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ON THE REVERSE EXTENDED HARDY'S INTEGRAL INEQUALITY

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Abstract: In this paper, a best generalization of the reverse extended Hardy's integral inequality is given by using the way of weight coefficient and the technique of real analysis. Some particular results are considered.

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1. Introduction

In 1920, Hardy [1] published the following Hardy's integral inequality: If $p > 1$, $f(x) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$, $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty (\frac{F(x)}{x})^p dx < (\frac{p}{p-1})^p \int_0^\infty f^p(x)dx, \tag{1}$$

where the constant factor $(\frac{p}{p-1})^p$ is the best possible (also cf. Theorem 327 in [2]). And in Theorem 328 of [2], it follows that

$$\int_0^\infty F^p(x)dx < p^p \int_0^\infty (xf(x))^p dx, \tag{2}$$

where the constant factor p^p is still the best possible and $F(x) = \int_x^\infty f(t)dt$.

In 1928, Hardy [3] gave the following best extensions of (1) and (2), by introducing a independent parameter λ: For $\lambda \neq 1$, define $F(x)$ as: $F(x) = \int_0^x f(t)dt$ $(\lambda > 1)$; $F(x) = \int_x^\infty f(t)dt$ $(\lambda < 1)$. Then

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$$\int_0^\infty x^{-\lambda}F^p(x)dx \quad < \quad (\frac{p}{|\lambda-1|})^p \int_0^\infty x^{-\lambda}(xf(x))^p dx \ (p>1); \tag{3}$$

$$\int_0^\infty x^{-\lambda}F^p(x)dx \quad > \quad (\frac{p}{|\lambda-1|})^p \int_0^\infty x^{-\lambda}(xf(x))^p dx \ (0<p<1), \tag{4}$$

where the constant factor $(\frac{p}{|\lambda-1|})^p$ is still the best possible (also cf. Theorem 330 and Theorem 347 in [2]). For $\lambda = p$, inequality (3) reduces to (1) and for $\lambda = 0$, (3) reduces to (2). One names (3) the extended Hardy's integral inequality and (4) the reverse extended Hardy's integral inequality. Inequalities (1)-(4) are important in analysis and its applications (cf. [2], [4]).

In 2004-2005, by introducing some parameters, Yang [5], [6] gave two best extensions of Hilbert's integral inequality. In this paper, by using the way of weight function and the technique of real analysis as [5], [6], one introduces some parameters and considers a best generalization of (4). That is

Theorem 1. *For $0 < p < 1$, $r > 0$, $f \geq 0$, $0 < \int_0^\infty x^{p(1+\frac{1-\lambda}{r})-1}f^p(x)dx < \infty$ $(\lambda \neq 1)$, define $F(x) = \int_0^x f(t)dt$ $(\lambda > 1)$; $F(x) = \int_x^\infty f(t)dt$ $(\lambda < 1)$. Then*

$$\int_0^\infty x^{\frac{p}{r}(1-\lambda)-1}F^p(x)dx > (\frac{r}{|1-\lambda|})^p \int_0^\infty x^{\frac{p}{r}(1-\lambda)-1}(xf(x))^p dx, \tag{5}$$

where the constant $(\frac{r}{|1-\lambda|})^p$ is the best possible. For $r = |1-\lambda|$, one has

$$\int_0^\infty x^{p\frac{1-\lambda}{|1-\lambda|}}F^p(x)dx > \int_0^\infty x^{p\frac{1-\lambda}{|1-\lambda|}-1}(xf(x))^p dx. \tag{6}$$

2. The Preliminary Theorems

Theorem 2. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, $\lambda > 1$, $f, g \geq 0$, $0 < \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1}f^p(t)dt < \infty$, $0 < \int_0^\infty t^{q(1-\frac{1-\lambda}{r})-1}g^q(t)dt < \infty$, then*

$$I := \int_0^\infty \int_y^\infty f(y)g(x)dxdy = \int_0^\infty \int_0^x f(y)g(x)dydx$$
$$> \frac{r}{\lambda-1}\{\int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1}f^p(y)dy\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1}g^q(x)dx\}^{\frac{1}{q}}, \tag{7}$$

where the constant factor $\frac{r}{\lambda-1}$ is the best possible.

