International Journal of Pure and Applied Mathematics

Volume 56 No. 2 2009, 235-257

CHARACTERIZATION OF PARTIAL DERIVATIVES WITH RESPECT TO BOUNDARY CONDITIONS FOR NONLOCAL BOUNDARY VALUE PROBLEMS FOR N-TH ORDER DIFFERENTIAL EQUATIONS

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Abstract: Under certain conditions, solutions of the nonlocal boundary value problem, $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$, $y(x_i) = y_i$ for $1 \leq i \leq n-1$, and $y(x_n) - \sum_{k=1}^{m} r_i y(\eta_i) = y_n$, are differentiated with respect to boundary conditions, where $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$, $r_1, \ldots, r_m, y_1, \ldots, y_n \in \mathbb{R}$.

AMS Subject Classification: 34B10, 34B15
Key Words: nonlinear boundary value problem, ordinary differential equation, nonlocal boundary condition, existence, uniqueness

1. Introduction

In this paper, we will be concerned with differentiating solutions of certain nonlocal boundary value problems with respect to boundary data for the n-th order ordinary differential equation,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad a < x < b, \tag{1}$$

satisfying

Received: August 31, 2009

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$$y(x_i) = y_i, \quad 1 \le i \le n-1, \quad y(x_2) - \sum_{k=1}^{m} r_k y(\eta_k) = y_n, \tag{2}$$

where $m \in \mathbb{N}$, $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$, and $y_1, \ldots, y_n, r_1, \ldots, r_m \in \mathbb{R}$, and where we assume:

(i) $f(x, u_1, \ldots, u_n) : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\frac{\partial f}{\partial u_i}(x, u_1, \ldots, u_n) : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $1 \le i \le n$, and

(iii) Solutions of initial value problems for (1) extend to (a,b).

We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1), (2), we will establish analogues of a result that Hartman [9] attributes to Peano concerning differentiation of solutions of (1) with respect to initial conditions. For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1), we will give much attention to the *variational equation for* (1) *along* $y(x)$, which is defined by

$$z^{(n)} = \sum_{k=1}^{n} \frac{\partial f}{\partial u_k}(x, y(x),\ y'(x), \ldots, y^{(n-1)}(x)) z^{(k-1)}. \tag{3}$$

Interest in multipoint boundary value problems for ordinary differential equations has been ongoing for several years, with much attention given to positive solutions. To see only few of these papers, we refer the reader to papers by Bai and Fang [1], Gupta and Trofimchuk [8], Ma [17], [18], Sukup [24] and Yang [25].

Likewise for equations on time scales, we suggest the manifold results in the papers [2]-[6], [9]-[14], [16], [19]-[23]. In fact, smoothness results have been given some consideration for (1), (2) when $n = 2$ and for specific and general values of m; see [7] and [15] as well as arbitrary n; see [12].

The theorem for which we seek an analogue, attributed to Peano by Hartman, can be stated in the context of (1) as follows:

Theorem 1. (Peano) *Assume that, with respect to* (1)*, conditions* (i)-(iii) *are satisfied. Let* $x_0 \in (a,b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2, \ldots, c_n)$ *denote the solution of* (1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i$, $1 \le i \le n$. *Then,*

(*a*) *for each* $1 \le i \le n$, $\frac{\partial y}{\partial c_i}(x)$ *exists on* (a,b), *and* $\alpha_i := \frac{\partial y}{\partial c_i}(x)$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions,*

$$\alpha_j^{(i-1)}(x_0) = \delta_{ij}, \quad 1 \le i, j \le n.$$

(b) $\frac{\partial y}{\partial x_0}(x)$ *exists on* (a,b)*, and* $\beta := \frac{\partial y}{\partial x_0}(x)$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions,*

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0), \quad 1 \le i \le n.$$

(c) $$\frac{\partial y}{\partial x_0}(x) = -\sum_{k=1}^{n} y^{(k)}(x_0)\frac{\partial y}{\partial c_k}(x).$$

In addition, our analogue of Theorem 1 depends on uniqueness of solutions of (1), (2), a condition we list as an assumption:

(iv) Given $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$, if $y(x_i) = z(x_i)$, $1 \le i \le n-1$, and $y(x_n) - \sum_{k=1}^{m} r_k y(\eta_k) = z(x_n) - \sum_{k=1}^{m} r_k z(\eta_k)$, where $y(x)$ and $z(x)$ are solutions of (1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (3) along solutions $y(x)$ of (1).

(v) Given $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$, and a solution $y(x)$ of (1), if $u(x_i) = 0$, $1 \le i \le n-1$, and $u(x_n) - \sum_{k=1}^{m} r_k u(\eta_k) = 0$, where $u(x)$ is a solution of (3) along $y(x)$, then $u(x) \equiv 0$.

2. An Analogue of Peano's Theorem for (1), (2)

In this section, we derive our analogue of Theorem 1 for the nonlocal boundary value problem (1), (2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions and parameters. Such continuity is an application of the Brouwer Invariance of Domain Theorem and was established in [13]. We state the Continuous Dependence Theorem here:

Theorem 2. (Continuous Dependence) *Assume* (i)-(iv) *are satisfied with respect to* (1)*. Let* $u(x)$ *be a solution of* (1) *on* (a,b)*, and let* $a < c < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < d < b$ *and* $r_1, \ldots, r_m \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for*

$$|x_i - t_i| < \delta, \quad 1 \le i \le n,$$

$$|\eta_i - \tau_i| < \delta, \quad |r_i - \rho_i| < \delta, \quad 1 \le i \le m,$$

$$|u(x_i) - y_i| < \delta, \quad 1 \le i \le n-1,$$

and

$$|u(x_n) - \sum_{k=1}^{m} r_k u(\eta_k) - y_n| < \delta,$$

there exists a unique solution $u_\delta(x)$ *of* (1) *such that*

$$u\delta(t_i) = y_i, \quad 1 \le i \le n-1,$$

$$u_\delta(t_n) - \sum_{k=1}^{m} \rho_k u_\delta(\tau_k) = y_n,$$

and for $1 \le j \le n$, $u_\delta^{(j-1)}(x)$ *converges uniformly to* $u^{(j-1)}(x)$ *as* $\delta \to 0$ *on* $[c,d]$.

