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CLASSICAL FUNCTION THEORY AND APPLIED PROOF THEORY

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Abstract: In this note, I discuss U. Kohlenbach's work *Applied Proof Theory. Proof Interpretations and their Use in Mathematics* [14] from a foundational and historical perspective. I put the emphasis on Kronecker's constructivist programme which is not mentioned by Kohlenbach and I show that Kronecker's idea of finitist foundations survives beyond Hilbert's programme in the foundations of mathematics. The idea of proof mining is meant to underline the project of extracting effective bounds from ineffective classical proofs in analysis. The logical tools used in that perspective are reviewed in a critical spirit.

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1. Introduction

It is not generally known that Kronecker is at the origin of fixed-point theory, but Brouwer and Hadamard acknowledged their debt to Kronecker's index or *indicatrix* [2]. Kronecker introduced the notion of winding number (*Windungszahl*) in his 1869 work on functions of several variables [17], following suit to Cauchy's results in the 1830's. In contemporary idiom it is called an index or a topological degree for a continuous function to itself on the closed unit ball D^n such that it has at least one fixed point. Brouwer's fixed-point theorem is not

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constructive, but the approximate fixed-point theorem is. Brouwer's statement at the end of [2] states that:

> Eine eindeutige und stetige Transformation eines n-dimensionalen Elementes in sich besitzt sicher einen Fixpunkt.

There are many extensions of Brouwer's work up to Schauder's fixed-point theorem on normed vector spaces and up to the general results of Krasnoselski and Browder. For sure, Brouwer, Poincaré and Hadamard have recognized their debt to Kronecker who used a notion of progression (*Fortgangsprinzip*) for continuous closed curves. Such a notion is also central to approximation theory, a field closely connected to fixed-point theory. Kronecker's approximation theory, like fixed-point theory, is a good candidate for constructivization in the sense of Kronecker, since it deals essentially with approximating polynomials to classical (e.g. trigonometric) functions. Kronecker dealt primarily with quadratic homogeneous polynomials, that is the theory of forms which was the main object of his arithmetical theory of algebraic quantities [16].

Take for example Chebyschev polynomials

$$f(x) \sim \sum_{i=0}^{\infty} c_i T_i(x)$$

for coefficients c and terms T. This would be a good starting point for applied proof theory, but Kohlenbach chooses a more general setting for fixed-point theory and approximation theory in concentrating on complete, separable metric spaces, that is so-called Polish spaces and their representation. Here enters the constructive or computational ingredient and the logical analysis of applied proof theory will proceed with the business of extracting bounds with moduli of uniform continuity on rational numbers. The logical methods that are employed in that enterprise were initiated by G, Kreisel, H. Luckhardt and others, but they are not all constructive in the proof mining project and I want to insist in the following on two such proof-theoretic methods that have proven successful in the tradition of Hilbert's metamathematics or theory of formal systems where Hilbert conceived of conceptual tools to analyse mathematical theories.

The tools in question are logical — to be distinguished from combinatorial or purely arithmetical and polynomial means. Extraction of bounds from existing "unbounded" proofs is performed using logical techniques stemming mainly from Herbrand's theorem and Gödel's functional (*Dialectica*) interpretation. Other notions like Kreisel's no-counterexample interpretation, modified realizability and Howard's majorazibilty could be seen as derived from those two primary sources.

The logical profit is banked on the informational content of the extractive procedure and this is of course more apparent in fully non-constructive classical analysis than it is, for instance, in number theory. In elementary number theory — elementary because it does not use analytical means or methods — logic is not involved in the unwinding of proofs, since number theory can do it on its own. For example, Euler's theorem on the infinity of primes (after Euclid's constructive proof) obtains with a minorization ($\log \log x$) on the divergent series of inverses of primes