Proof. By the reverse Hölder's inequality (cf. [7]), one obtains

$$I \quad = \quad \int_0^\infty \int_y^\infty [\frac{y^{(1+\frac{1-\lambda}{r})/q}}{x^{(1-\frac{1-\lambda}{r})/p}}f(y)][\frac{x^{(1-\frac{1-\lambda}{r})/p}}{y^{(1+\frac{1-\lambda}{r})/q}}g(x)]dxdy$$

$$\begin{aligned}
&\geq \{\int_0^\infty \int_y^\infty \frac{y^{(1+\frac{1-\lambda}{r})\frac{p}{q}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y)dxdy\}^{\frac{1}{p}}\{\int_0^\infty \int_y^\infty \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x)dxdy\}^{\frac{1}{q}} \\
&= \{\int_0^\infty [\int_y^\infty \frac{y^{(1+\frac{1-\lambda}{r})\frac{p}{q}}}{x^{1-\frac{1-\lambda}{r}}} dx] f^p(y)dy\}^{\frac{1}{p}}\{\int_0^\infty [\int_0^x \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} dy] g^q(x)dx\}^{\frac{1}{q}} \\
&= \frac{r}{1-\lambda}\{\int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x)dx\}^{\frac{1}{q}}. \qquad (8)
\end{aligned}$$

If (8) takes the form of equality, then there exist constants A and B, such that they are not all zero and

$$A\frac{y^{(1+\frac{1-\lambda}{r})\frac{p}{q}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y) = B\frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x) \text{ a.e. in } D,$$

where $D = \{(x, y)|0 < y \leq x, 0 < x < \infty\}$. It follows that $Ay^{p(1+\frac{1-\lambda}{r})} f^p(y) = Bx^{q(1-\frac{1-\lambda}{r})} g^q(x) = C$ a.e. in D, and C is a constant. Suppose that $A \neq 0$. Then for any $x > 0$, $y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) = C/(Ay)$ a.e. in $(0, x]$, which is equivalent to $y^{p(1+\frac{1-\lambda}{r})-1} f^p(y) = C/(Ay)$ a.e. in $(0, \infty)$. This contradicts the fact that $0 < \int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f^p(t)dt < \infty$. Hence one has (7).

For $0 < \varepsilon < \frac{p}{r}(\lambda-1)$, setting $f_\varepsilon(t) = g_\varepsilon(t) = 0, t \in (0, 1); f_\varepsilon(t) = t^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}}$, $g_\varepsilon(t) = t^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}}, t \in [1, \infty)$, then

$$\begin{aligned}
I_\varepsilon &= \varepsilon\int_0^\infty \int_0^x f_\varepsilon(y)g_\varepsilon(x)dydx \\
&\leq \varepsilon\int_1^\infty x^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}}(\int_0^x y^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}}dy)dx = \frac{rp}{p(\lambda-1)-r\varepsilon}; \qquad (9) \\
J_\varepsilon &= \varepsilon\{\int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} f_\varepsilon^p(t)dt\}^{\frac{1}{p}}\{\int_0^\infty t^{q(1-\frac{1-\lambda}{r})-1} g_\varepsilon^q(t)dt\}^{\frac{1}{q}} = 1. \quad (10)
\end{aligned}$$

If there exists $K \geq \frac{r}{\lambda-1}$, such that (7) is valid if one replaces $\frac{r}{\lambda-1}$ by K. In particular, by (9)-(10), $\frac{rp}{p(\lambda-1)-r\varepsilon} \geq I_\varepsilon > KJ_\varepsilon = K$, and for $\varepsilon \to 0^+$, it follows $\frac{r}{\lambda-1} \geq K$. Hence $K = \frac{r}{\lambda-1}$ is the best value. The theorem is proved. □

Theorem 3. *For $\lambda < 1$, the other is as the assumption of Theorem 2, one has*

$$\begin{aligned}
\widetilde{I} \; &:= \int_0^\infty \int_0^y f(y)g(x)dxdy = \int_0^\infty \int_x^\infty f(y)g(x)dydx \\
&> \frac{r}{1-\lambda}\{\int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x)dx\}^{\frac{1}{q}}, \; (11)
\end{aligned}$$

where the constant factor $\frac{r}{1-\lambda}$ is the best possible.

