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THE EXISTENCE OF COMPACT EXPONENTIAL ATTRACTING SET FOR DYNAMICAL SYSTEM

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Abstract: Using the measure of noncompactness, we obtain the existence of compact exponential attracting set for general dynamical system. Then we given a method for proving the existence of compact exponential attracting set.

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1. Introduction

The study of the asymptotic behavior that arise in many infinite dimensional dynamical systems is one of the most important problems of modern mathematical physics. For autonomous dissipative systems, one way is to analysis the existence and structure of its global attractor [1], [2], [5]. Many authors have paid much attention to this problem for a quite long time and have made a lot of successfully progress [4], [6], [3]. R. Teman [3] presents a general approach that is well suited to study equations arising in mathematics physics. In [4], the authors establish some necessary and sufficient conditions for the existence of the global attractor of an infinite dimensional dynamical system and give a new method for proving the existence of the global attractor. To verify the existence of a global attractor, one needs to show that:

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(1) the existence of an absorbing set, and

(2) condition (C),

where condition (C) is: for any bounded set B of X and for any $\varepsilon > 0$, there exist $t(B) > 0$ and a finite dimensional subspace X_1 of X, such that $\{\|PS(t)B\|\}$ is bounded and

$$\|(I-P)S(t)x\| < \varepsilon \quad \text{for } t \geq t(B),\ x \in B,$$

where $P : X \to X_1$, is a bounded projector.

In fact, for many equations, we find not only condition (C) holds true but also condition $(\widehat{C})$ (see Definition 3.5), i.e., the measure of noncompactness is exponential decay. By using the measure of noncompactness technique, we establish the existence of compact exponential attracting set for general dynamical system and includes the global attractor, and gives a method for proving the existence of compact exponential attracting set. As an examples, we consider reaction-diffusion equation and get the existence of compact exponential attracting set in H_0^1.

We will use the following notation: X is a complete metric space with the metric d, $B(X)$ is the set of all bounded subsets of X, we denote by $B(x,r)$ the neighborhood $\{y \in X : d(y,x) < r\}$ of radius $r > 0$ of x, and $B(A,r)$ the neighborhood $\{y \in X : d(y,A) < r\}$ of radius $r > 0$ of A.

2. Preliminaries

One parameter family of mappings $S(t) : X \to X (t \geq 0)$ is called the semigroup provided that:

(1) $S(0) = I$;

(2) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.

The couple $(S(t), X)$ is usually referred to as a dynamical system, $(S(n), X)$ is called the discrete dynamical system generated by $(S(t), X)$.

A set $\mathcal{A} \subset X$ is called a global attractor to $(S(t), X)$ if: (i) $\mathcal{A}$ is compact in X, (ii) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and (iii) for any $B \subset X$ that is bounded, $d(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$, where $d(B, \mathcal{A}) = \sup_{b\in B} \inf_{a \in \mathcal{A}} \| b - a \|_X$.

A set B is called a bounded absorbing set to $(S(t), X)$, if for any bounded set $B_0 \subset X$, there exists $t_0 = t_0(B_0)$ such that $S(t)B_0 \subset B$ for all $t \geq t_0$. A set E is called positively invariant w.r.t $S(t)$ if for all $t \geq 0$, $S(t)E \subset E$.

Theorem 2.1. (see [5]) *Continuous semigroup $S(t)$ has a global attractor $\mathcal{A}$*

if and only if $S(t)$ has a bounded absorbing set B and for an arbitrary sequence of points $x_n \in B$, the sequence $S(t_n)x_n$ has a subsequence converging in B.

In fact, we know that

$$\mathcal{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B}. \tag{2.1}$$

Next, we briefly review the basic concept about the Kuratowski measure of noncompact and recapitulate its basic property, which will be used to characterize the existence of compact exponential attracting set for dynamical system $(S(t), X)$.

Let B be a bounded subset of X. The Kuratowski measure of noncompactness $\alpha(B)$ of B is defined by $\alpha(B) = \inf\{\delta > 0 \mid B$ has a finite open cover of sets of diameter$\leq \delta\}$.

It has the following properties.