3. Main Result

We are now in a position to state the main result of this paper.

Theorem 3. *Assume conditions* (i)-(v) *are satisfied. Let* $u(x)$ *be a solution of* (1) *on* (a,b). *Let* $n \ge 2$, $m \in \mathbb{N}$, *and* $a < x_1 < x_2 < \cdots < x_{n-1} < \eta_1 < \cdots < \eta_m < x_n < b$ *and* $r_1, \ldots, r_m, u_1, \ldots, u_n \in \mathbb{R}$ *be given, so that*

$$u(x) = u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m),$$

where

$$u(x_i) = u_i,\ 1 \le i \le n-1, \qquad u(x_n) - \sum_{k=1}^{m} r_k u(\eta_k) = u_n.$$

Then,

(*a*) *for each* $1 \le i \le n$, $\frac{\partial u}{\partial u_i}(x)$ *exists on* (a,b). *Moreover, for each* $1 \le j \le n-1$, $y_j := \frac{\partial u}{\partial u_j}(x)$ *solves* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$y_j(x_i) = \delta_{ij},\ 1 \le i \le n-1, \quad y_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $y_n := \frac{\partial u}{\partial u_n}(x)$ *solves* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$y_n(x_i) = 0,\ 1 \le i \le n-1, \quad y_n(x_n) - \sum_{k=1}^{m} r_k y_n(\eta_k) = 1.$$

(*b*) *for each* $1 \le i \le n$, $\frac{\partial u}{\partial x_i}(x)$ *exists on* (a,b), *Moreover, for each* $1 \le j \le n-1$, $z_j := \frac{\partial u}{\partial x_i}(x)$ *solves* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$z_j(x_i) = -u'(x_i)\delta_{ij},\ 1 \le i \le n-1, \quad z_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = 0,$$

and $z_n := \frac{\partial u}{\partial x_n}(x)$ *solves* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$z_n(x_i) = 0,\ 1 \le i \le n-1, \quad z_n(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = -u'(x_n).$$

(*c*) *for* $1 \le j \le m$, $\frac{\partial u}{\partial \eta_j}(x)$ *exists on* (a, b), *and* $w_j := \frac{\partial u}{\partial \eta_j}(x)$ *is the solution of* (3) *along* $u(x)$ *satisfying*

$$w_j(x_i) = 0,\ 1 \le i \le n-1, \quad w_j(x_n) - \sum_{k=1}^{m} r_k w_j(\eta_k) = r_j u'(\eta_j).$$

(*d*) *for* $1 \le j \le m$, $\frac{\partial u}{\partial r_j}(x)$ *exists on* (a, b), *and* $v_j := \frac{\partial u}{\partial r_j}(x)$ *is the solution of* (3) *along* $u(x)$ *satisfying*

$$v_j(x_i) = 0,\ 1 \le i \le n-1, \quad v_j(x_n) - \sum_{k=1}^{m} r_k v_j(\eta_k) = u(\eta_j).$$

Proof. Before beginning the proof, we remark that occasionally we will suppress some limits of summation, arguments, or subscripts for the sake of space.

For part (a), let $1 \le j \le n-1$, and consider $\frac{\partial u}{\partial u_j}$, since the argument for $\frac{\partial u}{\partial u_n}$ is similar, we withhold its proof. In this case we designate, for brevity, $u(x, x_1, \dots, x_n, u_1, \dots, u_n, \eta_1, \dots, \eta_m, r_1, \dots, r_m)$ by $u(x, u_j)$.

Let $\delta > 0$ be as in Theorem 2, $0 < |h| < \delta$ be given, and define

$$y_{jh}(x) = \frac{1}{h}[u(x, u_j + h) - u(x, u_j)].$$

Note that $u(x_j, u_j + h) = u_j + h$, and $u(x_j, u_j) = u_j$, so that, for every $h \ne 0$,

$$\begin{aligned} y_{jh}(x_j) &= \frac{1}{h}[u_j + h - u_j] \\ &= 1. \end{aligned}$$

Also, for every $h \ne 0$, $1 \le i \le n-1$, $i \ne j$,

$$\begin{aligned} y_{jh}(x_i) &= \frac{1}{h}[u(x_i, u_j + h) - u(x_i, u_j)] \\ &= \frac{1}{h}[u_i - u_i] \\ &= 0, \end{aligned}$$

and for $h \ne 0$,

$$y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k) = \frac{1}{h}[u(x_n, u_j + h) - u(x_n, u_j)]$$

$$- \sum_{k=1}^{m} \frac{r_k}{h}[u(\eta_k, u_j + h) - u(\eta_k, u_j)]$$
$$= \frac{1}{h}[u_n - u_n]$$
$$= 0.$$

For $2 \leq i \leq n$, let

$$\beta_i = u^{(i-1)}(x_j, u_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, u_j + h) - \beta_i.$$