$$\sum_{p \leq x} 1/p \geq \log \log x - \log 2 (x \geq 2).$$

A similar example is given by Kohlenbach (see [14] pp. 15-16) and can serve as a motto for the extractive practice: in order to get the best computable results, define sharply the interval of values. It is not generally known (and not mentioned by Kohlenbach) that Kronecker, the pioneer of constructive mathematics, has insisted on such a procedure. In his criticism of Bolzano's theorem on intermediate values, Kronecker vilifies Bolzano for having used the crudest means (*den rohesten Mitteln*) to obtain an analytical result which cannot be applied to the roots of an entire function. In his refinement of Sturm's theorem on the change of signs in the real roots of an algebraic equation, Kronecker calls for the localization (*Isolierung*) of real roots in an algebraic equation with the help of multiple equations and inequations (see [5]). The idea of localization of roots could also be found in Selberg's elementary proof of Dirichlet's analytical proof on the distribution of primes in arithmetic progressions, where Selberg uses only elementary properties of the logarithmic function. In that context, Kohlenbach recalls some recent results by Avigad, Gerhardy and Towsner using logical analysis on the classical ergodic theorem in connection with the Green-Tao theorem on the existence of arbitrary long progressions in the distribution of primes. A more recent result by Kohlenbach (and Leustean) applies the functional interpretation to obtain more quantitative information on the mean ergodic theorem. As far as number theory is concerned, Kohlenbach mentions extractive variations of van der Waerden theorem (on arithmetic progressions in sets) — in this case the variation has produced a better bound, but no new information on the combinatorial content of the proof. A more informative result using a Herbrand analysis was obtained by H. Luckhardt on Roth's theorem on the rational approximations of an algebraic irrational number. This last result is certainly in line with a Kroneckerian spirit. Kohlenbach also mentions a more speculative potential mining of the Tanyama-Weil theorem in arithmetic geometry including Wiles's proof on Fermat's last theorem. Here one must add a reminder to the logician. André Weil has stressed the effectiveness of number-theoretic results by Fermat's method of infinite descent, for instance in the theory of finite fields, and Weil's own results have influenced considerably recent work in model theory. No mention is made of infinite descent in Kohlenbach's book and proof-theorists are warned that Weil did not deem the diagonal method to be a valid method of proof in number theory.

2. Herbrand's Theorem and Gödel's Functional (*Dialectica*) Interpretation

I give a brief treatment of Herbrand's formulation. Let A be a formula in prenex form, for instance

$$A \equiv \exists x \forall y \exists z \forall t B(x, y, z, t)$$

with B quantifier-free. Introduce two new function letters with f unary and g binary with terms $U_1 \dots U_n$, $W_1 \dots W_n$ then A is provable in predicate calculus in the form

$$A \equiv B(U_1, f(U_1), W_1, g(U_1, W_1)) \vee \dots \vee B(U_n, f(U_n), W_n, g(U_n, W_n)).$$

This disjunction is derivable in propositional calculus and may be used as a criterion of refutability in a *negative* interpretation (see [12] pp. 170 and ss.). The negation of A is

$$\neg A \equiv \forall x \exists y \forall z \exists t \neg B(x, y, z, t)$$

or

$$\neg A \equiv \neg B(x, f(x), z, g(x, z))$$

and while Herbrand tought of propositional formulas as refutable in an infinite or indefinite recursive domain (*champ infini*) [10], Kreisel has introduced the no-counterexample interpretation as a functional interpretation of higher type: the type recursive functionals are simply defined by

$$B_{x_1 \dots x_n}[F_1(f_1, \dots, f_n), \dots, F_n(F_1, \dots, F_n)]$$

with B open. For a true formula A, we have

$$B[F(f, g), f(F(f, g)), G(F, g), g(F(f, g), G(f, g))],$$

where the F's and the G's are obviously our new type recursive functionals.

This last formula A is true if there is no counterexample of the form

$$\neg B[x, f(x), z, g(x, z)]$$

with f and g being arguments of the higher-type recursive functionals F and $G \cdot H$; F and G are continuous and we cannot quantify over all such functionals — by diagonalization there is a recursive functional which is distinct from all recursive functionals. I contrast the no-counterexample interpretation with Gödel's own functional interpretation with which it has strong affinities.

Gödel introduced functionals (recursive functions of higher types) over all finite types as abstract objects beyond the (concrete) natural numbers. The *Dialectica* interpretation has been extended by Spector, Howard and Kreisel and others in the intuitionistic spirit of bar-induction and bar-recursion of finite type. Although Gödel was animated by intuitionistic motives, his proof for the consistency of Heyting arithmetic can be translated for Peano arithmetic where its constructive content is carried over.

Gödel states in a phrase reminiscent of Gentzen that the notion of accessibility (*Erreichbarkeit*) is an abstract concept which requires a kind of reflection on finite constructions [7]. Such a notion is the notion of a computable functional of finite type over the integers, which Gödel substitutes for the abstract notions of assertion and proof in intuitionistic mathematics. Formulas like

$$F' = \forall x \exists y A[x, y, z]$$

and

$$G' = \forall w \exists v B[w, v, u]$$

will be used to obtain a consistent interpretation of Heyting's arithmetic: for example, we have

$$(F \supset G)' = \forall y, w \exists V, Z[A(y, Z(yw), x) \supset B(V(y), w, u)]$$

and

$$\neg(F)' = \exists y \exists \bar{Z} \neg A(y, \bar{Z}(y), x),$$

where x, yz and w are finite sequences of variables of arbitrary type, u is a sequence of number variables while Y, V, Z and $\bar{Z}$ are second-order variables — A and B are quantifier-free. Those *generalized* formulas, as Shoenfield says (see [21] p. 218), constitute the functional interpretation. Gödel defines the finite types inductively with the following three clauses:

1) 0 is a finite type (the type of integers).

