International Journal of Pure and Applied Mathematics
Volume 56 No. 4 2009, 487-495

COMMON FIXED POINTS IN THE SET OF BEST APPROXIMATIONS

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Abstract: We prove a common fixed point result for noncommuting (f, g)-nonexpansive map T and then derive certain results on best approximation. Our results generalize the results of Al-Thagafi [1], Hussain et al [9], Jungck and Sessa [11], Khan, Hussain and Thaheem [12], Latif [16], [17], Sahab et al [22] and Singh [23].

AMS Subject Classification: 46B99
Key Words: p-normed space, best C-approximant, Banach operator pair, commuting maps, common fixed point, property (N)

1. Preliminaries

Let X be a linear space. A p-norm on X is a real valued function $\|\cdot\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$;

(ii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$;

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

for all $x, y \in X$ and all scalars λ. The pair $(X, \|\cdot\|_p)$ is called a p-normed space. It is a metric space with a translation invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of a normed space. It is well-known that the topology of every Hausdorff locally bounded

Received: August 24, 2007

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topological linear space is given by some p-norm, $0 < p \leq 1$ (see [15]). The spaces l_p and L_p, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that dual space X^* separates points of X (or equivalently X^* is total [21]) if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well-defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0,1]$, $0 < p < 1$, to the power p integrable functions, or $X = S[0,1]$, the space of measurable functions, then $X^* = \{0\}$ (see [12], [13], [21]). However, there are some non-locally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X.

Let C be a subset of p-normed space X, and let f, g be self maps of X. The set C is called starshaped with respect to $q \in C$ if for all $x \in C$ and for all $k, 0 \leq k \leq 1, kx + (1-k)q \in C$. The set C is said to be starshaped if it is starshaped with respect to one of its elements. Each convex set is necessarily starshaped. The map f defined on a q-starshaped set C is called affine if

$$f((1-k)q + kx) = (1-k)fq + kfx, \quad \text{for all} \quad x \in C.$$

A map $T : X \longrightarrow X$ is (f, g)-contraction [11, 13], if there exists a real number $k \in (0,1)$ such that

$$\|Tx - Ty\|_p \leq k \|fx - gy\|_p \qquad \text{for all } x, y \in X.$$

If in the above inequality $k = 1$, then T is called (f, g)-nonexpansive. Also if $f = g$, we say that T is f-nonexpansive. We denote the boundry of C by ∂C, closure of C by $cl(C)$, weak closure of C by $wcl(C)$ and the set of fixed points of T by $F(T)$.

A p-normed space X satisfies Opial's condition if, for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\|_p < \liminf_{n\to\infty} \|x_n - y\|_p$$

holds for all $y \neq x$. Every l_p $(1 < p < \infty)$ space satisfy Opial's condition. The map $T : M \to X$ is said to be demiclosed at 0 if, for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converges to $0 \in X$, then $0 = Tx$.

Using fixed point theory, Brosowski [2], Meinardus [18] have established some interesting results on invariant approximation in the setting of normed spaces. Jungck and Sessa [11] have also obtained some results in approximation theory in the setting of normed spaces. Their work has been extended, generalized and unified by many authors; for example, see [6], [7], [16], [17], [19], [20], [22].

For any $x_0 \in X$, define $d_p(x_0, C) = \inf_{u \in C} \|x_0 - u\|_p$,

$$P_C(x_0) = \left\{ y \in C : \|y - x_0\|_p = d_p(x_0, C) \right\}$$

the set of best C-approximants to x_0 and $D_C^f(x_0) = \{y \in C : f(y) \in P_C(x_0)\}$. Note that $P_C(x_0)$ contains $f\left(D_C^f(x_0)\right)$ and $g\left(D_C^g(x_0)\right)$. Assume that

$$D_C^{f,g}(x_0) = D_C^f(x_0) \cap D_C^g(x_0) \text{ and } D = P_C(x_0) \cap D_C^{f,g}(x_0).$$

Recently, Latif [16] has obtained the following result on common fixed point in best approximations, which generalize and extend the recent work of Al-Thagafi [1], Sahab, Khan and Sessa [22], Singh [23], etc.