By Theorem 2, for $2 \leq i \leq n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 1 for solutions of initial value problems for (1), viewing the solution $u(x)$ as the solution of an initial value problem, and denoting the solution $u(x) = y(x, x_j, u_j, \beta_2, \beta_3 \ldots, \beta_n)$, we have

$$y_{jh}(x) = \frac{1}{h}[y(x, x_j, u_j + h, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{jh}(x) =& \frac{1}{h}\big[y(x, x_j, u_j + h, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & + y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & - + \cdots \\ & - y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ & + y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)\big]. \end{aligned}$$

By Theorem 1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{jh}(x) =& \frac{1}{h}\alpha_1(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(u_j + h - u_j) \\ & + \frac{1}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ & + \cdots + \frac{1}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n), \end{aligned}$$

where $\alpha_k(x, y(\cdot))$, $1 \leq k \leq n$, is the solution of the variational equation (3) along $y(\cdot)$ satisfying,

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \leq i \leq n.$$

Furthermore, $u_j + \bar{h}$ is between u_j and $u_j + h$, and for $2 \leq i \leq n$, $\beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$. Now simplifying,

$$\begin{aligned} y_{jh}(x) =& \alpha_1(x, y(x, x_j, u_j + \bar{h}, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n)) \\ &+ \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n) \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n)). \end{aligned}$$

Thus, to show $\lim_{h\to 0} y_{jh}(x)$ exists, it suffices to show, for $2 \le i \le n$, $\lim_{h\to 0} \frac{\epsilon_i}{h}$ exists.

Now for $1 \le i \le n-1,\ i \ne j$,

$$0 = y_{jh}(x_i) = \alpha_1(x_i, y(\cdot)) + \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} 0 =& y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k, y(\cdot)) \\ =& \alpha_1(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot)) \\ &+ \frac{\epsilon_2}{h}\Big[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\Big] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\Big[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\Big]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$\begin{aligned} -\alpha_1(x_1, y(\cdot)) &= \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot)) \\ -\alpha_1(x_2, y(\cdot)) &= \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot)) \\ &\ \ \vdots \\ -\alpha_1(x_n, y(\cdot)) - &\sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot)) \\ &= \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \end{aligned}$$

$$+\cdots$$
$$+\frac{\epsilon_n}{h}\left[\alpha_n(x_n,y(\cdot))-\sum_{k=1}^{m}r_k\alpha_n(\eta_k,y(\cdot))\right].$$

Define the following matrices:

$$-\alpha := \begin{pmatrix} -\alpha_1(x_1,y(x,x_j,u_j+\bar{h},\beta_2+\epsilon_2,\dots,\beta_n+\epsilon_n)) \\ -\alpha_1(x_2,y(x,x_j,u_j+\bar{h},\beta_2+\epsilon_2,\dots,\beta_n+\epsilon_n)) \\ \vdots \\ -\alpha_1(x_n,y(x,x_j,u_j+\bar{h},\beta_2+\epsilon_2,\dots,\beta_n+\epsilon_n))- \\ \sum_{k=1}^{m} r_k\alpha_1(\eta_k,y(x,x_j,u_j+\bar{h},\beta_2+\epsilon_2,\dots,\beta_n+\epsilon_n)) \end{pmatrix}, \quad \epsilon := \begin{pmatrix} \frac{\epsilon_2}{h} \\ \frac{\epsilon_3}{h} \\ \vdots \\ \frac{\epsilon_n}{h} \end{pmatrix},$$

and

$$M(h) := \begin{pmatrix} \alpha_2(x_1,y(\cdot)) & \alpha_3(x_1,y(\cdot)) & \cdots & \alpha_n(x_1,y(\cdot)) \\ \alpha_2(x_2,y(\cdot)) & \alpha_3(x_2,y(\cdot)) & \cdots & \alpha_n(x_2,y(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n,y(\cdot))- & \alpha_3(x_n,y(\cdot))- & & \alpha_n(x_n,y(\cdot))- \\ \sum r_k\alpha_2(\eta_k,y(\cdot)) & \sum r_k\alpha_3(\eta_k,y(\cdot)) & \cdots & \sum r_k\alpha_n(\eta_k,y(\cdot)) \end{pmatrix}.$$

Then the system of equations written in its matrix form is
$$-\alpha = M(h)\epsilon.$$

Note that in the matrix $M(h)$, the solutions $y(\cdot)$ that each α is along are not identical. Thus we consider the matrix

$$M := \begin{pmatrix} \alpha_2(x_1,u(x)) & \alpha_3(x_1,u(x)) & \cdots & \alpha_n(x_1,u(x)) \\ \alpha_2(x_2,u(x)) & \alpha_3(x_2,u(x)) & \cdots & \alpha_n(x_2,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n,u(x))- & \alpha_3(x_n,u(x))- & & \alpha_n(x_n,u(x))- \\ \sum r_k\alpha_2(\eta_k,u(x)) & \sum r_k\alpha_3(\eta_k,u(x)) & \cdots & \sum r_k\alpha_n(\eta_k,u(x)) \end{pmatrix}.$$

We claim $\det(M)\neq 0$. Suppose to the contrary that $\det(M)=0$. Then there exist $p_2, p_3, \ldots, p_n \in \mathbb{R}$ not all zero such that

$$p_2 \begin{pmatrix} \alpha_2(x_1, u(x)) \\ \alpha_2(x_2, u(x)) \\ \vdots \\ \alpha_2(x_n, u(x)) - \\ \sum r\alpha_2(\eta, u(x)) \end{pmatrix} + \cdots + p_n \begin{pmatrix} \alpha_n(x_1, u(x)) \\ \alpha_n(x_2, u(x)) \\ \vdots \\ \alpha_n(x_n, u(x)) - \\ \sum r\alpha_n(\eta, u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix},$$

where the limits of summation and the subscripts of r and η have been suppressed.