Lemma 2.2. (see [2]) *Let $B,\ B_1,\ B_2 \in B(X)$. Then:*

(1) $\alpha(B) = 0 \Leftrightarrow \alpha(N(B,\varepsilon)) \leq 2\epsilon \Leftrightarrow \overline{B}$ is compact;

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \leq \alpha(B_2)$ whenever $B_1 \subset B_2$;

(4) $\alpha(B_1 \bigcup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(B) = \alpha(\overline{B})$;

(6) if B is a ball of radius ε then $\alpha(B) \leq 2\varepsilon$.

Lemma 2.3. *Let α be the measure of noncompactness. Assume that F_n is a sequence of bounded and closed subsets of X, satisfying:*

(1) $F_n \neq \emptyset$;

(2) $F_{n+1} \subset F_n,\ n = 1, 2, \cdots$, and

(3) $\alpha(F_n) \to 0$ as $n \to \infty$.

Then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.

3. Existence of Compact Exponential Attracting Set

Definition 3.1. (see [4]) A semigroup $S(t)$ is called ω-limit compact if for every bounded B of X and for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that $\alpha(\cup_{t\geq t_0} S(t)B) \leq \varepsilon$.

Lemma 3.2. (see [4]) *Assume the semigroup $S(t)$ is ω-limit compact, then for any sequence $t_n \in R^+, t_n \to \infty$, as $n \to \infty$, and any sequence $x_n \in B$, there exists a convergence subsequence of $\{S(t_n)x_n\}$ whose limit lies in $\omega(B)$, where B is bounded and $\omega(B)$ is ω-limit set of B defined by*

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B}.$$

Next, we first consider discrete dynamical system $(S(n), X)$.

Theorem 3.3. *Assume that B is a bounded absorbing set w.r.t discrete dynamical system $S(n)$ in X. The followings are equivalent:*

(1) The dynamical system $(S(n), X)$ is exponential decay of the measure of noncompactness, i.e., there exist $k_1, k_2 > 0$ such that

$$\alpha(\bigcup_{n\geq m} S(n)B) \leq k_1 e^{-k_2 m}.$$

(2) There exists a compact set E such that:

(i) $S(n)E \subset E$, and

(ii) $d(S(n)B, E) \leq k_1 e^{-k_2 n}$, for some $k_1, k_2 > 0$.

Obviously, semigroup $S(n)$ is $\omega-limit$ compact if $S(n)$ is exponential decay of the measure of noncompactness.

Proof. ((1)⇒(2)) $S(n)$ is exponential decay of the measure of noncompactness w.r.t B, by Lemma 3.2 and Theorem 2.1, we get

$$\mathcal{A} = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n\geq m} S(n)B}, \tag{3.1}$$

is the global attractor for $S(n)$. Using (3) of Lemma 2.2 and the assumption of (1), we find $\alpha(S(m)\mathcal{B}) \leq \alpha(\bigcup_{n\geq m} S(n)B) \leq k_1 e^{-k_2 m}$, by the definition of the measure of noncompactness, $\forall m \in \mathbb{N}$, there exist finite points $x_{m_i} \in S(m)B$, and $S(m)B \subset \bigcup B(x_{m_i}, k_1 e^{-k_2 m})$. Let $E_m = \{x_{m_i}\}$, setting

$$E_0 = \cup_{m,k\geq 0} S(m)E_k,$$

and

$$E = \mathcal{A} \cup E_0.$$

We claim that E is the compact set we are looking for.

Since $S(n)\mathcal{A} = \mathcal{A}, S(n)E_0 \subset E_0$, we get $S(n)E \subset E$. In addition we find E_0 is a countable set, for arbitrary sequence of $x_n \in E$, by Lemma 3.2. There exists a convergence subsequence that has an accumulation point in $\mathcal{A}$. Furthermore, $\alpha(S(m)B) \leq k_1 e^{-k_2 m}$, so we get $d(S(m)B, E) \leq k_1 e^{-k_2 m}$.

