorem 4.1 are satisfied, and therefore T has at least three fixed points, that is, the BVP (1)-(2) has at least three positive solutions u, v and w such that

$$\| u \| < d_0, \quad d_1 < \min_{t\in[a,\sigma^3(b)]} v(t), \quad \| w \| > d_0, \quad \min_{t\in[a,\sigma^3(b)]} w(t) < d_1. \qquad \square$$

5. Examples

Now, we give some examples to illustrate the main results.

Example 1. Consider the following boundary value problem

$$\begin{aligned} y^{\Delta\Delta\Delta} + y^2(1+9e^{-8y}) &= 0, \\ \frac{9}{2}y(0) - 5y(1) &= 0, \\ 2y^{\Delta}(0) - 3y^{\Delta}(1) &= 0, \\ y^{\Delta\Delta}(0) - 2y^{\Delta\Delta}(\sigma(1)) &= 0. \end{aligned} \tag{16}$$

It is easy to see that all the conditions of Theorem 3.2 hold. It follows from Theorem 3.2, the BVP (16) has at least one positive solution.

Example 2. Consider the following boundary value problem

$$\begin{aligned} y^{\Delta\Delta\Delta} + \frac{100(y+1)}{16(y^2+1)} &= 0, \\ y(0) - 3y(1) &= 0, \\ y^{\Delta}(0) - 2y^{\Delta}(1) &= 0, \\ y^{\Delta\Delta}(0) - 2y^{\Delta\Delta}(\sigma(1)) &= 0. \end{aligned} \tag{17}$$

A simple calculation shows that $\gamma = 0.1$. If we choose $d_0 = \frac{1}{2}$, $d_1 = 19$, then the conditions (13)-(15) are satisfied. Therefore, it follows from Theorem 4.2 that the BVP (17) has at least three positive solutions.

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