Let

$$w(x, u(x)) := p_2\alpha_2(x, u(x)) + p_3\alpha_3(x, u(x)) + \cdots + p_n\alpha_n(x, u(x)).$$

Then

$$w(x_i, u(x)) = 0,\ 1 \le i \le n-1,$$

and

$$w(x_n, u(x)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, u(x)),$$

which when coupled with hypothesis (v) yields $p_2 = p_3 = \cdots = p_n = 0$. This is a contradiction to the choice of p_i's. Hence $\det(M) \neq 0$ which means M has an inverse. Hence, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse, and therefore, we can solve for each $\epsilon_i/h,\ 2 \le i \le n$, using Crammer's rule:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|} \times$$

$$\begin{vmatrix} \alpha_2(x_1) & \cdots & \alpha_{i-2}(x_1) & -\alpha_1(x_1) & \cdots & \alpha_n(x_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n) - & & \alpha_{i-2}(x_n) - & -\alpha_1(x_n) + & & \alpha_n(x_n) - \\ \sum r\alpha_2(\eta) & \cdots & \sum r\alpha_{i-2}(\eta) & \sum r_k\alpha_1(\eta) & \cdots & \sum r\alpha_n(\eta) \end{vmatrix},$$

where each solution $\alpha_i,\ 1 \le i \le n$, is along its particular $y(\cdot)$.

Note as $h \to 0,\ \det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1,\ \epsilon_i(h)/h \to \det(M_i)/\det(M) := A_i$ as $h \to 0$, where M_i is the $n-1 \times n-1$ matrix found by replacing the appropriate column of the matrix defining M by

$$\mathrm{col}\Big[-\alpha_1(x_1, u(x)), \ldots, -\alpha_1(x_n, u(x)) + \sum_{k=1}^{m} r_k \alpha_1(\eta_k, u(x)) \Big].$$

Now let $y_j(x) = \lim_{h\to 0} y_{jh}(x)$, and note by construction of $y_{jh}(x)$,

$$y_j(x) = \frac{\partial u}{\partial u_j}(x).$$

Furthermore,

$$\begin{aligned} y_j(x) = \lim_{h\to 0} y_{jh}(x) =& \alpha_1(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n)) \\ &+ A_2\alpha_2(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n)) \\ &+ \cdots \\ &+ A_n\alpha_n(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n)) \\ =& \alpha_1(x, u(x)) + \sum_{i=2}^{n} A_i\alpha_i(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3) along $u(x)$. In addition,

$$y_j(x_i) = \lim_{h\to 0} y_{jh}(x_i) = \delta_{ij},\ 1 \le i \le n-1,$$

and

$$y_j(x_n) - \sum_{k=1}^{m} r_k y_j(\eta_k) = \lim_{h\to 0}\left[y_{jh}(x_n) - \sum_{k=1}^{m} r_k y_{jh}(\eta_k)\right] = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_j}$.

For part (b), let $1 \le j \le n-1$, and consider $\frac{\partial u}{\partial x_j}$, since the argument for $\frac{\partial u}{\partial x_n}$ is similar, we omit its proof. This time we designate $u(x, x_1, \dots, x_n, u_1, \dots, u_n, \eta_1, \dots, \eta_m, r_1, \dots, r_m)$ by $u(x, x_j)$.

Let $\delta > 0$ be as in Theorem 2, let $0 < |h| < \delta$ be given, and define

$$z_{jh}(x) = \frac{1}{h}[u(x, x_j + h) - u(x, x_j)].$$

Note that for $h \ne 0$,

$$\begin{aligned} z_{jh}(x_j) =& \frac{1}{h}[u(x_j, x_j + h) - u(x_j, x_j)] \\ =& \frac{1}{h}[u(x_j, x_j + h) - u(x_j + h, x_j + h) \\ &+ u(x_j + h, x_j + h) - u_1] \\ =& -\frac{1}{h}[u(c_{x_j,h}, x_j + h)\cdot h] \\ =& -u'(c_{x_j,h}, x_j + h), \end{aligned}$$

where $c_{x_j,h}$ lies between x_j and $x_j + h$.

Also, for $1 \leq i \leq n-1,\ i \neq j$, and $h \neq 0$,

$$\begin{aligned} z_{jh}(x_i) =& \frac{1}{h}[u(x_i, x_j + h) - u(x_i, x_j)] \\ =& \frac{1}{h}[u_i - u_i] \\ =& 0. \end{aligned}$$

In addition,

$$\begin{aligned} z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k) =& \frac{1}{h}[u(x_n, x_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, x_j + h) \\ & - \{u(x_n, x_j) - \sum_{k=1}^{m} r_k u(\eta_k, x_j)\}] \\ =& \frac{1}{h}[u_n - u_n] \\ =& 0, \end{aligned}$$

for every $h \neq 0$.

