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U-CYCLES IN N-PERSON TU-GAMES WITH EQUAL-SIZED OBJECTIONABLE FAMILIES OF COALITIONS

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Abstract: It has recently been proven that the non-existence of certain types of cycles of pre-imputation, fundamental cycles, is equivalent to the balancedness of a TU-game (see [3]). In some cases, the class of fundamental cycles can be narrowed and a characterization theorem may still be obtained. In this paper, we deal with n-person TU-games for which the only coalitions with non-zero value, aside from the grand coalition, are all coalitions of the same size $k \leq n$, which also form a balanced family of coalitions. This class of games includes those studied in previous papers where the non-zero value coalitions are the family of coalitions with $n-1$ players. The main result obtained in this framework is that it is always possible to find a U-cycle, a certain type of fundamental cycle, provided the game under consideration is non-balanced and n and k are relatively prime. A computational procedure to get the cycle is provided as well. In many situations, these cycles turn out to be maximal U-cycles, an even more restricted class of fundamental cycles.

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1. Introduction

The Shapley-Bondareva Theorem (see [2], [15]) is the most well-known characterization of games with transferable utility (TU-games) with non-empty core (balanced games). Recently, one of the authors proved an alternative equivalence result (see [4]). While Shapley-Bondareva's result belongs to the subject of the duality theory, that rests on very different grounds. The key is the notion of cycle of pre-imputations or fundamental cycle, which is a finite sequence of pre-imputations where each pair of neighboring elements are interrelated to each other through a transfer of some amount of utility from members of a certain coalition to the members of the complementary coalition. This transfer is carried out with the understanding that individual gains or losses within any coalition are proportional to the number of members in the coalition.

In general, fundamental cycles are difficult to obtain. However, a subclass of the fundamental cycles, the maximal U-cycles, are strongly related to a transfer scheme (see [3]) designed originally to approximate a point in the core of a balanced TU-game. In fact, some computational experiences show that maximal U-cycles appear as limit cycles when the transfer scheme is run on a non-balanced game, although there is no formal proof of this fact in the general case. There is no general proof that maximal U-cycles, or even U-cycles do exist in every non-balanced game either. Nevertheless, a quite complete set of results has been obtained for the subclass of n-person games which only admit non-zero value for the grand coalition and those having $n-1$ players (see [6], [7]). The latter is a balanced family of coalitions. The objective of this paper is to study the existence problem for U-cycles in a class of games which enlarges the class of games studied in these papers. Here, aside from the grand coalition, the only coalitions admitting non-zero value belong to a balanced family of coalitions, $\mathcal{B}_{n,k}$ having two properties: 1) all the coalitions have the same size k, and 2) all of them are generated from one of these coalitions by the application of a shift operator (Section 3). We study here the case n and k relatively prime.

It is already known that each U-cycle has associated a balanced family of coalitions, called its support which is also an objecting family of coalitions (see Theorem 1). Here we deal with an inverse problem, namely, if given an objecting balanced set of coalitions, there exits a U-cycle having it as its support. It is always true that the existence of a U-cycle (maximal U-cycle) in a game indicates that the game is non-balanced. A positive answer to the main question addressed in this paper would provide a converse statement in terms of cycles which could be computable in the case that they are maximal. Maximal U-

cycles are also related to a particular type of dynamic solution (see [16]) recently introduced in the framework of TU-games (see [7], [8]). In some cases, the dynamic solution is the union of all regular maximal U-cycles (Section 2). From this point of view, apart from the existence problem, the description of the whole set of regular maximal U-cycles is also an interesting issue. That problem has been completely solved in the case of n-person games which only admit non-zero value for the grand coalition and those having $n-1$ players in [7]. The results proved in this paper provide the key facts to develop a parallel analysis for the class of games studied here.

We would like to mention that many of the concepts supporting the notion of cycles, as well as that of dynamic solution, are much in the spirit of some of the ongoing literature on coalition formation processes (see [12], [13], [14]). Hopefully, maximal U-cycles will provide a new way to address to this important issue in cooperative game theory.

The paper is organized as follows; preliminaries and some notation are set forth in the next section. There, cycles of pre-imputation are defined and it is shown that the existence of a certain class of cycles (U-cycles) in a TU-game implies the non-balancedness of it (Corollary 4). In Section 3 we indicate how to construct a U-cycle and show that an initial point of the cycle is characterized as the solution of a linear system. However, the solution has to satisfy an additional requirement. This result is obtained for a very symmetric game, although it is shown later that it is enough to get a similar result for the more general class of games that are dealt with in this paper. The next two sections are somewhat technical and are dedicated to prove that this linear system has indeed a solution that satisfies the required extra condition. In Section 4 we introduce some basic concepts about circulant matrices and use them to prove some auxiliary results that are later needed. In particular, the balanced families of coalitions worked with here depend on n, the number of players, and a parameter k, their size. We prove that the families are minimal balanced if and only if n and k are relatively prime. In Section 5 we demonstrate that the set of appropriate solutions of the system stated in Section 3 coincides with the set of solutions of a related linear system. We prove the consistency of this latter system in the case that n and k are relatively prime. Moreover, it is shown that, in this case, the solution is unique. The final section is devoted to proving some maximal properties of the cycles having as starting point, the unique solution of the linear system studied in the previous sections.

2. Preliminaries

A TU-game is a pair $(N, v\ )$, where $N = \{1, 2, ..., n\}$, represents the set of players and v the characteristic function. We assume that v is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(N) = 1, v(\Phi) = 0$, and $v(\{i\}) = 0$ for each $i \in N$. The elements in $\mathcal{P}(N)$ are called coalitions.

The set of pre-imputations is defined by $E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i\in N} x_i = 1\}$ and the set of imputations by $A = \{x \in E : x_i \geq 0 \text{ for all } i \in N\}$.

Given a coalition $S{\in}\mathcal{P}(N)$ and a pre-imputation x, the excess of the coalition S with respect to x is defined by $e(S, x) = v(S) - x(S)$, where $x(S) = \sum_{i\in S} x_i$ if $S{\neq}\Phi$ and 0 otherwise. The excess of coalition S represents the aggregate gain (or loss, if negative) to its members if they depart from an agreement that yields x in order to form their own coalition. The core of a game (N, v) is defined by $C = \{x \in E : e(S, x) \leq 0 \text{ for all } S \in P(N)\}$.