((2)⇒(1)). Since E is compact, hence there exist $x_1, x_2, ..., x_l \in E$, such that $E \subset \bigcup_{i=1}^{l} B(x_i, k_1 e^{-k_2 n})$, therefore we have

$$S(n)B \subset \bigcup_{i=1}^{l} B(x_i, 2k_1 e^{-k_2 n}).$$

By (3.1) we know that $\mathcal{A} = \lim_{k\to\infty} \overline{\bigcup_{m\geq k} S(m)B}$, for $\varepsilon = k_1 e^{-k_2 n}$. There exists K, $\forall k \geq K$, so we get

$$\textstyle\bigcup_{m\geq k} S(m)\mathcal{B} \subset B(\mathcal{A}, k_1 e^{-k_2 n}).$$

$\mathcal{A}$ is compact, so there exist finite points $x_i' \in \mathcal{A}$ such that

$$\mathcal{A} \subset \bigcup B(x_i', k_1 e^{-k_2 n}),$$

and

$$\textstyle\bigcup_{m\geq k} S(m)\mathcal{B} \subset \bigcup B(x_i', 2k_1 e^{-k_2 n}).$$

By (4) of Lemma 2.2, we find $\alpha(\bigcup_{m\geq n} S(m)B) \leq 2k_1 e^{-k_2 n}$, i.e., $S(n)$ is exponential decay of the measure of noncompactness. □

Theorem 3.4. *Assume B be a bounded absorbing set for $S(t)$ in X, and there exists integer number $M > 0$, for $t_0 \geq M$, $\alpha(\cup_{t\geq t_0}) S(t)B) \leq k_1 e^{-k_2 t_0}$ for some $k_1, k_2 > 0$, i.e., $S(t)$ is exponential decay of the measure of noncompactness. Then there exists a compact set E, for any bounded $B \in B(X)$, there exist η_1, η_2 and $T > 0$ such that*

$$d(S(t)B, E) \leq \eta_1 e^{-\eta_2 t}, \quad for \ \ t \geq T.$$

Proof. By the proof of Theorem 3.3, we know that there exists a compact set E and $k_1, k_2 > 0$ such that

$$d(S(n)B, E) \leq k_1 e^{-k_2 n}, \ for \ n \geq M. \tag{3.2}$$

Since B is bounded absorbing set of $S(t)$, so we get there exists $T_1 > 0$, for $\forall t \geq T_1$,

$$S(t)B \subset B.$$

Let $T = M + T_1 + 1$, for $\forall t \geq T$. There exists a unique $T_0 \in [T_1, T_1 + 1)$ such that $t - T_0 = [t - T_0]$, where [ ] denote maximum integer number. By (3.2) we must have

$$d(S(t)B, E) = d(S(t - T_0)S(T_0)B, E) \leq k_1 e^{-k_2(t-T_0)} \leq k_1 e^{k_2(T_1+1)} e^{-k_2 t}.$$

Setting $\eta_1 = k_1 e^{k_2(T_1+1)}, \eta_2 = k_2$, we obtain

$$d(S(t)B, E) \leq \eta_1 e^{-\eta_2 t}.$$

For any bounded set B, we take T_1 which satisfies $S(t)B \subset B$ for $\forall t \geq T_1$.

The same conclusion also holds true.

We present now a method to verify that the semigroup $S(t)$ is exponential decay of the measure of noncompactness.

Definition 3.5. (Condition $(\widehat{C})$) Let X be a convex Banach space, for any bounded set B of X. There exist k_1, k_2 and $T > 0$, and for any finite dimension subspace X_1 of X, such that

(i) $P_m(\bigcup_{t\geq t_0} S(t)B)$ is bounded, and

(ii) $\| (I - P_m)(\bigcup_{t\geq 2t_0} S(t)x) \| \leq k_1 e^{-k_2 t_0} + k(m)$, $\forall x \in B$, for all $t_0 \geq T$. Where $P_m : X \to X_1$ is a bounded projector, m is the dimension of X_1, $\| \cdot \|$ denote the norm in X and $k(s)$ is real-valued function satisfying

$$\lim_{s\to\infty} k(s) = 0.$$

Theorem 3.6. *If a semigroup $S(t)$ satisfies condition $(\widehat{C})$ then it implies that the semigroup $S(t)$ is exponential decay of the measure of noncompactness.*