Next, for $2 \leq i \leq n$, let

$$\beta_i = u^{(i-1)}(x_j, x_j),$$

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, x_j + h) - \beta_i,$$

and

$$\epsilon_1 = \epsilon_1(h) = u(x_j, x_j + h) - u_j.$$

By Theorem 2, for $1 \leq i \leq n$, $\epsilon_i \to 0$ as $h \to 0$. As in part (a), we employ the notation of Theorem 1 for solutions of initial value problems for (1). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \ldots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} z_{jh}(x) =& \frac{1}{h}[y(x, x_j, u_j + \epsilon_1, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2, \ldots, \beta_n)] \\ =& \frac{1}{h}[y(x, x_j, u_j + \epsilon_1, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & + y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ & - + \cdots \\ & - y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ & + y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \end{aligned}$$

$$- y(x, x_j, u_j, \beta_2, \dots, \beta_n)].$$

Applying the Mean Value Theorem and Theorem 1,

$$\begin{aligned} z_{jh}(x) =& \frac{1}{h}[\epsilon_1 \alpha_1(x, y(x, x_j, u_j + \bar{\epsilon}_1, \beta_2 + \epsilon_2, \dots, \beta_n + \epsilon_n)) \\ &+ \cdots \\ &+ \epsilon_n \alpha_n(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n + \bar{\epsilon}_n))], \end{aligned}$$

where, for $1 \le i \le n,\ \bar{\epsilon}_i$ lies between β_i and $\beta_i+\epsilon_i$, and for $1 \le k \le n,\ \alpha_k(x, y(\cdot))$ is the solution of (3) along $y(\cdot)$ satisfying,

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Hence, to show $\lim_{h\to 0} z_{jh}(x)$ exists, it suffices to show for $1 \le i \le n,\ \lim_{h\to 0} \frac{\epsilon_i}{h}$ exists. From above,

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon_1}{h} &= \lim_{h\to 0} z_{jh}(x_j) \\ &= - \lim_{h\to 0} u'(c_{x_j,h}, x_j + h) \\ &= - u'(x_j). \end{aligned}$$

Now, by construction, for $1 \le i \le n-1, i \ne j$,

$$0 = z_{jh}(x_i) = \frac{\epsilon_1}{h}\alpha_1(x_i, y(\cdot)) + \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} 0 =& z_{jh}(x_n) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k, y(\cdot)) \\ =& \frac{\epsilon_1}{h}\left[\alpha_1(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_1(\eta_k, y(\cdot))\right] + \\ & \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$u'(x_j)\alpha_1((x_1, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot))$$

$$u'(x_j)\alpha_1((x_2, y(\cdot)) = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot))$$

$$\vdots$$

$$\begin{aligned} u'(x_j)\Big[\alpha_1(x_n, y(\cdot)) - &\sum_{k=1}^{m} r_k\alpha_1(\eta_k, y(\cdot))\Big] \\ &= \frac{\epsilon_2}{h}\Big[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k\alpha_2(\eta_k, y(\cdot))\Big] \\ &\quad + \cdots \\ &\quad + \frac{\epsilon_n}{h}\Big[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k\alpha_n(\eta_k, y(\cdot))\Big], \end{aligned}$$

which we can represent as a matrix equation $u'(x_j)\alpha = M(h)\epsilon$, similar to the matrix equation from part (a).

At this point, we omit the part of the proof where we solve show $M(h)$ has nonzero determinant as it is nearly identical to the method used in part (a). Instead, we simply provide the formula for each $\epsilon_i/h,\ 2 \le i \le n$:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$
$$\begin{vmatrix} \alpha_2(x_1) & \cdots & \alpha_{i-2}(x_1) & u'(c_{x_j,h})\alpha_1(x_1) & \cdots & \alpha_n(x_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n)- & & \alpha_{i-2}(x_n)- & u'(c_{x_j,h})\times & & \alpha_n(x_n,)- \\ \sum r\alpha_2 & \cdots & \sum r\alpha_{i-2} & [\alpha_1(x_n) - \sum r\alpha_1] & \cdots & \sum r\alpha_n \end{vmatrix},$$

where each solution $\alpha_i,\ 1 \le i \le n$, is along its particular $y(\cdot)$. As a result of continuous dependence, we are able to take the limit for each $\epsilon_i/h,\ 2 \le i \le n$. Denote $\lim_{h\to 0} \epsilon_i/h := B_i,\ 2 \le i \le n$.

Now let $z_j(x) = \lim_{h\to 0} z_{jh}(x)$, and note by construction of $z_{jh}(x)$,

$$z_j(x) = \frac{\partial u}{\partial x_j}(x).$$

Furthermore,

$$\begin{aligned} z_j(x) = \lim_{h\to 0} z_{jh}(x) = &- u'(x_j)\alpha_1(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &+ B_2\alpha_2(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &+ \cdots \end{aligned}$$

$$+ B_n\alpha_n(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n))$$
$$= - u'(x_j)\alpha_1(x, u(x)) + \sum_{i=2}^{n} B_i\alpha_i(x, u(x)),$$

which is a solution of the variational equation (3) along $u(x)$.

In addition, from above observations, $z_j(x)$ satisfies the boundary conditions

$$z_j(x_i) = \lim_{h\to 0} z_{jh}(x_i) = -\delta_{ij}u'(x_j),\ 1 \le i \le n-1,$$

and

$$z_j(x_n) - \sum_{k=1}^{m} r_k z_j(\eta_k) = \lim_{h\to 0}\left[z_{jh}(x_2) - \sum_{k=1}^{m} r_k z_{jh}(\eta_k)\right] = 0.$$

This completes the proof for $\frac{\partial u}{\partial x_j}$.