Theorem 1.1. *Let f, g and T be self-maps of a p-normed space X such that $x_0 \in F(f) \cap F(g) \cap F(T)$ and $C \in X$ with $T(\partial C \cap C) \subset C$. Let T be a (f, g)-nonexpansive map and continuous on $D \cup \{x_0\}$ such that $cl(T(D))$ is compact, and let f and g be continuous, surjective, affine and commute with T on D. If D is closed and starshaped with respect to $q \in F(f) \cap F(g)$, then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \phi$.*

The aim of this paper is to establish a general common fixed point theorem for Banach operator pairs in the setting of locally bounded topological vector spaces. As application, we derive some results on the existence of best approximations. Our results unify and extend the results of Al-Thagafi [1], Habiniak [4], Hicks and Humphries [5], Jungck and Sessa [11], Khan et al [12], [13], Latif [16], [17], Sahab, Khan and Sessa [22], and Singh [23].

2. Common Fixed Points and Best Approximations

The ordered pair (T, f) of two selfmaps of a metric space (X, d) is called a Banach operator pair, if the set $F(f)$ is T-invariant, namely, $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but not conversely, in general; see [3], [6]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (cf. Example 1 in [3]). If the selfmaps T and f of X satisfy

$$d(fTx, Tx) \leq kd(fx, x),$$

for all $x \in X$ and $k \geq 0$, then (T, f) is a Banach operator pair; in particular, when $f = T$ and X is a normed space, the above inequality can be rewritten as

$$\|T^2x - Tx\| \leq k\|Tx - x\| \text{ for all } x \in X.$$

We also recall the following common fixed point result due to Hussain [6].

Lemma 2.1. *Let C be a nonempty subset of a metric space (X,d), and (T,f) and (T,g) be Banach operator pairs on C. Assume that $cl(T(C))$ is complete, $F(f)\cap F(g)$ is nonempty and closed and T is (f,g)-contraction. Then $C\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

The following result properly contains Theorems 3.2 and 3.3 of [3] and extends and improves Theorem 2.2 of [1], [20], Theorem 4 in [4] and Theorem 6 of [11].

Theorem 2.2. *Let C be a nonempty subset of a p-normed space X and T, f and g be self-maps of C. Suppose that $F(f)\cap F(g)$ is closed and q-starshaped with $q\in F(f)\cap F(g)$. If (T,f) and (T,g) are Banach operator pairs and T is (f,g)-nonexpansive on C, then $C\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$, provided one of the following conditions holds;*

(i) $cl(T(C))$ is compact,

(ii) X is complete, $wcl(T(C))$ is weakly compact, f, g are weakly continuous and either $f-T$ is demiclosed at 0 or X satisfies the Opial's condition.

Proof. Since $F(f)\cap F(g)$ is q-starshaped with $q\in F(f)\cap F(g)$, we can define $T_n : F(f)\cap F(g)\to F(f)\cap F(g)$ by $T_nx=(1-k_n)q+k_nTx$ for all $x\in M$ and a fixed sequence of real numbers $k_n(0<k_n<1)$ converging to 1. As (T,f) and (T,g) are Banach operator pairs and $F(f)\cap F(g)$ is closed, so $clT(F(f)\cap F(g))\subseteq F(f)\cap F(g)$. Since (T,f) is a Banach operator pair, for $x\in F(f)\cap F(g)$ we have $Tx\in F(f)\cap F(g)$, and hence $T_nx=(1-k_n)q+k_nTx\in F(f)\cap F(g)$ by the fact that $F(f)\cap F(g)$ is q-starshaped with $q\in F(f)\cap F(g)$. Thus for each $n\geq 1$, (T_n,f) is a Banach operator pair on $F(f)\cap F(g)$. Similarly, (T_n,g) is a Banach operator pair on $F(f)\cap F(g)$. By (f,g)-nonexpansiveness of T, we have

$$\|T_nx-T_ny\|_p=(k_n)^p\|Tx-Ty\|_p\leq(k_n)^p\|fx-gy\|_p,$$

for each $x,y\in C$ and $0<k_n<1$.

(i) As $clT(F(f)\cap F(g))\subseteq cl(T(C))$ and $cl(T(C))$ is compact, for each $n\in N$, $cl(T_n(F(f)\cap F(g)))$ is compact and hence complete. By Lemma 2.1, for each $n\geq 1$, there exists $x_n\in C$ such that $x_n=fx_n=gx_n=T_nx_n$. The compactness of $cl(T(C))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m\to z\in cl(T(F(f)\cap F(g)))=cl(F(f)\cap F(g))=F(f)\cap F(g)$ as $m\to\infty$. Since $k_m\to 1$, $x_m=T_mx_m=(1-k_m)q+k_mTx_m\to z$. Moreover,

$$\|Tx_m-Tz\|_p\leq\|fx_m-gz\|_p=\|x_m-z\|_p.$$

Taking the limit as $m\to\infty$, we get $z=Tz$. Thus $C\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.