Proof. For any bounded set B of X, from (2) and (6) of Lemma 2.2, and by condition $(\widehat{C})$, we get

$$\alpha(\bigcup_{t\geq t_0} S(t)B) \leq \alpha(P_m(\bigcup_{t\geq t_0} S(t)B)) + \alpha((I-P_m)(\bigcup_{t\geq t_0} S(t)B)$$
$$= \alpha((I-P_m)(\bigcup_{t\geq t_0} S(t)B) = \alpha((I-P_m)(\bigcup_{t\geq 2(t_0/2)} S(t)B) \leq k_1 e^{-\frac{k_2}{2}t_0} + k(m).$$

Since $k(m) \to 0$, for $\varepsilon_0 = k_1 e^{-\frac{k_2}{2}t_0}$, there exists $M > 0$, $\forall m > M$. We have

$$|k(m)| < k_1 e^{-\frac{k_2}{2}t_0}.$$

Hence, $\alpha(\bigcup_{t\geq t_0} S(t)B) \leq 2k_1 e^{-\frac{k_2}{2}t_0}$ for $\forall m > M$, that is, $S(t)$ is exponential decay of the measure of noncompactness. □

By Theorem 3.4 and Theorem 3.6, we have the following theorem.

Theorem 3.7. *Let X be a complete metric space and $S(t)$ be a semigroup in X. Then there is a compact set E exponential attracting every bounded set B of X if the following conditions hold true:*

(i) $S(t)$ satisfies condition $(\widehat{C})$, and

(ii) there is a bounded absorbing set $B \subset X$.

Remark 3.8. In fact if $S(t)$ satisfies condition $(\widehat{C})$ that implies that $S(t)$ is ω-limit compact, by Theorem 2.1. We get $S(t)$ has a global attractor $\mathcal{A}$, from Theorem 3.3, knowing that $\mathcal{A} \subset E$.

4. Example

We consider the following nonlinear reaction diffusion equation:

$$u_t - \triangle u + f(u) = g(x), \tag{4.1}$$

$$u|_{\partial\Omega} = 0, \tag{4.2}$$

$$u(x,0) = u_0, \tag{4.3}$$

where Ω is a bounded smooth domain in R^n, $g(x) \in L^2(\Omega)$, f is a $\mathcal{C}^1$ function and there exist $p \geq 2$, $c_i > 0$, $i = 1, ..., 5$ such that

$$c_1|u|^p - c_2 \leq f(u)u \leq c_3|u|^p + c_4, \tag{4.4}$$

$$f_u(u) \geq -c_5, \tag{4.5}$$

for all $u \in R$.

For convenience, let $|\cdot|_p$ be the norm of $L^p(\Omega)(p \geq 1)$, and c_i the arbitrary positive constants, which may be different from line to line and even in the same line. We denote $H = L^2(\Omega)$ with scalar product $(\cdot)$ and norm $|\cdot|$, let $((\cdot))$ and $\|\cdot\|$ denote the scalar product and norm of $H_0^1(\Omega)$ and $((u,v)) = \int_\Omega \nabla u \nabla v dx$ for all $u, v \in H_0^1$.

For this initial boundary value problem, we know from [5], that for any initial date $u_0 \in L^2$ and any $T > 0$, there exists a unique solution $u(t) \in \mathcal{C}([0,T); H(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^p(0,T; L^p(\Omega))$.

Thanks to the existence theorem, the initial boundary value problem is equivalent to a semigroup $S(t)$ defined by

$$S(t) : H \to H. \tag{4.6}$$

Lemma 4.1. (see [6]) *The semigroup $S(t)$ has a bounded absorbing set in $H_0^1(\Omega)$ and $L^{2p-2}(\Omega)$ respectively, that is, for any bounded subset B in $L^2(\Omega)$, there exists a positive constant T, such that*

$$|u(t)|_{2p-2}^{2p-2} \leq M \text{ for any } u_0 \in B \text{ and } t \geq T \tag{4.7}$$

and

$$\|u(t)\|^2 \leq M \text{ for any } u_0 \in B \text{ and } t \geq T, \tag{4.8}$$

where M is a positive constant independent of B, $u(t) = S(t)u_0$.

Next we verify that $S(t)$ satisfies condition $(\widehat{C})$ in H_0^1.