For (c), we fix $1 \le j \le m$, and this time we designate $u(x, x_1, \dots, x_n, u_1, \dots, u_n, \eta_1, \dots, \eta_m, r_1, \dots, r_m)$ by $u(x, \eta_j)$. Let $\delta > 0$ be as in Theorem 2, let $0 < |h| < \delta$ be given, and define

$$w_{jh}(x) = \frac{1}{h}[u(x, \eta_j + h) - u(x, \eta_j)].$$

Note that for $h \ne 0$,

$$\begin{aligned} w_{jh}(x_j) - &\sum_{k=1}^{m} r_k w_{jh}(\eta_k) \\ =&\frac{1}{h}\Big[u(x_j, \eta_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j + h) \\ &- u(x_j, \eta) + \sum_{k=1}^{m} r_k u(\eta_k, \eta_j)\Big] \\ =&\frac{1}{h}\Big[u(x_j, \eta_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, \eta_j + h) \\ &- r_j u(\eta_j + h, \eta_j + h) + r_j u(\eta_j + h, \eta_j + h) - u_n\Big] \\ =&\frac{r_j}{h}[u(c_{\eta_{j,h}}, \eta_j + h) \cdot h] \\ =&r_j u'(c_{\eta_{j,h}}, \eta_j + h), \end{aligned}$$

where $c_{\eta_j,h}$ lies between η_j and $\eta_j + h$. Also, for $1 \le i \le n-1$ and $h \ne 0$

$$\begin{aligned} w_{jh}(x_i) =&\frac{1}{h}[u(x_i, \eta_j + h) - u(x_i, \eta_j)] \\ =&\frac{1}{h}[u_i - u_i] \end{aligned}$$

$$=0.$$

Next, for $2 \le i \le n$, let

$$\beta_i = u^{(i-1)}(x_j, \eta_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, \eta_j + h) - \beta_i.$$

By Theorem 2, for $2 \le i \le n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. We employ the notation of Theorem 1 for solutions of initial value problems for (1). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \dots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} w_{jh}(x) =& \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \dots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2, \dots, \beta_n)] \\ =& \frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \dots, \beta_n + \epsilon_n) \\ & - y(x, x_j, u_j, \beta_2, \dots, \beta_n + \epsilon_n) \\ & + y(x, x_j, u_j, \beta_2, \dots, \beta_n + \epsilon_n) \\ & - + \cdots \\ & - y(x, x_j, u_j, \beta_2, \dots, \beta_n)]. \end{aligned}$$

Then, by the Mean Value Theorem and Theorem 1,

$$\begin{aligned} w_{jh}(x) =& \frac{1}{h}[\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \dots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ & + \cdots \\ & + \alpha_n(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n)] \\ =& \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \dots, \beta_n + \epsilon_n)) \\ & + \cdots \\ & + \frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \dots, \beta_n + \bar{\epsilon}_n)), \end{aligned}$$

where, for $2 \le i \le n$, $\bar{\epsilon}_i$ lies between β_i and $\beta_i + \epsilon_i$, and, for $1 \le k \le n$, $\alpha_k(x, y(\cdot))$ is the solution of (3) along $y(\cdot)$ satisfying

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Thus, to show $\lim_{h \to 0} w_{jh}(x)$ exists, it suffices to show, for $2 \le i \le n$, $\lim_{h \to 0} \frac{\epsilon_i}{h}$ exists. Now for $1 \le i \le n-1$, $i \ne j$,

$$0 = w_{jh}(x_i) = \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \frac{\epsilon_3}{h}\alpha_3(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} r_j u'(c_{\eta_j,h}, \eta_j + h) =& w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k, y(\cdot)) \\ =& \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &+ \frac{\epsilon_3}{h}\left[\alpha_3(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot))\right] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$\begin{aligned} 0 &= \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot)), \\ 0 &= \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot)), \\ &\vdots \\ r_j u'(c_{\eta_j,h}, \eta_j + h) =& \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right], \end{aligned}$$

which we can represent as a matrix equation $\alpha = M(h)\epsilon$, similar to the matrix equation from part (a).

As was done in part (b), we omit proof that $M(h)$ has nonzero determinant. Instead, we simply provide the formula for each
$\epsilon_i/h,\ 2 \le i \le n:$

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$

$$\left|\begin{array}{cccccc} \alpha_2(x_1) & \cdots & \alpha_{i-2}(x_1) & 0 & \cdots & \alpha_n(x_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n)- & & \alpha_{i-2}(x_n)- & & & \alpha_n(x_n)- \\ \sum r\alpha_2 & \cdots & \sum r\alpha_{i-2} & r_j u'(c_{\eta_j,h}) & \cdots & \sum r\alpha_n \end{array}\right|,$$

where each solution α_i, $2 \le i \le n$, is along its particular $y(\cdot)$. As a result of continuous dependence, we are able to take the limit for each ϵ_i/h, $2 \le i \le n$. Denote $\lim_{h\to 0} \epsilon_i/h := C_i$, $2 \le i \le n$.

Now let $w_j(x) = \lim_{h\to 0} w_{jh}(x)$, and note by construction of $w_{jh}(x)$,

$$w_j(x) = \frac{\partial u}{\partial \eta_j}(x).$$

Furthermore,

$$\begin{aligned} w_j(x) &= \lim_{h\to 0} w_{jh}(x) \\ &= \sum_{i=2}^{n} C_i\alpha_i(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &= \sum_{i=2}^{n} C_i\alpha_i(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3) along $u(x)$.

In addition, from above observations, $w_j(x)$ satisfies the boundary conditions

$$w_j(x_i) = \lim_{h\to 0} w_{jh}(x_i) = 0,\ 1 \le i \le n-1,$$

and

$$w_j(x_n) - \sum_{k=1}^{m} r_k w_j(\eta_k)) = \lim_{h\to 0}\left[w_{jh}(x_n) - \sum_{k=1}^{m} r_k w_{jh}(\eta_k)\right] = r_j u'(\eta_j).$$

This completes the proof for $\frac{\partial u}{\partial \eta_j}$.