2) if s and t are finite types, then $s \times t$ (their Cartesian product) is a finite type.

3) if s and t are finite types, then $s \to t$ is also a finite type.

Remark. The third clause means that there is a mapping from functionals of type s to functionals of type t.

3. The Continuation of Hilbert's Programme

Herbrand's theorem and Gödel's functional interpretation were devised as a response to to Hilbert's programme. Herbrand wanted a finitist consistency proof for arithmetic and his theorem for predicate logic is in the line of the Hilbert-Ackermann consistency theorem which says:

> An open theory T is inconsistent iff there is a quasi-tautology which is a disjunction of negations of nonlogical axioms of T.

An open theory T has its nonlogical axioms quantifier-free and a quasi-tautology is a tautology derived from equality axioms. Herbrand's theorem holds for quasi-tautologies in classical first-order predicate calculus with equality and it is in this form that it is employed in applied proof theory. Gödel proposed his quantifier-free *Dialectica* interpretation to overcome the finitist framework by introducing functionals as abstract types beyond the natural numbers or the concrete objects of a Hilbertian formal system. The shift of emphasis initiated by Kreisel and the course taken by Kohlenbach consist in focusing on the existential quantifier in formulas in order to "scoop in" the parameters of a given proof in functional terms. There seems to be a task more modest than Hilbert's grand design of the consistency problem couched in the universal quantifier — the existential quantifier can be more easily eliminated! Here again a reminder: Hilbert had conceived a polynomial calculus (the epsilon calculus) subjected to infinite descent in order to reduce ε-formulas — infinite descent is called *"die Methode der Zurückführung"* in Hilbert and Bernays (see [12], p. 190), as I have shown above, and it is natural to think that both Herbrand and Gödel had such a calculus in mind and hoped to further it by other means, that is a theory of ascending types versus the number-theoretic method of infinite descent — this could be the main cleavage between number theory and logic (and the cumulative hierarchy of set theory by the same token) but infinite descent is still at work in ZF with von Neumann's axiom of foundation inspired by Mirimanoff's idea of finite descent for ordinals (*ensembles ordinaires*) (see [19] and [5], p. 83), that is transitive sets. With von Neumann ordinals arose the identification (confusion) of transfinite induction with infinite descent in axiomatic set theory, in spite Mirimanoff's cautious stance [19]. At any rate, Herbrand's and Gödel's efforts have resulted in a proof theory searching for polynomial bounds, but it should be noted that this proof theory is not interested in the computational complexity of proofs for logical calculi (the work of Cook, Urquhart, Krajicek, Pudlák and others) but in extracting bounds for proofs in classical analysis. If the business of proof mining is not devoid of tricks, as Kreisel has admitted, the main objective is the transformation of non-effective proofs into effective ones. Effective may mean computable, but sometimes effective proofs are hard to compute or hardly computable and the constructive content is far from being apparent; nonetheless, the objective is most of the time to retrieve more information from a non-constructive proof.

The paradigm theory is here again number theory and polynomial arithmetic. What Kronecker called general arithmetic (*allgemeine Arithmetik*) for his theory of forms or homogeneous polynomials is in more ways than one the mother theory of contemporary arithmetic-algebraic geometry where a given result of finiteness does not yield an explicit calculation for solutions (or the number of points). Among the logical methods used by applied proof theory, it is the functional interpretation which is privileged and it comes equipped with a logical (hereditarily inductive) relation of majorazibility $\geq$ (due to W.A. Howard) which is in a sense the logical counterpart of number-theoretic minorization. The relation

$$x^* \text{ majorizes } x \, (x^* \geq_0 x)$$

is defined for closed terms by induction on functionals of finite types; as Kohlenbach points out it is a structural property of the closed terms which eschews normalization (see [14], p. 60) and he privileges a monotone or step-by-step functional interpretation. As is well known, the logical background of the functional interpretation was intuitionistic, since constructions over and above formal objects (concrete representations of natural numbers) were allowed in. E. Bishop wanted to reduce those constructions to their numerical content, but the logical formalisms that have evolved from intuitionistic motivations pervade the techniques of applied proof theory. For instance, the fan rule that is mentioned and used by Kohlenbach (see [14] p. 111) is a descendant of Brouwer's fan theorem which is a finiteness law for spreads (*spreiding*) in an arborescent structure whose branches are assigned natural numbers; this is the equivalent of the axiom of choice and it is linked with the (weak) König's lemma on a finitely-branched infinite tree. Brouwer derived from the fan principle his famous theorem stating that "Every total function in the real interval $[0,1]$ is uniformly continuous". Kleene's realizability interpretation for intuitionistic logic with numerical realizers or witnesses for functions, Kreisel's modified realizability, Gödel's negative translation from intuitionistic logic to classical logic, Markov's principle of double negation elimination for the existential quantifier, bar-recursion associated with the fan rule, all those notions are called to duty in the prospection procedures. Modified realizability was put forward to escape

