FIXED POINTS IN SIMILARITY TRANSFORMATIONS

Yutaka Nishiyama
Department of Business Information
Faculty of Information Management
Osaka University of Economics
2, Osumi Higashiyodogawa, Osaka, 533-8533, JAPAN
e-mail: nishiyama@osaka-ue.ac.jp

Abstract: A new method of constructing fixed points in congruence transformations is introduced, and a detailed explanation of fixed points in similarity transformations then follows. Constructions in Euclidean geometry generally require the use of a compass and ruler, but the newly developed method required the use of only a ruler. The new construction can also be applied to affine transformations.

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1. Random Dot Patterns

In 1982, the author discovered a new method of constructing fixed points in congruence transformations [3]. Constructions in Euclidean geometry generally require the use of a compass and ruler, but the newly developed method required the use of only a ruler. The method was also applicable to the fixed points of similarity transformations. This paper gives a detailed description of the application of the method to similarity transformations.

Assume that two pieces of square, congruent origami paper are placed randomly on a desktop (Figure 1). It is possible to align these pieces of paper...
using some combination of translations and rotations (or symmetric dis
placements), and doing so is referred to as a congruence transformation. It is known,
however, that for all congruence transformations there exists at least one point
that will not move as a result of the transformation. Such immobile points are
called fixed points, and a rotation of a figure using a fixed point as the axis
of rotation can be used to align the two figures. In other words, the previous
combination of translation and rotation can be replaced by a single rotation.

The construction of a fixed point generally requires the use of a compass
and ruler. Taking the vertices of a square as \(A, B, C, D\), and its vertices after
transformation as \(A', B', C', D'\), the fixed point is constructed by finding the
point of intersection formed by the perpendicular bisectors of the line segments
\(AA'\) and \(BB'\). Doing so is not problematic, but in 1982 I stumbled upon a
method for finding these fixed points using only a ruler. The method involves
connecting the intersections of opposing sides of the two squares, as indicated
in Figure 1. The point of intersection of the two lines is a fixed point, making
this a very simple method of construction of such points. One can place the
squares in congruence by pressing down on this fixed point with the point of
a compass or the tip of a pen, and then rotating the upper square. I invite
readers who have never done so to try this for themselves. If you do not have
access to origami paper, then copier paper can be cut into squares to produce
the same effect.

\[\text{Figure 1: The fixed point in a congruence transformation}\]

I discovered this new construction method when working with random dot
patterns like those shown in Figure 2(1). Copying such a random dot pattern
onto an overhead projector sheet, placing the sheet on top of the original pat-
tern, and then rotating it by a small amount produces concentric circles like
those shown in Figure 2(2). The moment at which the concentric circles ap-
PEAR is quite obvious. Placing one's fingertip on the center of the concentric circles and rotating the upper sheet will again bring two patterns into perfect alignment and the circles disappear. In other words, the center of rotation is a fixed point. The random dot patterns used were created with about twenty lines of Visual Basic code, using the built-in RND function to place 2000 points randomly within a square with sides 20-cm in length.

![Random dot patterns](image)

Figure 2: A random dot pattern

2. Diagrams with Differing Scales

The main topic of this paper, however, is the fixed points of similarity transformations, not congruence transformations. Figure 3(1) depicts a rectangle, onto which has been placed a similar rectangle (assuming, for the sake of discussion, a scaling factor of 50%). We would now like to find the fixed point for this case. Coxeter's *Introduction to Geometry* gives the example as follows [1]: “If two maps of the same country on different scales are drawn on tracing paper and superposed, there is just one place that is represented by the same spot on both maps. (It is understood that one of the maps may be turned over before it is superposed on the other.)”

This is an example showing the guaranteed existence of a fixed point after a similarity transformation composed of scaling and rotation. Readers who are not familiar with this example should try it by copying a reduced diagram onto a clear overhead projector sheet and placing it on top of the original diagram.