The core of a game may be an empty set. The Shapley-Bondareva Theorem characterizes the subclass of TU-games with a non-empty core. A central role is played by balanced families of coalitions. A family of non-empty coalitions $\mathcal{B} \subseteq \mathcal{P}(N)$ is *balanced* if there exists a set of positive real numbers $(\lambda_S)_{S\in\mathcal{B}}$ satisfying $\sum\limits_{\substack{S\in\mathcal{B}\\ S\ni i}} \lambda_S = 1$, for all $i \in N$. The numbers $(\lambda_S)_{S\in\mathcal{B}}$ are called the balancing weights for $\mathcal{B}$. $\mathcal{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Equivalently, if $\chi_S \in \mathbb{R}^n$ denotes the characteristic vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 if $i \in N \backslash S$, the family $\mathcal{B}$ is balanced if there exists a family of positive balancing weights $(\lambda_S)_{S\in\mathcal{B}}$, such that

$$\sum_{S\in\mathcal{B}} \lambda_S.\chi_S = \chi_N. \tag{1}$$

A well-known result establishes that

$$\sum_{S\in\mathcal{B}} \lambda_S.x\,(S) = x\,(N) \tag{2}$$

for all balanced family of coalitions. A game (N, v) is balanced if

$$\sum_{S\in\mathcal{B}} \lambda_S.v(S) \leq v(N) \tag{3}$$

for all balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. As usual, the quantity $\sum_{S\in\mathcal{B}} \lambda_S.v(S)$ is called the worth of $\mathcal{B}$ with respect to the set of balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. The Shapley-Bondareva Theorem states that a game (N, v) has a non-empty core if and only if it is balanced. An *objectionable* family is a balanced family not satisfying (3).

In what follows, the notion of U-transfer will play a central role. Given $x \in E$ and a proper coalition S, the *U-transfer* from $N\backslash S$ to S with respect to x is the pre-imputation y defined by

$$y = x + e(S, x).\beta_S. \tag{4}$$

Here $\beta_S = \frac{\chi_S}{|S|} - \frac{\chi_{N\backslash S}}{|N\backslash S|}$ if S is a proper coalition and the zero vector of $\mathbb{R}^n$ in the other case. $|S|$ indicates the number of players in S. The vector β_S describes a transfer of one unit of utility from the members of $N\backslash S$ to the members of S. The U-transfer is maximal if $e(S, x) \geq e(T, x)$ for all $T \in \mathcal{P}(N)$.

Now we introduce some kinds of cycles of pre-imputations and state, without proof, several results proved in [4], [5] and [6].

A *cycle* $\mathbf{c}$ in a TU-game (N, v) is a finite sequence of pre-imputations $(x^k)_{k=1}^m$, $m > 1$, along with two associated sequences, one of positive real numbers $(\mu_k)_{k=1}^m$, and the other of non-empty proper coalitions of N (not necessarily all different) $(S_k)_{k=1}^m$ satisfying the neighboring transfer properties

$$x^{k+1} = x^k + \mu_k.\beta_{S_k} \text{ for all } k = 1, \ldots, m, \tag{5}$$

and

$$x^{m+1} = x^1, \tag{6}$$

as well.

A cycle is *fundamental* if $\mu_k \leq e(S_k, x^k)$ for all $k = 1, ..., m$.

A cycle is a *U-cycle* if $\mu_k = e(S_k, x^k)$ for all $k = 1, ..., m$.

A U-cycle is *maximal* if for all $k = 1, ..., m, e(S_k, x^k) \geq e(S, x^k)$ for all coalition S. A maximal U-cycle is *regular* if for all $k = 1, ..., m, e(S_k, x^k) > e(S, x^k)$ for all $S \neq S_k$.

Given that cycle $\mathbf{c} = (x^k)_{k=1}^m$, we denote the sequence $(S_k)_{k=1}^m$ by supp($\mathbf{c}$). Also, we refer to the numbers $(\mu_k)_{k=1}^m$ as the *transfer amounts.*

The following result is a special case of a similar one proved in [4], Theorem 1 for cycles.

Theorem 1. *Let* $\mathbf{c} = (x^k)_{k=1}^m$ *be a U-cycle in a TU-game* (N, v)*. Then the coalitions in supp(*$\mathbf{c}$*)=* $(S_k)_{k=1}^m$ *form a balanced family of coalitions with weights given by*

$$\lambda_{S_k} = \frac{n}{|S_k| \, . \, |N\backslash S_k|} . \frac{e(S_k, x^k)}{\sum_{q=1}^m \frac{e(S_q, x^q)}{|N\backslash S_q|}} \tag{7}$$

for all $k = 1, \ldots, m$.

The existence of cycles in a TU-game is strongly related to the non-existence of points in the core of the game. The results proved in [4] allow us to state the following theorem.

Theorem 2. *If there exists a U-cycle (maximal U-cycle)* $\mathbf{c}$ *in a TU-game* (N, v), *then the game is non-balanced.*

One of the open questions in this subject is the existence of U-cycles (maximal U-cycles) in non-balanced TU-games. Up to now, only partial results have been obtained.

3. Construction of a Basic U-Cycle

The aim of the following sections is to prove a converse of Theorem 2. We have not been successful in proving it in a general framework. In [5] we get an existence result of U-cycles in non-balanced monotonic 3-person games. This result was extended later to n-person games, in which the only permissible coalitions with non-zero value are the grand coalition and those with $n-1$ players (see [6]). The U-cycles exhibited in these papers have associated objectionable families of coalitions consisting of equal-sized coalitions with $n-1$ players in each one. Our goal here is to show that for some classes of objectionable families of coalitions $\mathcal{B}$ with coalitions of the same size, we are able to construct a U-cycle whose support has all of its coalitions belonging to that family.

In the first part of this section we describe how to construct a basic U-cycle for a quite particular situation. We end the section dealing with a more general situation.

Now we introduce some terminology. We will make an extensive use of the *linear shift operator* $\mathbf{T}$ on $\mathbb{C}^n$ ($\mathbb{C}$ stands for the set of complex numbers) which is defined by

$$\mathbf{T}((z_1, ..., z_{n-1}, z_n)) = (z_n, z_1, ..., z_{n-1}).$$

$\mathbf{T}^k$ will stand for k successive applications of the operator $\mathbf{T}$. As usual, $\mathbf{T}^0$ will represent the identity operator.

For $k < n$, let $K = \{1, ..., k\}$. The family of coalitions $\mathcal{B}_{n,k}$ is defined as follows: let S_1 be the coalition having the vector χ_K as its characteristic vector, and for each $2 \leq i \leq n$, let S_i be the coalition having $\chi_{S_i} = \mathbf{T}^{i-1}(\chi_K)$ as its characteristic vector. It is easy to see that $\mathcal{B}_{n,k}$ is a balanced family of coalitions with $(\frac{1}{k})_{i=1}^n$ as a set (probably not unique) of balancing weights. For the rest of the paper we will consider $n \geq 3$, and $1 \leq k < n$. The case $\mathcal{B}_{n,n-1}$

has extensively been studied in [6] and [7].

We start proving two properties, where the subscript of the coalitions involved will be considered mod(n).