Markov's principle

$$\neg\neg\exists x A(x,y) \rightarrow \exists x A(x,y)$$

independent from intuitionistic principles and not realizable in the typed lambda-calculus of modified realizability. But the full functional calculus on all finite types, that is the full-blown system HA^{ω} of Heyting arithmetic can accomodate Markov's principle; Kohlenbach even shows (Chapter 7) that an extensional Heyting arithmetic can be made semi-intuitionistic by adopting highly non-constructive principles. Meanwhile, the functional interpretation assigns to each formula $A(a)$ of HA^{ω} a formula

$$A \equiv \exists x \forall y A(x,y,a)$$

where A is quantifier-free and x, y are variables of finite type. The idea is to extract a computational witness (a realizer) as a closed term by more or less constructive means. This is a task Kohlenbach accomplishes with a vast array of technical means and an extensive logico-mathematical machinery — for instance, chapter 11 treats "The functional interpretation of full classical analysis" via a variety of comprehension and choice principles. A nice application is on Polish (complete and separable) metric spaces where a function (type-1 object) represents a unique element X for a computable enumerative procedure that picks a closed term as a bound. Other examples in function approximation theory abound.

Whatever the benefits of the functional interpretation in the process of proof transformations, one should not forget that Gödel had in mind what I call the "internal consistency" of arithmetic in contrast with the external or outer consistency, the so-called ω-consistency or 1-consistency. Gödel was apparently not satisfied with Gentzen's recourse to transfinite induction and had not rejected an inner treatment of the consistency problem. In a 1972 note (see [5]), Gödel draws the attention to a remark he had made in 1966 (see [8] on "outer" or ω-consistency saying that "perhaps it has not received sufficient notice". Gödel's insistence on outer consistency refers to Hilbert's own characterization of a formal system as *"äusseres Handeln"* or external treatment of internal or contentual inference *"inhaltliches Schliessen"*. The very idea of outer consistency means that internal consistency of various systems of arithmetic cannot be proven within the concrete finite ressources afforded by the external treatment of a formal system. Consistency or ω-consistency must be assumed from without in order to justify the transfinite axioms, as Gödel says. That does not mean however that Hilbert's consistency program is doomed to failure, only that it must be pursued by more internal — or more abstract — means. Gödel's *Dialectica* interpretation is an attempt in that direction — see my Abstract in the *Bulletin of Symbolic Logic* [6]

4. Concluding Remarks

The fact that the name of Kronecker is rarely mentioned in connection with the tradition of numerical functional analysis and its proof theory should not detract one's attention from its importance in contemporary mathematics, for instance in algebraic-arithmetic geometry, as André Weil has stressed. Langlands' programme is one example of the deep influence of Kronecker's *Jugendtraum* programme for the arithmetization of algebra in algebraic geometry, as Langlands himself confessed [18].

Admittedly, number theory has been the main arena for constructive proofs and to give one major case one has only to mention Selberg's (and Erdös') constructive elementary proof of Dirichlet's analytical theorem on the infinity of primes in arithmetic progressions. From a different angle, the applied proof theory of Kohlenbach and others reaches out to ergodic theory and non-expansive mappings, a central theme in T. Tao's recent work on the existence of arbitrary long progressions in the distribution of primes. If one acknowledges that the logical methods of applied proof theory derive essentially from Hilbert's metamathematical programme in the foundations of mathematics, one could also recognize that Kronecker's finitist arithmetization programme — which had a profound impact on Hilbert's ideas (see Gauthier [5]) — is at the foundation of today's computational constructive mathematics. I could also say, on a more amusing tone, that the road from Kronecker to applied proof theory goes from the winding number of function theory to the "unwinding" of proofs in the proof mining project.

The search for effective proofs and the extraction of bounds in classical results does not look at replacing classical analysis, as Brouwer once dreamed with his intuitionistic counterexamples, but rather at reinforcing the mathematical content of the classical results by exploring their internal logic (Hilbert's *inhaltiche Logik*) and the constructive computational content it makes manifest.

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