Figure 3(2) shows one method of finding the location of the fixed point. First, the diagram in Figure 3(1) is copied at a 50% reduction. That copy is then aligned with the smaller rectangle. This reduction copying and alignment
process is then repeated multiple times, and the result is that the rectangle will approach the fixed point. The reduced rectangle will of course converge towards the fixed point, but this method is not necessary to determine its location. It can also be constructed using only the relationship between the two rectangles in Figure 3(1).

![Figure 3: The fixed point in a similarity transformation](image)

To do so, when taking two similar rectangles we need to consider the similarity transformation not only of the points on their sides, but within the space that they enclose as well. To that end, draw three vertical and four horizontal lines within the rectangles, as shown in Figure 4(1). Corresponding lines are also created in the reduced rectangles. Next, draw lines connecting the vertices of the original rectangle with the corresponding vertices of the reduced rectangle. The vertices will move as a result of the similarity transformation as indicated by the arrows, but there will be a point that does not move at all. This point is the fixed point of the similarity transformation (Figure 4(2)).

Performing this operation by hand is time consuming, so I created a program using Visual Basic that allows an increase in the number of vertices. The result is the diagram shown in Figure 5. The location of the fixed point is clear in the diagram.

### 3. Construction Using a Compass and Ruler

Figure 5 makes the location of the fixed point clear, and one can also see that there is only a single fixed point. Let us search, however, for a method of
precisely locating the position of the fixed point.

Similarity transformations are performed through some combination of three simultaneous transformations: translation, rotation, and scaling. Congruence transformations, however, are performed using only translations and rotations. Figure 6 shows the case where there are no rotations. Here, the relationship between the original rectangle $ABCD$ and the scaled rectangle $A'B'C'D'$ is indicated by the corresponding sides $AB$ and $A'B'$, $BC$ and $B'C'$, $CD$ and $C'D'$, and $DA$ and $D'A'$. Connecting the corresponding vertices results in a single intersection, $O$, and this point is called the center of similitude, or the homothetic center. The relationship between the scaling from $AB$ to $A'B'$ can be understood from the homothetic center $O$. The ratio of the lengths of $AB$ and $A'B'$ are called the homothetic ratio.

The situation indicated by Figure 6 is very important to the fixed point of a similarity transformation, so the relationship between the positions should be well understood. As we saw in Figures 3 and 4, the actual similarity transformation adds a rotation to the translation and scaling, so let us consider how to construct the fixed point in such a situation. Generally known methods involve the use of a compass and ruler, and here I will introduce two such methods.

Let $P$ be the point of intersection between sides $AB$ and $A'B'$. In the case where those sides do not intersect, we will consider the intersection created by extension of those sides. Draw the circle formed by the points $P$, $A$, and $A'$, and the circle formed by points $P$, $B$, and $B'$. The fixed point will be the point $O$ formed by the intersection of the circles. Figure 7 shows an example construction, and one can see that there is a similarity relationship between

Figure 4: Adding arrows between corresponding vertices
△ OAB and △ OA′B′, scaling with O as the center and rotating so that side AB moves to side A′B′. This construction was performed according to the description in Reference [2].

The second construction uses Apollonian circles (Figure 8). Taking the vertices A and A′ on rectangles ABCD and A′B′C′D′, the locus of points at a distance of the homothetic ratio (AB : A′B′) from those vertices forms an Apollonian circle O1 with a diameter on the line formed by vertices A and A′. Similarly, the locus of points at a distance of the homothetic ratio from vertices B and B′ forms an Apollonian circle O2 with a diameter on the line formed.
by those vertices. The intersection $O$ of these two Apollonian circles is the fixed point. Also, $OA : OA' = AB : A'B'$ and $OB : OB' = AB : A'B'$. This construction was performed according to the description in reference [1].

4. Construction Using Only a Ruler

The above are standard methods of finding the fixed points, but we opened this discussion with the question as to the possibility of applying the method of construction used for congruence transformations to similarity transformations, and indeed this is possible. The method for doing so is shown in Figure 9.