Lemma 3. *Given the family $\mathcal{B}_{n,k}$, and a game (N,v) with $v(S_i)=1$ for all $S_i \in \mathcal{B}_{n,k}$:*

(i) $\mathbf{T}^k(\beta_{S_i}) = \beta_{S_{k+i}}$ *for all* $i=1,...,n$;

(ii) $e(S_{k+i}, \mathbf{T}^k(x)) = e(S_i, x)$ *for all* $S_i \in \mathcal{B}_{n,k}, x \in E$.

Proof. To get property (i) we recall that $\chi_{S_i} = \mathbf{T}^{i-1}(\chi_{S_1}), i=1,...,n$. Therefore, $\mathbf{T}^k(\chi_{S_i}) = \mathbf{T}^{k+i-1}(\chi_{S_1}) = \chi_{S_{k+i}}$ for all $i=1,...,n$. Furthermore, it is easy to show that $\beta_S = \frac{n}{|S|.|N\backslash S|}.\chi_S - \frac{1}{|N\backslash S|}.\chi_N$ for each proper coalition S. Then,

$$\mathbf{T}^k(\beta_{S_i}) = \frac{n}{|S_i|\,.\,|N\backslash S_i|}.\mathbf{T}^k(\chi_{S_i}) - \frac{1}{|N\backslash S_i|}.\mathbf{T}^k(\chi_N)$$
$$= \frac{n}{|S_{k+i}|\,.\,|N\backslash S_{k+i}|}.\chi_{S_{k+i}} - \frac{1}{|N\backslash S_{k+i}|}.\chi_N = \beta_{S_{k+i}}.$$

To prove (ii) we note that

$$e(S_{k+i}, \mathbf{T}^k(x)) = 1 - \sum_{j=k+i}^{2.k+i-1} \mathbf{T}^k(x)_j = 1 - \sum_{j=i}^{k+i-1} x_j = e(S_i, x).$$

All the indexes in the above summations are also considered mod(n). □

Theorem 4. *Let (N,v) be a TU-game with its characteristic function satisfying $v(N)=v(S_i)=1$ for all $S_i \in \mathcal{B}_{n,k}$, and for some $1 \le k < n$. If $\tilde{x}$ is a solution of the following linear system:*

$$\mathbf{T}^k(x) = x + e(S_1, x).\beta_{S_1}, \tag{8}$$

then $\tilde{x}$ is the starting point of a U-cycle $\tilde{\mathbf{c}}$ provided $\tilde{x} \in E$ and $e(S_1, \tilde{x}) > 0$. Moreover, all the coalitions in the support of $\tilde{\mathbf{c}}$ belong to $\mathcal{B}_{n,k}$. As before, $S_1 = \{1,...,k\} \in \mathcal{B}_{n,k}$.

Proof. We first point out that $\mathcal{B}_{n,k}$ is an objectionable family in this game. Indeed, its worth is $\frac{n}{k} > 1$ when considered with respect to the set of balancing weights $(\frac{1}{k})_{i=1}^n$. Now we will show that $\tilde{\mathbf{c}} = (\mathbf{T}^{i.k}(\tilde{x}))_{i=0}^{\tilde{n}-1}$ is a U-cycle for some $\tilde{n} \le n$. First, we note that $\mathbf{T}^{n.k}(\tilde{x}) = \mathbf{T}^0(\tilde{x}) = \tilde{x}$. Let $\tilde{n}$ be the first integer satisfying $\tilde{n}.k = 0$ mod(n). Then, $\mathbf{T}^{\tilde{n}.k}(\tilde{x}) = \tilde{x}$. Besides, from (8) and the fact that $\tilde{x} \in E$ it follows that $\mathbf{T}^k(\tilde{x}) \in E$. Thus, since $\mathbf{T}^k(\tilde{x}) = \tilde{x} + e(S_1, \tilde{x}).\beta_{S_1}$, and $\mathbf{T}^{2.k}(\tilde{x}) = \mathbf{T}^k(\mathbf{T}^k(\tilde{x}))$, we get that

$$\mathbf{T}^{2.k}(\tilde{x}) = \mathbf{T}^k(\tilde{x} + e(S_1, \tilde{x}).\beta_{S_1}) = \mathbf{T}^k(\tilde{x}) + e(S_1, \tilde{x}).\mathbf{T}^k(\beta_{S_1})$$

$$= \mathbf{T}^k(\tilde{x}) + e(S_{k+1}, \mathbf{T}^k(\tilde{x})).\beta_{S_{k+1}},$$

the last equality being obtained by using properties (i) and (ii) Lemma 3. Because of the latter condition and since $e(S_1, \tilde{x}) > 0$, $e(S_{k+1}, \mathbf{T}^k(\tilde{x})) > 0$ too. Thus, $\mathbf{T}^{2.k}(\tilde{x})$ is a U-transfer from $\tilde{x}$. Now, an inductive argument shows that $\mathbf{T}^{i.k}(\tilde{x}) = \mathbf{T}^{(i-1).k}(\tilde{x}) + e(S_{(i-1).k+1}, \mathbf{T}^{(i-1).k}(\tilde{x})).\beta_{S_{(i-1).k+1}}, i = 3, ..., \tilde{n}$. These relationships assure that we get pre-imputations at each stage. The inductive argument also shows that $e(S_{(i-1).k+1}, \mathbf{T}^{(i-1).k}(\tilde{x})) > 0$, for all $i = 3, ..., \tilde{n}$ indicating that all these pre-imputations are U-transfers too. The subindexes of the coalitions are considered mod(n). □

Remark 5. For the cycle $\tilde{\mathbf{c}}$ obtained in the previous result we have that supp($\tilde{\mathbf{c}}$)= $\{S_1, S_{k+1}, ..., S_{(\tilde{n}-1).k+1}\} \subseteq \mathcal{B}_{n,k}$. The equality holds if and only if k, n are relatively prime. The consistency of system (8) under this condition will be proven later.

The next two results stress on the fact that considering $v(S_i) = 1$ for all $S_i \in \mathcal{B}_{n,k}$, which is a not too restrictive condition.

Lemma 6. *Let (N, v) a game with $v(N) = 1, v(S_i) = \mu_i$ for all $S_i \in \mathcal{B}_{n,k}, n$ and k relatively prime. Then, there is a one-to-one correspondence between the U-cycles of pre-imputations in this game having $\mathcal{B}_{n,k}$ as their support[1], and those of a game $(N, \bar{v})$ with $\bar{v}(N) = 1, \bar{v}(S_i) = \mu$ for all $S_i \in \mathcal{B}_{n,k}$, with $\mu = \frac{1}{n}.\sum_{i=1}^{n} \mu_i$, and whose support is also $\mathcal{B}_{n,k}$.*

Proof. Let $\bar{\mathbf{c}} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^k)$ be a U-cycle in the game $(N, \bar{v})$ with supp($\bar{\mathbf{c}}$) equal to $\mathcal{B}_{n,k}$. Let $t = (t_1, t_2, ..., t_n)$ be the unique solution of the linear system $\mathbf{E}t' = b'$, where $\mathbf{E}$ is the square matrix of order n having χ_{S_i} as its i-th row, and $b = (\mu_1 - \mu, \mu_2 - \mu, ..., \mu_n - \mu)$. The non-singularity of matrix $\mathbf{E}$ is a consequence of the minimal balancedness of the family $\mathcal{B}_{n,k}$ whenever n and k are relatively prime. This fact will be proven in the next section in Theorem 14. Associating the following matrix product in two different ways, we get that

$$(1, 1, ..., 1)\mathbf{E}t' = k \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} \mu_k - n.\mu = 0.$$

Thus, we conclude that $\sum_{k=1}^{n} t_k = 0$.