We set $A = -\Delta$, since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ..., \quad \lambda_j \to +\infty, \text{ as } j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^1(\Omega)$ which are orthonormal in H such that

$$Ae_j = \lambda_j e_j \quad \forall j \in N.$$

Let H_m =span$\{e_1, e_2, ..., e_m\}$ in H and $P_m : H \to H_m$ is a orthogonal projector. For any $u \in H$ we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2.$$

Lemma 4.2. *Assume $g(x) \in L^2(\Omega)$ and f satisfies (4.4) and (4.5), and let $S(t)$ be the semigroup associated with (4.1)-(4.3). Then $S(t)$ satisfies condition $(\widehat{C})$ in H_0^1, that is, for any bounded subset B in $L^2(\Omega)$, there exist $k_1, k_2, T > 0$ and $k(m)$ such that*

$$\|(I - P_m) \cup_{t \geq 2t_0} u(t)\|^2 \leq k_1 e^{-k_2 t_0} + k(m), \text{ for any } u_0 \in B,$$

and

$$\lim_{m \to \infty} k(m) = 0,$$

providing that $t_0 \geq T$.

Combining Lemma 4.1 and Lemma 4.2, by using Theorem 3.7, we have the following result.

Theorem 4.3. *Assume $g(x) \in L^2(\Omega)$ and f satisfies (4.4) and (4.5), and let $S(t)$ be the semigroup associated with (4.1)-(4.3). Then $S(t)$ has a compact exponential attracting set E in $H_0^1(\Omega)$, i.e., for any bounded $B \subset X$, there exist η_1, η_2 and $T > 0$ such that*

$$d(S(t)B, E) \leq \eta_1 e^{-\eta_2 t}, \text{ for } t \geq T.$$

Proof of Lemma 4.2. By Lemma 4.1, for any bounded subset B in H, there exist positive constants T and M, such that

$$|u(t)|_{2p-2}^{2p-2} \leq M, \ \|u(t)\| \leq M \text{ for any } u_0 \in B \text{ and } t \geq T. \tag{4.9}$$

Taking inner product of (4.1) with $-\Delta u_2$ in H, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2\|^2 + |\Delta u_2|^2 \leq |f(u)||\Delta u_2| + |g(x)||\Delta u_2|, \tag{4.10}$$

since

$$|f(u)||\Delta u_2| \leq \frac{|\Delta u_2|^2}{4} + |f(u)|^2, \qquad |g(x)||\Delta u_2| \leq \frac{|\Delta u_2|^2}{4} + |g(x)|^2,$$

(4.10) implies that

$$\frac{d}{dt}\|u_2\|^2 + |\Delta u_2|^2 \leq 2(|f(u)|^2 + |g(x)|^2).$$

Using (4.4), we find

$$|f(u)|^2 = \int_\Omega |f(u)|^2 dx \le c(|u|_{2p-2}^{2p-2} + 1),$$

thanks to (4.9) and Poincaré inequality, we get

$$\frac{d}{dt}\|u_2\|^2 + \lambda_m \|u_2\|^2 \le c, \text{ for } t \ge T,$$

applying the Gronwall's lemma, we have

$$\begin{aligned} \|u_2(t)\|^2 &\le e^{-\lambda_m(t-t_0)}\|u_2(t_0)\|^2 + \frac{c}{\lambda_m} \\ &\le e^{-\lambda_1 t_0}\|u(t_0)\|^2 + \frac{c}{\lambda_m}, \end{aligned}$$

for $t \ge 2t_0$, and $t_0 > T$.

Obviously, condition $(\widehat{C})$ hold true. □

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References

[1] D.N. Cheban, *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*, World Scientific Publishing (2004).

[2] V. Chepyzhov, M. Vishik, *Attractors for Equations of Mathematics Physics*, Volume 49 of American Mathematical Society Colloquium Publications, AMS, Providence, RI (2002).

[3] S. Lu, H. Wu, C. Zhong, Attractor for nonautonomous 2D-Navier-Stokes equations with normal external forces, *Disc. Cont. Dyn. Syst.*, **13** (2005), 701-719.

[4] Q. Ma, S. Wang, C. Zhong, Necessary and sufficient condition for the existence of global attractors for semigroups and applications, *Indiana Univ. Math. J.*, **51** (2002), 1541-1559.

[5] R. Teman, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York (1997).

[6] C. Zhong, M.H. Yang, C.Y. Sun, The existenc of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations, *J. Diff. Eqns.*, **223** (2006), 367-399.