(ii) The weak compactness of $wcl(T(C))$ implies that $wcl(T_n(F(f)\cap F(g)))$ is weakly compact and hence complete due to completeness of X. From Lemma 2.1, for each $n \geq 1$, there exists $x_n \in F(f)\cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. Moreover, we have $\|x_n - Tx_n\|_p \to 0$ as $n \to \infty$. The weak compactness of $wcl(T(C))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in wcl(T(C))$ as $m \to \infty$. The weak continuity of f and g implies that $F(f)\cap F(g)$ is weakly closed. Since $\{Tx_m\}$ is a sequence in $T(F(f)\cap F(g))$, therefore $y \in wcl(T(F(f)\cap F(g))) \subseteq wcl(F(f)\cap F(g)) = F(f)\cap F(g)$. Also we have, $x_m - Tx_m \to 0$ as $m \to \infty$.

Suppose $id - T$ is demiclosed at 0, then $y = Ty$. Thus $C\cap F(T)\cap F(f)\cap F(g) \neq \emptyset$.

Suppose X satisfies the Opial's condition. If $y \neq Ty$, then

$$\begin{aligned}\liminf d_p(x_m, y) &< \liminf d_p(x_m, Ty)\\ &\leq \liminf d_p(x_m, Tx_m) + \liminf d_p(Tx_m, Ty)\\ &= \liminf d_p(Tx_m, Ty) \leq \liminf d_p(fx_m, gy) = \liminf d_p(x_m, y),\end{aligned}$$

which is a contradiction. Thus $y = Ty$ and $C\cap F(T)\cap F(f)\cap F(g) \neq \emptyset$. □

Corollary 2.3. (see [16], Theorem 2.2) *Let X be a p-normed space and C a closed subset of X which is starshaped with respect to q. Let f, g and T be continuous self-maps of C such that $T(C) \subset f(C)\cap g(C)$, T commutes with f and g, and $q \in F(f)\cap F(g)$. If $cl(T(C))$ is compact, f and g are affine, and T is (f,g)-nonexpansive then, $C\cap F(T)\cap F(f)\cap F(g) \neq \emptyset$.*

Proof. As f and g are continuous and C is closed, then $F(f)$ and $F(g)$ are closed and so $F(f)\cap F(g)$. Using the commutativity of T with f and g we obtain, $T(F(f)) \subseteq F(f)$ and $T(F(g)) \subseteq F(g)$. Thus (T,f) and (T,g) are Banach operator pairs. Since f and g are affine and $q \in F(f)\cap F(g)$, so $F(f)\cap F(g)$ is q-starshaped. The desired conclusion follows now from Theorem 2.1. □

Corollary 2.4. (see [1], Theorem 2.1) *Let C be a nonempty closed and q-starshaped subset of a normed space X and T and f be self-maps of C such that $T(C) \subseteq f(C)$. Suppose that T commutes with f and $q \in F(f)$. If $cl(T(C))$ is compact, f is continuous and linear and T is f-nonexpansive on C, then $C\cap F(T)\cap F(f) \neq \emptyset$.*

Assume that $D_C^{f,g}(x_0) = D_C^{f}(x_0)\cap D_C^{g}(x_0)$ and $D = P_C(x_0)\cap D_C^{f,g}(x_0)$.

Theorem 2.5. *Let f, g and T be self-maps of a p-normed (resp. Banach) space X such that $x_0 \in F(f)\cap F(g)\cap F(T)$ and $C \subset X$ with $T(\partial C\cap C) \subset C$. Suppose that (T,f) and (T,g) are Banach operator pairs, T is (f,g)-*

nonexpansive on $D \cup \{x_0\}$ and $D_0 := D \cap F(f) \cap F(g)$ is closed and q-starshaped. If $cl(T(D))$ is compact (resp. $wcl(T(D))$ is weakly compact, f, g are weakly continuous and either $id - T$ is demiclosed at 0 or X satisfies the Opial's condition), then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$.