We now define $x^j = \bar{x}^j + t$ for all $j = 1, 2, ..., n$. All these vectors are then pre-imputations. We claim that $\mathbf{c} = (x^j)_{j=1}^n$ is a U-cycle of pre-imputations in

[1] The assertion supp($\mathbf{c}$)=$\mathcal{B}_{n,k}$ means that the family of coalitions appearing in the sequence supp($\mathbf{c}$) coincides with $\mathcal{B}_{n,k}$.

the game (N,v) with supp($\mathbf{c}$) $= \mathcal{B}_{n,k}$. To prove this, we first note that, for each $j = 1,...,n, S_i \in \mathcal{B}_{n,k}$, we have that

$$\begin{aligned} e(S_i, x^j) &= v(S_i) - x^j(S_i) = \mu_i - \bar{x}^j(S_i) - t(S_i) \\ &= \mu - \bar{x}^j(S_i) + (\mu_i - \mu - t(S_i)) = \mu - \bar{x}^j(S_i) = e(S_i, \bar{x}^j). \quad (9) \end{aligned}$$

Here we point out that $e(S,x)$ stands for $v(S)-x(S)$ and $e(S,\bar{x}) = \bar{v}(S)-\bar{x}(S)$. Thus, if

$$\bar{x}^2 = \bar{x}^1 + e(S_{i_1}, \bar{x}^1)\beta_{S_{i_1}},$$

then

$$x^2 = \bar{x}^2 + t = \bar{x}^1 + t + e(S_{i_1}, \bar{x}^1)\beta_{S_{i_1}} = x^1 + e(S_{i_1}, x^1)\beta_{S_{i_1}},$$

which proves that x^2 is a U-transfer obtained from x^1. An inductive argument shows that, in general, x^j is a U-transfer obtained from x^{j-1}, for $j = 3,...,n$. It is also easy to see that $x^{n+1} = x^1$. Therefore, $\mathbf{c}$ is a U-cycle in (N,v). The converse is proven similarly by using the inverse transformation $\bar{x}^i = x^i - t$. □

Lemma 7. *Let (N,v) a game with $v(N) = 1$ and $v(S) = \mu$ for all $S \in \mathcal{B}_{n,k}, \mu > \frac{k}{n}$. Let $(N,\bar{v})$ another game with $\bar{v}(N) = 1$ and $\bar{v}(S) = 1$ for all $S \in \mathcal{B}_{n,k}$. Then, there is a one to one correspondence between the U-cycles of (N,v) whose support is included in $\mathcal{B}_{n,k}$ and those of $(N,\bar{v})$ whose support is also included in $\mathcal{B}_{n,k}$.*

Proof. We note that the condition $\mu > \frac{k}{n}$ guarantees that $\mathcal{B}_{n,k}$ is an objecting family in $(N,\bar{v})$. Now, let $\bar{\mathbf{c}} = (\bar{x}^j)_{j=1}^m$ be a U-cycle in the game $(N,\bar{v})$ with supp($\bar{\mathbf{c}}$)$= (S_{i_j})_{j=1}^m$ included in $\mathcal{B}_{n,k}$. Let us define

$$x^j = \frac{n}{n-k}.(1-\mu).(\frac{1}{n}.\chi_N) + (1 - \frac{n}{n-k}.(1-\mu)).\bar{x}^j, \quad (10)$$

$j = 1,...,m$. Since $e(S,x^j) = (1 - \frac{n}{n-k}.(1-\mu)).e(S,\bar{x}^j)$ for all $S \in \mathcal{B}_{n,k}, j = 1,...,m$, the condition $\mu > \frac{k}{n}$ guarantees that the quantities $e(S,x^j)$ and $e(S,\bar{x}^j)$, $S \in \mathcal{B}_{n,k}$, are ordered in the same way and that they have the same sign. On the other hand, a simple calculation shows that

$$\begin{aligned} x^j + e(S_{i_j}, x^j).\beta_{S_{i_j}} &= \frac{n}{n-k}.(1-\mu).(\frac{1}{n}.\chi_N) + (1 - \frac{n}{n-k}.(1-\mu)).\bar{x}^j \\ &\quad + (1 - \frac{n}{n-k}.(1-\mu)).e(S_{i_j}, \bar{x}^j).\beta_{S_{i_j}} \\ &= \frac{n}{n-k}.(1-\mu).(\frac{1}{n}.\chi_N) + (1 - \frac{n}{n-k}.(1-\mu)).\bar{x}^{j+1} = x^{j+1} \end{aligned}$$

for all $j = 1,...,m$. From these relationships is easy to conclude that $\mathbf{c} = (x^j)_{j=1}^m$ is a U-cycle in the game (N,v) with supp($\mathbf{c}$)=supp($\mathbf{c}$). The converse is proven in a similar way using the inverse transformation of (10). We stress on the fact that $e(S, x^j)$ is computed with reference to the game (N, v) while $e(S, \bar{x}^j)$ is computed with reference to the game $(N, \bar{v})$. $\square$

In this result we do not require n and k being relatively prime, while the following is a direct consequence of Lemma 6 and Lemma 7.

Corollary 8. *The transformations (9) and (10) define a one-to-one correspondence between the U-cycles of two non-balanced games whose supports coincide with $\mathcal{B}_{n,k}$ provided n and k are relatively prime.*

The remaining sections are aimed to prove that system (8) always has a solution satisfying the conditions of Theorem 4. Many of the proofs we present here depend on results about circulant matrices although some techniques from graph theory could be used as well.

4. Circulant Matrices

Many properties of circulant matrices are well known and easily derived (see [9], [11]). Here we state without proof some results we will need later.

For any vector $z \in \mathbb{C}^n$, the *circulant matrix* $circ(z)$ is the $n \times n$ matrix whose k-th row is given by the vector of $\mathbf{T}^{k-1}(z), k = 1, ..., n$. Clearly, the identity matrix $\mathbb{I}^n = circ(e^1)$, where $e^1 = \chi_{\{1\}}$. In general, $e^i = \mathbf{T}^{i-1}(e^1), i = 1, ..., n$.

The following result describes the eigenstructure of any circulant matrix.