**Theorem 1.** First, find the point of intersection between corresponding sides of rectangle $ABCD$ and rectangle $A'B'C'D'$. In cases where the sides do not intersect, consider extensions of those sides. Take the intersection of sides $AB$ and $A'B'$ as $P$, that of sides $CD$ and $C'D'$ as $Q$, that of sides $DA$ and $D'A'$ as $R$, and that of sides $BC$ and $B'C'$ as $S$. The fixed point of the similarity transformation is then the intersection $O$ of the lines $PQ$ and $RS$. This is the most elegant construction of the point that does not involve the use of a compass.

![Figure 9: Construction without using a compass (solution 3)](image)

**Proof.** I will now demonstrate why the constructed point $O$ is the fixed point. To do so, it is sufficient to show that the fixed point must lie on both the line $PQ$ and on the line $RS$. First, assume that the fixed point is some
point $O$ on the line $PQ$. As shown in Figure 10, the ratio $OH_1 : OH_2$ of the length of the perpendicular segments drawn from point $O$ to the sides $AB$ and $A'B'$ is the homothetic ratio $(BC : B'C')$. Similarly, the ratio $OH_3 : OH_4$ of the length of the perpendicular segments drawn from point $O$ to the sides $CD$ and $C'D'$ is also the homothetic ratio $(BC : B'C')$.

Rotate rectangle $A'B'C'D'$ anti-clockwise about the point $O$ so that the corresponding sides are made parallel, and draw a line from point $O$ to each vertex $A', B', C', D'$. A reverse calculation of the homothetic ratio on the extended lines gives a scaled rectangle $A''B''C''D''$ of the original rectangle. Comparing the original rectangle $ABCD$ with the rectangle $A''B''C''D''$ that we just constructed, we see that sides $AB$ and $A''B''$ lie on the same line, and sides $CD$ and $C''D''$ lie on the same line (Figure 11). In other words, taking some point lying on line $PQ$ as the center of rotation, the upper and lower sides are aligned. The corresponding sides will not be brought into congruence under a congruence transformation, but they are brought into a “superimposed” positional relationship like that of a similarity transformation shown in Figure 6.

We next perform the same transformation, this time taking some point $O$ on line $RS$ as the center of rotation (Figure 12). In this case, too, the ratio of the lengths of the perpendicular segments drawn from point $O$ to $DA$ and $D'A'$ is the homothetic ratio. The ratio of the perpendicular segments drawn from point $O$ to $BC$ and $B'C'$ is also the homothetic ratio.

Figure 13 shows rotating the rectangle $A'B'C'D'$ anti-clockwise about the
Figure 12: Taking the center of rotation on the line $RS$

Figure 13: Aligning the right and left sides

Figure 14: Aligning the top and bottom and the right and left sides
point $O$ so that the four corresponding sides are made parallel. Drawing segments from point $O$ to each of the vertices of the rectangle $A'B'C'D'$, and applying the homothetic ratio to reconstruct the original rectangle results in $A''B''C''D''$. From this diagram, we see that sides $B''C''$ and $BC$, and sides $D''A''$ and $DA$ lie on the same line. In other words, bringing the point of rotation to line $RS$ aligns the right and left sides.

From Figures 11 and 13, we see that to align both the top and bottom and the right and left sides, the point of rotation must lie on both line $PQ$, and moreover on line $RS$. In other words, if the intersection $O$ of lines $PQ$ and $RS$ is taken as the point of rotation, then the top and bottom and the right and left sides will align simultaneously. This point of rotation is therefore the fixed point of the similarity transformation. □

This paper presents a method of constructing fixed points using only a ruler. The method can be applied not only to congruence transformations and similarity transformations, but also to affine transformations. Because affine transformations are linear transformations by matrices, they allow transformations of both length and angle. Construction of the fixed point described by two arbitrary triangles should also be possible using this method. For details, see reference [4].

References