Proof. Let $y \in D_0$. Then $\|y - x_0\|_p = d_p(x_0, C)$ since $y \in P_C(x_0)$. Note that for any $k \in (0,1)$

$$\|kx_0 + (1-k)y + x_0\|_p = (1-k)^p \|y - x_0\|_p < d_p(x_0, C).$$

It follows that the line segment $\{kx_0 + (1-k)y : 0 < k < 1\}$ and the set C are disjoint. Thus y is not in the interior of C and so $y \in \partial C \cap C$. Since $T(\partial C \cap C) \subset C, Ty$ must be in C. Also since $fy \in P_C(x_0), x_0 \in F(T) \cap F(g)$ and T is (f,g)-nonexpansive on $D_0 \cup \{x_0\}$, we have

$$\|Ty - x_0\|_p = \|Ty - Tx_0\|_p \leq \|fy - gx_0\|_p = \|fy - x_0\|_p = d_p(x_0, C),$$

and hence $Ty \in P_C(x_0)$. Moreover, since T commutes with f on D_0 and, $x_0 \in F(g) \cap F(T)$, we have

$$\|fTy - x_0\|_p = \|Ty - Tx_0\|_p \leq \|fy - gx_0\|_p = \|fy - x_0\|_p = d_p(x_0, C),$$

and similarly, since T commutes with g on D_0 and $x_0 \in F(f) \cap F(T)$, we get

$$\|gTy - x_0\|_p \leq \|gy - x_0\|_p = d_p(x_0, C).$$

Thus fTy and gTy are in $P_C(x_0)$ and so $Ty \in D_C^{f,g}(x_0)$. Thus the definition of D_0 implies that $Ty \in D_0$. Consequently T maps D_0 into D_0. For $n \in N$, define the maps T_n from D_0 into D_0 as in the proof of Theorem 2.2. As in the proof of Theorem 2.2, there exists $x_n \in D_0$ such that $x_n \in F(T_n) \cap F(f) \cap F(g)$ for each n. Now, we complete the proof as that of Theorem 2.2. ☐

Remark 2.6. Theorems 4.1 and 4.2 of Chen and Li [3] and Theorem 1.1 are particular cases of Theorem 2.5.

Corollary 2.7. (see [17], Theorem 2.2) *Let T and f be selfmaps on an Opial p-normed space X and C a subset of X such that $T(\partial C) \subset C$, $x_0 \in F(f) \cap F(T)$. Suppose that $D = P_C(x_0)$ is nonempty, weakly compact and q-starshaped. Assume that f is affine, continuous, $D = fD$, $fq = q$ and T is f-nonexpansive on $D \cup \{X_0\}$. If T and f are commuting maps, then $P_C(x_0) \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.8. (see [11], Theorem 7) *Let f and T be selfmaps of a Banach space X with $u \in F(f) \cap F(T)$ and $C \subset X$ with $T(\partial M) \subset M$. Suppose that $D = P_C(x_0)$ is q-starshaped with $q \in F(f)$, $f(D) = D$, f is affine and continuous in the weak and strong topology on D. If f and T are commuting on D and T is f-nonexpansive on $D \cup \{u\}$, then $P_C(x_0) \cap F(T) \cap F(f) \neq \emptyset$ provided either: (i) D is weakly compact, and $(f - T)$ is demiclosed; or (ii) D is weakly compact and X satisfies Opial's condition.*

Definition 2.9. A subset C of a linear space X is said to have the property (N) with respect to T [9, 10] if,

(i) $T : C \to C$,

(ii) $(1 - k_n)q + k_n Tx \in C$, for some $q \in C$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in C$.

Hussain et al [9] noted that each q-starshaped set C has the property (N) but converse does not hold, in general.

Remark 2.10. (1) All the results of this paper (Theorems 2.2-2.5) remain valid, provided the q-starshapedness of the sets $F(f) \cap F(g)$ and D_0, is replaced by the property (N). Consequently, recent results due to Hussain, O'Regan and Agarwal [9], Khan et al [12] and Khan and Khan [14] are improved.

(2) All results of the paper (Theorems 2.2-2.5) remain valid in the setup of a metrizable locally convex topological vector space (tvs) (X, d), where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$ (recall that d_p is translation invariant and satisfies $d_p(\alpha x, \alpha y) \leq \alpha^p d_p(x, y)$ for any scalar $\alpha \geq 0$). Consequently, Theorem 2.2 (i)-(ii)-Theorem 3.3 (i)-(ii) due to Hussain and Khan [8] and Theorems 3.5 and 4.2 due to Hussain, O'Regan and Agarwal [9] are improved.

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