Theorem 9. *Let $M = circ(z)$ for a vector $z \in \mathbb{C}^n$, and w a primitive n-th root of unity. For each integer $l, 0 \leq l \leq n-1$, let the number ψ_l be defined by*

$$\psi_l = \sum_{k=1}^{n} z_k . w^{(k-1).l}$$

and $u^l \in \mathbb{C}^n$ the vector defined by

$$u^l = \frac{1}{\sqrt{n}} .(1, w^l, ..., w^{(n-1).l}).$$

Then ψ_l is an eigenvalue for M with u^l as an associated eigenvector.

The reader is referred to [10], Theorem 1 for a proof of this result. From now on, we will take $w = e^{\frac{2\pi}{n} i}$ (in this context, $i = \sqrt{-1}$).

Remark 10. We point out that the eigenvalues of a circulant matrix not need to be all distinct. We note also that all circulant matrix M can be diagonalized by the matrix

$$C_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2.(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2.(n-1)} & \cdots & w^{(n-1).(n-1)} \end{pmatrix},$$

namely, $M = C_n.\Lambda.C_n^{-1}$, ψ_l being the $(l+1)$-th diagonal element of the diagonal matrix $\Lambda, 0 \leq l \leq n-1$.

Remark 11. C_n turns to be a unitary matrix ($C_n.C_n^* = C_n^*.C_n = I_n$). Here $\bar{C}_n^*$ denotes the conjugate transpose of C_n ($C_n^* = \bar{C}_n'$). C_n^* is usually called the *Fourier matrix of order* n, and denoted by F_n.

From Remark 10 it is easy to prove the following result whose proof can be found in [11], Theorem 3.1.

Theorem 12. *Let $M_1 = circ(z^1), M_2 = circ(z^2)$ two $n \times n$ circulant matrices with eigenvalues*

$$\psi_l^1 = \sum_{k=1}^{n} z_k^1.w^{(k-1).l}, \psi_l^2 = \sum_{k=1}^{n} z_k^2.w^{(k-1).l},$$

$l = 0, ..., n-1$. Then, $M_1 + M_2$ is also a circulant matrix having $\psi_l^1 + \psi_l^2, l = 0, ..., n-1$ as its eigenvalues.

To determine if a circulant matrix with real entries is non-singular, the next result is useful (see [10], Theorem 3).

Theorem 13. *Let $z = (z_1, ..., z_n) \in \mathbb{R}^n$. If the entries of z form a weakly monotone sequence (that is, a non-decreasing or non-increasing sequence) of non-negative or non-positive real numbers, then the matrix $M = circ(z)$ is singular if and only if for some integer $d|n, d \geq 2$, the vector z consists of $\frac{n}{d}$ consecutive constant blocks of length d. As usual, $d|n$ indicates that d divides n.*

We are ready to prove some results which we will use later. The next one establishes when the balanced family of coalitions $\mathcal{B}_{n,k}$ is minimal balanced.

Theorem 14. *The family $\mathcal{B}_{n,k}$, is minimal balanced if and only if n and k are relatively prime.*

Proof. First we assume that $\mathcal{B}_{n,k}$ is minimal balanced. If there exists $1 < d \leq k$ such that $d|k$ and $d|n$, it is easy to see that the proper subfamily of

$\mathcal{B}_{n,k}$ having coalitions with characteristic vectors given by $\mathbf{T}^{d.j}(\chi_K), j = 1, .., \frac{n}{d}$ is also balanced with $(\frac{d}{k})_{j=1}^{\frac{n}{d}}$ as a set of balancing weights. But this contradicts the minimal balancedness property assumed for $\mathcal{B}_{n,k}$.

To prove the converse, we note that a balanced family of coalitions is minimal if and only if it has a unique set of balancing weights, namely, if system (1) has a unique solution. To this end, let us point out that the matrix of this system, in the case $\mathcal{B} = \mathcal{B}_{n,k}$, is $E = circ(\chi_K)$. The entries of χ_K form a non-increasing sequence of real numbers. Besides, since n and k are relative prime, there is no $d|n, d \geq 2$ such that χ_K can be partitioned into $\frac{n}{d}$ consecutive constant blocks of length d. Otherwise, the set of the first k components (all equal to 1) should be partitioned exactly in s blocks of length d, and the set of the last $n-k$ components (all equal to 0) should be partitioned exactly into t blocks of length d. Thus, $k = s.d$ and $n - k = t.d$. Therefore d would be a common divisor for both n and k,which is a contradiction. Thus, Theorem 13 guarantees that E is non-singular and system (1) has a unique solution. □

Now, let $P^k = circ(\mathbf{T}^k(e^1))$, $P^{n,k} = P^k - I^n$, and $\tilde{P}^{n,k}$ the restriction of $P^{n,k}$ to its $n-1$ first rows, and let

$$C_{1,1}^{n,k} = \begin{pmatrix} \tilde{P}^{n,k} \\ \chi_N \end{pmatrix} \tag{11}$$

(the use of the notation $C_{1,1}^{n,k}$ will become clear in the next section).

Proposition 15. *a) The rank of $P^{n,k}$ is $n-1$ if and only if n and k are relatively prime.*

b) The rank of $C_{11}^{n,k}$ is n if and only if n and k are relatively prime.

Proof. Because of Theorem 12, a) is proven provided it is shown that P^k has 1 as an eigenvalue with multiplicity one if and only if n and k are relatively prime. But this follows directly from Theorem 9 which shows that the eigenvalues of P^k are $e^{2.\pi.\frac{k.l}{n}i}, l = 0, ..., n-1$.

To prove b) let us consider the linear system

$$C_{1,1}^{n,k}.\lambda' = 0', \tag{12}$$

$\lambda = (\lambda_1, ..., \lambda_n)$. The first $n-1$ equations of this system can be written as

$$\lambda_i = \lambda_{k+i},$$

$i = 1, ..., n-1$, the second subscript being considered mod(n). The assignation $i \to k+i$ defines a permutation of N. Therefore, it can be decomposed as a product of d disjoint cycles. Besides, $d = 1$ if and only if k, n are relatively prime. Because of this observation, if $d = 1$, the solution is forced to satisfy

$\lambda_1 = ... = \lambda_n$. Finally, since this solution must satisfy the n-th equation of the system under consideration, the common value for its entries must be zero, indicating that the columns of $C_{1,1}^{n,k}$ are linearly independent.

If k, n are not relatively prime ($d > 1$), a solution of the linear system (12) is forced to be constant over those components whose subscript belongs to the same cycle in the decomposition. In this situation, the n-th equation admits several non-zero solutions satisfying that piecewise constant property. □

Remark 16. The first part of the above result is a particular case of the more general statement which indicates that the rank of $P^{n,k}$ always coincides with the multiplicity of the eigenvalue 1 of P^k. On the other hand, it can be shown that $dim(Ker(C_{1,1}^{n,k})) = d - 1$, where $d = (k, n)$ is the greatest common divisor of n and k.