Proof. By the reverse Hölder's inequality (cf. [7]), one obtains

$$\begin{aligned}
\widetilde{I} &= \int_0^\infty \int_0^y [\frac{y^{(1+\frac{1-\lambda}{r})/q}}{x^{(1-\frac{1-\lambda}{r})/p}} f(y)][\frac{x^{(1-\frac{1-\lambda}{r})/p}}{y^{(1+\frac{1-\lambda}{r})/q}} g(x)]dxdy \\
&\geq \{\int_0^\infty \int_0^y \frac{y^{(1+\frac{1-\lambda}{r})\frac{p}{q}}}{x^{1-\frac{1-\lambda}{r}}} f^p(y)dydx\}^{\frac{1}{p}} \{\int_0^\infty \int_0^y \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} g^q(x)dydx\}^{\frac{1}{q}} \\
&= \{\int_0^\infty [\int_0^y \frac{y^{(1+\frac{1-\lambda}{r})\frac{p}{q}}}{x^{1-\frac{1-\lambda}{r}}} dx] f^p(y)dy\}^{\frac{1}{p}} \{\int_0^\infty [\int_x^\infty \frac{x^{(1-\frac{1-\lambda}{r})\frac{q}{p}}}{y^{1+\frac{1-\lambda}{r}}} dx] g^q(x)dy\}^{\frac{1}{q}} \\
&= \frac{r}{1-\lambda} \{\int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy\}^{\frac{1}{p}} \{\int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x)dx\}^{\frac{1}{q}}.
\end{aligned}$$

Similarly, one has (11). For $\varepsilon > 0$, setting $f_\varepsilon(t), g_\varepsilon(t)$ as Theorem 2, one has

$$\begin{aligned}
\widetilde{I}_\varepsilon &= \varepsilon \int_0^\infty \int_x^\infty f_\varepsilon(y) g_\varepsilon(x) dydx \\
&\leq \varepsilon \int_1^\infty x^{-1+\frac{1-\lambda}{r}-\frac{\varepsilon}{q}} (\int_x^\infty y^{-1-\frac{1-\lambda}{r}-\frac{\varepsilon}{p}} dy)dx = \frac{pr}{p(1-\lambda)+r\varepsilon}; \\
\widetilde{J}_\varepsilon &= \varepsilon \{\int_0^\infty t^{p(1+\frac{1-\lambda}{r})-1} {f_\varepsilon}^p(t)dt\}^{\frac{1}{p}} \{\int_0^\infty t^{q(1-\frac{1-\lambda}{r})-1} {g_\varepsilon}^q(t)dt\}^{\frac{1}{q}} = 1.
\end{aligned}$$

If there exists $K \geq \frac{r}{1-\lambda}$, such that (11) is still valid if one replaces $\frac{r}{1-\lambda}$ by K. In particular, one has $\frac{pr}{p(1-\lambda)+r\varepsilon} \geq \widetilde{I}_\varepsilon > K\widetilde{J}_\varepsilon = K$, and for $\varepsilon \to 0^+$, it follows $\frac{r}{1-\lambda} \geq K$. Hence $K = \frac{r}{1-\lambda}$ is the best value of (11). The theorem is proved. □

3. The Equivalent Forms

Theorem 4. *As the assumption of Theorem 2, then*

$$J := \int_0^\infty x^{\frac{p}{r}(1-\lambda)-1} (\int_0^x f(y)dy)^p dx > (\frac{r}{\lambda-1})^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy, \quad (12)$$

where the constant $(\frac{r}{\lambda-1})^p$ *is the best possible; (12) is equivalent to (7).*