It remains to verify part (d). Fix $1 \le j \le m$ as before and consider $\frac{\partial u}{\partial r_j}$. Again, let $\delta > 0$ be as in Theorem 2, $0 < |h| < \delta$ be given, denote $u(x, x_1, \ldots, x_n, u_1, \ldots, u_n, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$
by $u(x, r_j)$, and define

$$v_{jh}(x) = \frac{1}{h}[u(x, r_j + h) - u(x, r_j)].$$

Note that for $h \neq 0$,

$$\begin{aligned} v_{jh}(x_j) - &\sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\ =&\frac{1}{h}\Big[u(x_j, r_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) \\ &- u(x_j, r_j) + \sum_{k=1}^{m} r_k u(\eta_k, r_j)\Big] \\ =&\frac{1}{h}\Big[u(x_j, r_j + h) - \sum_{k=1}^{m} r_k u(\eta_k, r_j + h) \\ &- hu(\eta_j, r_j + h) + hu(\eta_j, r_j + h) - u_n\Big] \\ =&u(\eta_j, r_j + h). \end{aligned}$$

Also, for $1 \leq i \leq n-1$ and $h \neq 0$

$$\begin{aligned} v_{jh}(x_i) =&\frac{1}{h}[u(x_i, r_j + h) - u(x_i, r_j)] \\ =&\frac{1}{h}[u_i - u_i] \\ =&0. \end{aligned}$$

Now, for $2 \leq i \leq n$, let

$$\beta_i = u^{(i-1)}(x_j, r_j),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i-1)}(x_j, r_j + h) - \beta_i.$$

By Theorem 2, for $2 \leq i \leq n$, $\epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. We employ the notation of Theorem 1 for solutions of initial value problems for (1). Viewing the solution $u(x)$ as the solution of an initial value problem, $u(x) = y(x, x_j, u_j, \beta_2, \beta_3, \ldots, \beta_n)$, and using a telescoping sum, we have

$$\begin{aligned} v_{jh}(x) =&\frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &- y(x, x_j, u_j, \beta_2, \ldots, \beta_n)] \\ =&\frac{1}{h}[y(x, x_j, u_j, \beta_2 + \epsilon_2, \ldots, \beta_n + \epsilon_n) \\ &- y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &+ y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \epsilon_n) \\ &- + \cdots \end{aligned}$$

$$- y(x, x_j, u_j, \beta_2, \ldots, \beta_n)].$$

By the Mean Value Theorem and Theorem 1,

$$\begin{aligned} v_{jh}(x) =& \frac{1}{h}[\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n))(\beta_2 + \epsilon_2 - \beta_2) \\ &+ \cdots \\ &+ \alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n))(\beta_n + \epsilon_n - \beta_n)] \\ =& \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_j, u_j, \beta_2 + \bar{\epsilon}_2, \ldots, \beta_n + \epsilon_n)) \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\alpha_n(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n + \bar{\epsilon}_n)), \end{aligned}$$

where for $2 \le i \le n$, $\beta_i + \bar{\epsilon}_i$ lies between β_i and $\beta_i + \epsilon_i$ and, for $1 \le k \le n$, $\alpha_k(x, y(\cdot))$ is the solution of (3) along $y(\cdot)$ satisfying

$$\alpha_k^{(i-1)}(x_j) = \delta_{ik},\ 1 \le i \le n.$$

Therefore, to show $\lim_{h \to 0} v_{jh}(x)$ exists, it suffices to show, for $2 \le i \le n$, $\lim_{h \to 0} \frac{\epsilon_i}{h}$ exists.

Now for $1 \le i \le n-1$, $i \ne j$,

$$0 = v_{jh}(x_i) = \frac{\epsilon_2}{h}\alpha_2(x_i, y(\cdot)) + \frac{\epsilon_3}{h}\alpha_3(x_i, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_i, y(\cdot)),$$

and

$$\begin{aligned} u(\eta_j, r_j + h) =& v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k) \\ =& \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &+ \frac{\epsilon_3}{h}\left[\alpha_3(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_3(\eta_k, y(\cdot))\right] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

Hence, we have a system of $n-1$ equations with $n-1$ unknowns (note the x_jth equation is omitted):

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_1, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_1, y(\cdot)),$$

$$0 = \frac{\epsilon_2}{h}\alpha_2(x_2, y(\cdot)) + \cdots + \frac{\epsilon_n}{h}\alpha_n(x_2, y(\cdot)),$$

$$\vdots$$

$$\begin{aligned} u(\eta_j, r_j + h) =& \frac{\epsilon_2}{h}\left[\alpha_2(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_2(\eta_k, y(\cdot))\right] \\ &+ \cdots \\ &+ \frac{\epsilon_n}{h}\left[\alpha_n(x_n, y(\cdot)) - \sum_{k=1}^{m} r_k \alpha_n(\eta_k, y(\cdot))\right]. \end{aligned}$$

which we can represent as a matrix equation $\alpha = M(h)\epsilon$, similar to the matrix equation from part (a).

As was done in parts (b) and (c), we omit proof that $M(h)$ has nonzero determinant. Instead, we simply provide the formula for each $\epsilon_i/h,\ 2 \le i \le n$:

$$\frac{\epsilon_i(h)}{h} = \frac{1}{|M(h)|}\times$$
$$\begin{vmatrix} \alpha_2(x_1) & \cdots & \alpha_{i-2}(x_1) & 0 & \alpha_i(x_1) & \cdots & \alpha_n(x_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_2(x_n)- & & \alpha_{i-2}(x_n)- & & \alpha_i(x_n)- & & \alpha_n(x_n,)- \\ \sum r\alpha_2 & \cdots & \sum r\alpha_{i-2} & u(\eta_j) & \sum r\alpha_i & \cdots & \sum r\alpha_n \end{vmatrix},$$

where each solution $\alpha_i,\ 2 \le i \le n$, is along its particular $y(\cdot)$. As a result of continuous dependence, we are able to take the limit for each $\epsilon_i/h,\ 2 \le i \le n$. Denote $\lim_{h\to 0} \epsilon_i/h := D_i,\ 2 \le i \le n$.