Proof. First we assume that $\mathcal{B}_{n,k}$ is minimal balanced. If there exists $1 < d \leq k$ such that $d|k$ and $d|n$, it is easy to see that the proper subfamily of $\mathcal{B}_{n,k}$ having coalitions with characteristic vectors given by $\mathbf{T}^{d.j}(\chi_K), j = 1, .., \frac{n}{d}$ is also balanced with $(\frac{d}{k})_{j=1}^{\frac{n}{d}}$ as a set of balancing weights. But this contradicts the minimal balancedness property assumed for $\mathcal{B}_{n,k}$.

To prove the converse, we note that a balanced family of coalitions is minimal if and only if it has a unique set of balancing weights, namely, if system (1) has a unique solution. To this end, let us point out that the matrix of this system, in the case $\mathcal{B} = \mathcal{B}_{n,k}$, is $E = circ(\chi_K)$. The entries of χ_K form a non-increasing sequence of real numbers. Besides, since n and k are relative prime, there is no $d|n, d \geq 2$ such that χ_K can be partitioned into $\frac{n}{d}$ consecutive constant blocks of length d. Otherwise, the set of the first k components (all equal to 1) should be partitioned exactly in s blocks of length d, and the set of the last $n - k$ components (all equal to 0) should be partitioned exactly into t blocks of length d. Thus, $k = s.d$ and $n - k = t.d$. Therefore d would be a common divisor for both n and k, which is a contradiction. Thus, Theorem 13 guarantees that E is non-singular and system (1) has a unique solution. □

5. The Existence of an Initial Point for a Cycle

As we mention at the end of Section 3, we still have to prove that system (8) has a solution x which is a pre-imputation satisfying $e(S_1, x) > 0$. Let $C^{n,k}$ the matrix of order $n+1$ be defined by

$$C^{n,k} = \left(\begin{array}{cc} \tilde{P}^{n,k} & \tilde{\beta}'_{S_{n-k+1}} \\ \chi_N & 0 \\ \chi_{S_1} & 1 \end{array} \right). \tag{13}$$

Here $\tilde{\beta}_{S_{n-k+1}}$ stands for the restriction of $\beta_{S_{n-k+1}}$ to its $n-1$ first entries. We first state the following equivalence result.

Proposition 17. *Let x be a pre-imputation satisfying system (8). Then, $\lambda = (x_1, ..., x_n, e(S_1, x))$ satisfies the linear system*

$$C^{n,k}.\lambda' = \left(\begin{array}{c} e^{n\prime} \\ 1 \end{array} \right). \tag{14}$$

Conversely, if $\lambda = (\lambda_1, ..., \lambda_n, \lambda_{n+1})$ satisfies system (14), then $x = (\lambda_1, ..., \lambda_n)$ is a pre-imputation which satisfies system (8) and $e(S_1, x) = \lambda_{n+1}$.

Proof. Let us assume first that a pre-imputation $x = (x_1, ..., x_n)$ satisfies system (8). Then it also satisfies the related linear system $P^k(x)' = x' + \lambda_{n+1}.\beta'_{S_1}$ in the $n+1$ variables $x_1, ..., x_n, \lambda_{n+1}$, with $\lambda_{n+1} = e(S_1, x)$. Rearranging the equations of this last system by putting the first $k-$equations in the last k positions, and then, deleting the n-th equation, we obtain the $n-1$ first equations of system (14). It is worth noting that this new arrangement makes that the $(n+1)$-th column of the system can be identified with $\tilde{\beta}'_{S_{n-k+11}}$. The last two equations of system (14) indicate that x is a pre-imputation and that

$$\lambda_{n+1} = e(S_1, x) = 1 - \chi_{S_1}.x'.$$

The converse is a direct consequence that the last row of the matrix

$$\left(\begin{array}{cc} P^{n,k} & \beta'_{S_{n-k+1}} \end{array} \right)$$

is a linear combination of the first $n-1$ rows. □

The results we will prove next are the key to providing starting points for cycles associated to families of coalitions $\mathcal{B}_{n,k}$ with n and k relatively prime. A well-known result in multivariate analysis establishes that, for a partitioned definite positive matrix

$$C = \left(\begin{array}{cc} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{array} \right),$$

$$\det(C) = \det(C_{2,2} - C_{2,1}.(C_{1,1})^{-1}.C_{1,2}). \det(C_{1,1}) \tag{15}$$

provided $C_{1,1}$ is non-singular (see [1], Appendix I). With almost the same proof given in [1], it can be shown that the condition of being definite positive can be dispensed with.

Theorem 18. *The matrix $C^{n,k}$ is non-singular if and only if n and k are relatively prime.*

Proof. We first assume that n and k are relatively prime. We will apply (15) to $C^{n,k}$ defined in (13) partitioned in such a way that it has $C^{n,k}_{1,1}$ as the matrix defined by (11) and $C^{n,k}_{2,2} = 1$ as its main diagonal blocks. Then, $C^{n,k}_{1,2} = \begin{pmatrix} \tilde{\beta}'_{S_{n-k+1}} \\ 0 \end{pmatrix}$, and $C^{n,k}_{2,1} = \chi_{S_1}$. The non-singularity of $C^{n,k}$ follows from the non-singularity of $C^{n,k}_{1,1}$ proved in Proposition 15, and from the quite surprising fact that always

$$C^{n,k}_{2,1}.(C^{n,k}_{1,1})^{-1}.C^{n,k}_{1,2} = \frac{1}{2}.$$

A proof of this claim is provided in Lemma 20 below.

To prove the converse, let us assume that $(k, n) = d \geq 2$. If $d > 2$, the rank of $C^{n,k}_{11}$ is less that $n-1$ (see Remark 16) and the rank of

$$\begin{pmatrix} C^{n,k}_{1,1} & C^{n,k}_{1,2} \end{pmatrix} \tag{16}$$

is at most $n-1$. This implies that the rank of $C^{n,k}$ is at most n, so, it is singular. If $d = 2$, we will show that the linear system

$$C^{n,k}_{1,1}.\lambda' = C^{n,k}_{1,2} \tag{17}$$

is consistent, which implies once more, that the rank of (16) is at most, $n-1$. To see this, first let us assume that $C^{n,k}_{1,2} = (r_1, ..., r_n)'$. From the first equation of the system, we get that

$$\lambda_{k+1} = \lambda_1 + r_1$$

and from the $(s.k+1)$-th equation, and after successive substitutions we get that

$$\lambda_{(s+1).k+1} = \lambda_1 + \sum_{j=0}^{s} r_{j.k+1} \tag{18}$$

$s = 1, ..., \frac{n}{2} - 2$. All the subindexes involved are considered mod(n).

In a similar way we obtain that

$$\lambda_{(s+1).k+k} = \lambda_k + \sum_{j=0}^{s} r_{j.k+k} \tag{19}$$

for $s = 0, ..., \frac{n}{2} - 2$. From the last equation of system (17) it follows that the solution has to satisfy the relation

$$\frac{n}{2}.(\lambda_1+\lambda_k)+\sum_{s=1}^{\frac{n}{2}-1}\left(\sum_{j=0}^{s-1} r_{j.k+1}+\sum_{j=0}^{s-1} r_{j.k+k}\right)=0 \qquad (20)$$

which always has a non-trivial solution λ_1, λ_k. Besides, the solution of system (17) also has to satisfy the equation $(\frac{n}{2}-1).k+1$, which is possible only if $\sum_{j=0}^{\frac{n}{2}-1} r_{j.k+1}=0$. To verify this condition in the case we are dealing with, we first note that the indexes involved are all the $\frac{n}{2}$ odd ones. Therefore, $r_{j.k+1}=-\frac{1}{n-k}$ for the $\frac{n-k}{2}$ odd indexes associated with the first $n-k$ components of $\hat{\beta}_{S_{n-k+1}}$, and $r_{j.k+1}=\frac{1}{k}$ for the $\frac{k}{2}$ odd indexes associated with the $k-1$ last components of $\hat{\beta}_{S_{n-k+1}}$. We point out here that, in this case, the component of $\beta_{S_{n-k+1}}$ deleted to get $\hat{\beta}_{S_{n-k+1}}$, has associated an even index. Thus, $\sum_{j=0}^{\frac{n}{2}-1} r_{j.k+1}=-\frac{n-k}{2}.\frac{1}{n-k}+\frac{k}{2}.\frac{1}{k}=0$. This guarantees that each non-trivial solution of (20) generates a non-trivial solution for (16) showing that the rank of that matrix is less than n. □

Remark 19. To get (18) and (19) we have taken into account that $d=2$, and a decomposition argument similar to that already used during the proof of part b) in Proposition 15, to show that the subscripts of the components of the solution λ can be grouped in the two subsets of indexes considered in these relationships.

Lemma 20. *Let $C^{n,k}$ be partitioned as indicated above. Then*

$$C_{2,1}^{n,k}.(C_{1,1}^{n,k})^{-1}.C_{1,2}^{n,k}=\frac{1}{2}.$$

Proof. Let us call $\lambda=C_{2,1}^{n,k}.(C_{1,1}^{n,k})^{-1}$. Then, λ is the solution of the linear system $\lambda.C_{1,1}^{n,k}=C_{2,1}^{n,k}$. It is easy to verify that the vector $\lambda=(\lambda_1,...,\lambda_n)$ with $\lambda_i=\frac{i}{n}-1, i=1,...,n-1$ and $\lambda_n=\frac{k}{n}$ is the solution of that system. Therefore, and after a few simple arithmetic manipulations we get that

$$C_{2,1}^{n,k}.(C_{1,1}^{n,k})^{-1}.C_{1,2}^{n,k}=\lambda.\tilde{\beta}'_{S_{n-k+1}}=-\frac{1}{n-k}.\sum_{i=1}^{n-k}(\frac{i}{n}-1)+\frac{1}{k}.\sum_{i=n-k+1}^{n-1}(\frac{i}{n}-1)$$

$$=-\frac{1}{n-k}.\frac{(n-k).(n-k+1)}{2n}+\frac{n-k}{n-k}+\frac{1}{k}.\sum_{i=1}^{k}(\frac{n-k+i}{n}-1)$$

$$= -\frac{n-k+1}{2n} + 1 + \frac{1}{k}.k.\frac{n-k}{n} - \frac{1}{k}.k + \frac{1}{k}.\frac{k.(k+1)}{2.n}$$
$$= \frac{1}{2.n}.(-(n-k+1) + 2.(n-k) + k + 1) = \frac{1}{2}. \quad \square$$

Remark 21. The vector λ in the previous proof is always a solution of the linear system $\lambda.C_{1,1}^{n,k} = C_{2,1}^{n,k}$ whether $C_{1,1}^{n,k}$ is non-singular or not.

To show that the component λ_{n+1} of the unique solution of (14) is positive, we first prove the following auxiliary result.

Proposition 22. *If n and k are relatively prime, then* $\det(C_{1,1}^{n,k}) = (-1)^{n-1}.n$. *Besides, if* $\hat{C}_{n+1}^{n,k} = \begin{pmatrix} C_{1,1}^{n,k} & e^{n\prime} \\ C_{2,1}^{n,k} & 1 \end{pmatrix}$, *then* $\det(\hat{C}_{n+1}^{n,k}) = (-1)^{n-1}(n-k)$.

Proof. We already know that $C_{1,1}^{n,k}$ is non-singular (see Proposition 15). We now prove that $\det(C_{1,1}^{n,k}) = -(1)^{n-1}.n$. To see this, we will use (15) again with $C_{1,1} = \tilde{P}_n^{n,k}$, the matrix obtained from $\tilde{P}^{n,k}$ by deleting the n-th column, $C_{2,2} = 1, C_{2,1} = \tilde{\chi}'_{\{1,\ldots,n-1\}}$, and $C_{1,2} = \tilde{e}^{k+1}$. As before, $\tilde{\chi}'_{\{1,\ldots,n-1\}}$, and $\tilde{e}^{k+1}$ stand for the restriction to the first $n-1$ entries of $\chi'_{\{1,\ldots,n-1\}}$, and e^{k+1} respectively. The subscripts of these vectors are taken mod(n). Since $C_{1,1} = -I^{n-1} + T, T$ having 0 as the unique eigenvalue with multiplicity $n-1$ (see Appendix for an outline of a proof of this claim), it is easy to see that $\det(C_{1,1}) = (-1)^{n-1}$. Besides, to compute $C_{2,1}.(C_{1,1})^{-1}.C_{1,2}$ we only need the $(k+1)$-th column of $(C_{1,1})^{-1}$ which is $-\tilde{\chi}_{\{1,\ldots,n-1\}}$, as it can be verified by direct substitution. Therefore,

$$C_{2,1}.(C_{1,1})^{-1}.C_{1,2} = -(n-1)$$

and

$$C_{22} - C_{2,1}.(C_{1,1})^{-1}.C_{1,2} = n.$$

Thus, $\det(C_{1,1}^{n,k}) = (-1)^{n-1}.n$.

The second part is proven in a similar way. In this case, the n-th column of $(C_{1,1}^{n,k})^{-1}$ is $\frac{1}{n}.\tilde{\chi}'_N$. Consequently,

$$\det(\hat{C}^{n,k}) = (1 - \frac{k}{n}).(-1)^{n-1}.n = (-1)^{n-1}.(n-k). \quad \square$$

Corollary 23. *If n and k are relatively primes, then,* $\det(C^{n,k}) = (-1)^{n-1}.\frac{n}{2}$, *and the linear system (14) has a unique solution with* $\lambda_{n+1} > 0$.