Now let $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, and note by construction of $v_{jh}(x)$,

$$v_j(x) = \frac{\partial u}{\partial r_j}(x).$$

Furthermore,

$$\begin{aligned} v_j(x) &= \lim_{h\to 0} v_{jh}(x) \\ &= \sum_{i=2}^{n} D_i \alpha_i(x, y(x, x_j, u_j, \beta_2, \ldots, \beta_n)) \\ &= \sum_{i=2}^{n} D_i \alpha_i(x, u(x)), \end{aligned}$$

which is a solution of the variational equation (3) along $u(x)$.

In addition, from above observations, $w_j(x)$ satisfies the boundary conditions

$$v_j(x_i) = \lim_{h\to 0} v_{jh}(x_i) = 0,\ 1 \le i \le n-1,$$

and

$$v_j(x_n) - \sum_{k=1}^{m} r_k v_j(\eta_k)) = \lim_{h\to 0}\left[v_{jh}(x_n) - \sum_{k=1}^{m} r_k v_{jh}(\eta_k)\right] = u(\eta_j).$$

This completes the proof for $\frac{\partial u}{\partial r_j}$. □

References

[1] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, *J. Math. Anal. Appl.*, **281** (2003), 76-85.

[2] A. Datta, Differences with respect to boundary points for right focal boundary conditions, *J. Difference Equations Appl.*, **4** (1998), 571-578.

[3] J. Ehme, Differentiation of solutions of boundary value problems with respect to nonlinear boundary conditions, *J. Differential Equations*, **101** (1993), 139-147.

[4] J. Ehme, P. W. Eloe, J. Henderson, Differentiability with respect to boundary conditions and deviating argument for functional-differential systems, *Differential Equations Dynam. Systems*, **1** (1993), 59-71.

[5] J. Ehme, J. Henderson, Functional boundary value problems and smoothness of solutions, *Nonlinear Anal.*, **26** (1996), 139-148.

[6] J. Ehme, B. Lawrence, Linearized problems and continuous dependence for finite difference equations, *Panamer. Math. J.* **10** (2000), 13-24.

[7] J. Ehrke, J. Henderson, C. Kunkel, Q. Sheng, Boundary data smoothness for solutions of nonlocal boundary value problems for second order differencial equations, *J. Math Anal. Appl.*, **333** (2007), 191-203.

[8] C.P. Gupta, S.I. Trofimchuk, Solvability of a multi-point boundary value problem and related *a priori* estimates, *Canad. Appl. Math. Quart.*, **6** (1998), 45-60.

[9] P. Hartman, *Ordinary Differential Equations,* Wiley, New York (1964).

[10] J. Henderson, Right focal point boundary value problems for ordinary differential equation and variational equations, *J. Math. Anal. Appl.*, **98** (1984), 363-377.

[11] J. Henderson, Disconjugacy, disfocality and differentiation with respect to boundary conditions, *J. Math. Anal. Appl.*, **121** (1987), 1-9.

[12] J. Henderson, B. Hopkins, E. Kim, J. Lyons, Boundary data smoothness for solutions of nonlocal boundary value problems for nth order differential equations, *Involve*, **1**, No. 2 (2008), 167-181.

[13] J. Henderson, B. Karna, C. C. Tisdell, Uniqueness implies existence for multipoint boundary value problems for second order equations, *Proc. Amer. Math. Soc.*, **133** (2005), 1365-1369.

[14] J. Henderson, B. Lawrence, Smooth dependence on boundary matrices, *J. Difference Equations Appl.*, **2** (1996), 161-166.

[15] J. Henderson, C. C. Tisdell, Boundary data smoothness for solutions of three point boundary value problems for second order ordinary differential equations, *Z. Anal. Anwendungen*, **23** (2004), 631-640.

[16] B. Lawrence, A variety of differentiability results for a multi-point boundary value problem, *J. Comput. Appl. Math.*, **141** (2002), 237-248.

[17] R. Ma, Existence theorems for a second-order three-point boundary value problems, *J. Math. Anal. Appl.*, **212** (1997), 430-442.

[18] R, Ma, Existence and uniqueness of solutions to first-order three-point boundary value problems, *Appl. Math. Lett.*, **15** (2002), 211-216.

[19] A.C. Peterson, Comparison theorems and existence theorems for ordinary differential equations, *J. Math. Anal. Appl.*, **55** (1976), 773-784.

[20] A.C. Peterson, Existence-uniqueness for ordinary differential equations, *J. Math. Anal. Appl.*, **64** (1978), 166-172.

[21] A.C. Peterson, Existence-uniqueness for focal point boundary value problems, *SIAM J. Math. Anal.*, **12** (1981), 173-185.

[22] A.C. Peterson, Existence and uniqueness theorems for nonlinear difference equations, *J. Math. Anal. Appl.*, **125** (1987), 185-191.

[23] J. Spencer, Relations between boundary value functions for a nonlinear differential equation and its variational equation, *Canad. Math. Bull.*, **18** (1975), 269-276.

[24] D. Sukup, On the existence of solutions to multipoint boundary value problems, *Rocky Mtn. J. Math.*, **6** (1976), 357-375.

[25] B. Yang, *Boundary Value Problems for Ordinary Differential Equations*, Ph.D. Dissertation, Mississippi State University, Mississippi State, MS (2002).