Proof. Since $\det(C^{n,k}) = \det(C_{2,2}^{n,k} - C_{2,1}^{n,k}.(C_{1,1}^{n,k})^{-1}.C_{1,2}^{n,k}).\det(C_{1,1}^{n,k})$, taking into account the results proved in Lemma 20 and Proposition 21 we get that

$\det(C^{n,k}) = \frac{1}{2}.(-1)^{n-1}.n$, which proves the first part of this corollary.

The uniqueness of the solution of system (14) follows from the non-singularity of the matrix of coefficients $C^{n,k}$. On the other side, the $(n+1)$-th entry of the solutions λ can be computed by Cramer's rule which gives, according to the results of Proposition 22,

$$\lambda_{n+1} = 2.\frac{n-k}{n} \tag{21}$$

which is a positive number. □

6. Maximality of the Basic *U*-Cycle

The U-cycle studied in the preceding sections has an important maximality property, which, in many cases implies that it is a maximal regular U-cycle. We first prove this auxiliary key result.

Lemma 24. *Let n and k be relatively prime and $\lambda = (\lambda_1, ..., \lambda_n, \lambda_{n+1})$ be the unique solution of the linear system (14). If $x = (\lambda_1, ..., \lambda_n)$, then $x(S_1) < x(S_i)$ for all $i = 2, ..., n$.*

Proof. From the $n-k$ first equations of system (14) we get that $\lambda_i = \lambda_{k+i} - \frac{1}{n-k}.\lambda_{n+1}, i = 1, ..., n-k$, and because of (21), we obtain that $\lambda_i < \lambda_{k+i}, i = 1, ..., n-k$. Since $x(S_2) = x(S_1) + \lambda_{k+1} - \lambda_1$, and taking into account the former inequalities with $i = 1$, we get that $x(S_2) > x(S_1)$. A similar argument shows that $x(S_i) > x(S_{i-1}), i \leq n-k+1$. On the other hand, from the last $k-1$ equations, and with the same argument, we show that $x(S_{n-k+1}) > ... > x(S_n)$. Finally, since x is also the solution of system (8), and taking into account that the last equation of that system implies that the inequality $\lambda_n > \lambda_k$ holds too, we conclude that $x(S_n) > x(S_1)$. This completes the proof. □

The following is an immediate consequence that we state without proof.

Corollary 25. *Let n and k be relatively prime and (N, v) a TU-game with its characteristic function satisfying $v(N) = v(S) = 1$ for all $S \in \mathcal{B}_{n,k}$. Then, $e(S_1, x) > e(S_i, x), i = 2, ..., n$.*

Corollary 25 has some important consequences. If we consider games for which $v(S) = 0$ for all $S \notin \mathcal{B}_{n,k}, S \neq N$ and if the solution x of system (8) is an imputation, then the U-cycle having x as a starting point (see Theorem 4) is a regular maximal U-cycle. This is the key fact needed to generalize Theorem 12 in [6] which would enlarge the class of games for which the emptiness of the core can be characterized in terms of the existence of maximal U-cycles.

On the other hand, as we mentioned in Introduction, regular maximal U-cycles can be interpreted as part of a special kind of dynamic solution defined in an appropriate way. Corollary 25 is also a basic result to completely describe the dynamic solution in the class of games studied in this paper in a similar way as was done in [7].

We close mentioning two facts. During the proof of Lemma 24 we have exhibited some partial ordering between the components of the solution of the linear system (14). Some numerical experience performed for different combinations of the pairs (n, k) always showed that $x_1 < x_2 < ... < x_n$. Therefore, to guarantee that x is an imputation it would be enough to find out conditions for x_1 to be non-negative.

Maximal U-cycles can have a rather complex structure even if the minimal family of coalitions acting as its support is quite simple as the following example shows. Let a 5-person game have the following characteristic function: $v(N) = v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = v(\{4,5\}) = 1$, and $v(S) = 0$ otherwise. If $\mathcal{B} = \{S_1, S_2, S_3, S_4\}$, with $S_1 = \{1,2\}, S_2 = \{2,3\}, S_3 = \{1,3\}, S_4 = \{4,5\}$, the game has a maximal U-cycle whose support has 26 elements (all of them belonging to $\mathcal{B}$) which is $(S_2, S_3, S_4, S_1, S_3, S_4, S_2, S_4, S_3, S_1, S_4, S_2, S_4, S_1, S_3, S_4, S_2, S_3, S_4, S_1, S_4, S_3, S_2, S_4, S_1, S_4)$.

It is worth noting that there also exists a U-cycle, although not a maximal one, whose support is (S_1, S_2, S_3, S_4).

Similar results to those demonstrated in this paper whenever n and k are not relatively prime, will appear elsewhere.

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Appendix I

In this appendix we outline the proof that the matrix T mentioned in Lemma 16 has 0 as its only eigenvalue with multiplicity $n-1$. To this end, we point out that the matrix T is obtained from $circ(e^{k+1})$ by deleting the n-th row and column. Therefore, T is a square matrix of order $n-1$ having its $(n-k)$-th row (and its k-th column) with all their entries equal to zero. All the other rows have only one non-zero entry with value 1. Since n and k are relatively prime, $n - i.k \neq k$ for $i = 1, ..., n-1$ (we recall we are taking $k > 1$). Let us denote by $i \rightarrow \rho(i), i = 1, ..., n$, the function which assigns to the row i of $cir(e^{k+1})$ the position of its entry 1. It is easy to see that $\rho(i) = i + k \text{ mod}(n)$. In particular, $\rho(n-k) = n$. We now want to show that the characteristic polynomial of T is $(-1.\lambda)^{n-1}$. For this, it is enough to prove that the only non-zero term in the development of $\det(T - \lambda I^{n-1})$ is the product of the element in the main diagonal of $T - \lambda I^{n-1}$ which is $(-1.\lambda)^{n-1}$. Any term to be considered has the form (except for the sign) $m_\tau = m_{1,\tau(1)}.m_{2,\tau(2)}...m_{n-1,\tau(n-1)}$, where τ is a permutation of the set $\{1, ..., n-1\}$. It is clear that, any $m_\tau \neq 0$ must have $m_{n-k,\tau(n-k)} = -\lambda$. Thus, $\tau(n-k) = n-k$. Because the $(n-k)$-th column of T is non-zero, it has an entry equal to 1 in the position $i = \rho^{-1}(n-k) = n - 2.k$, mod$(n)$. Then, $m_{i,\tau(i)} = (-\lambda)$, for if it were not the case, there would be two elements belonging to the same column in m_τ. Then, $\tau(i) = i$. We stress the fact that $n - 2.k$ is always less than n unless $n - k = k$. This point guarantees that the process we are describing can be restricted to the submatrix T of $circ(e^{k+1})$. The proof is completed by using an inductive argument which shows that $\tau(i) = i$ for $i = 1, ..., n-1$.