Proof. If $J = 0$, then (12) is naturally valid; if $J > 0$, then there exists $n_0 \in \mathbf{N}$, such that for $n \geq n_0$, $\int_{\frac{1}{n}}^n t^{p(1+\frac{1-\lambda}{r})-1} [f(t)]_n^p dt > 0$ and $J_n := \int_{\frac{1}{n}}^n x^{\frac{p}{r}(1-\lambda)-1} (\int_{\frac{1}{n}}^x [f(y)]_n dy)^p dx > 0$, where

$$[f(t)]_n = \begin{cases} \frac{1}{n}, \ f(t) < \frac{1}{n}, \\ f(t), \ \frac{1}{n} \leq f(t) \leq n, \\ n, \ f(t) > n. \end{cases} \quad (13)$$

One sets $g_n(x) := x^{\frac{p}{r}(1-\lambda)-1}(\int_{\frac{1}{n}}^{x}[f(y)]_n dy)^{p-1}, x \in [\frac{1}{n}, n]$, and uses (8) to obtain

$$\begin{aligned} \infty &> \int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx = J_n = \int_{\frac{1}{n}}^{n}\int_{\frac{1}{n}}^{x}[f(y)]_n g_n(x)dydx \\ &> \frac{r}{\lambda-1}\{\int_{\frac{1}{n}}^{n} y^{p(1+\frac{1-\lambda}{r})-1}[f(y)]_n^p dy\}^{\frac{1}{p}}\{\int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx\}^{\frac{1}{q}}; \\ 0 &< \int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx = J_n > (\frac{r}{\lambda-1})^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy. \end{aligned}$$

Hence $0 < \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g_\infty^q(x)dx < \infty$, and by (7), for $n \to \infty$, both inequalities above still preserve the strict sign-inequalities. And one has (12).

By the reverse Hölder's inequality, one has

$$I = \int_0^\infty [x^{\frac{1-\lambda}{r}-\frac{1}{p}}\int_0^x f(y)dy][x^{\frac{1}{p}-\frac{1-\lambda}{r}} g(x)]dx \\ \geq J^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x)dx\}^{\frac{1}{q}}. \quad (14)$$

On the other hand, suppose that (12) is valid. Then by (14), one has (7), which is equivalent to (12). One conforms that the constant factor in (12) is the best possible. Otherwise, one can cause up with a contradiction by (14) that the constant factor in (7) is the best possible. This proves the theorem. □

Theorem 5. *As the assumption of Theorem 3, one has*

$$\widetilde{J} := \int_0^\infty x^{\frac{p}{r}(1-\lambda)-1}(\int_x^\infty f(y)dy)^p dx \\ > (\frac{r}{1-\lambda})^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy, \quad (15)$$

where the constant $(\frac{r}{1-\lambda})^p$ *is the best possible; (15) is equivalent to (11).*

Proof. If $\widetilde{J} = 0$, then (15) is naturally valid; if $\widetilde{J} > 0$, then there exists $n_0 \in \mathbf{N}$, such that for $n \geq n_0, \int_{\frac{1}{n}}^{n} t^{p(1+\frac{1-\lambda}{r})-1}[f(t)]_n^p ddt > 0$ and $\widetilde{J}_n := \int_{\frac{1}{n}}^{n} x^{\frac{p}{r}(1-\lambda)-1}(\int_x^n [f(y)]_n dy)^p dx > 0$, where $[f(y)]_n$ is defined by (13). One sets $g_n(x) := x^{\frac{p}{r}(1-\lambda)-1}(\int_x^n [f(y)]_n dy)^{p-1}, x \in [\frac{1}{n}, n]$ and use (11) to obtain

$$\begin{aligned} \infty &> \int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx = \widetilde{J}_n = \int_{\frac{1}{n}}^{n}\int_x^n [f(y)]_n g_n(x)dydx \\ &> \frac{r}{1-\lambda}\{\int_{\frac{1}{n}}^{n} y^{p(1+\frac{1-\lambda}{r})-1}[f(y)]_n^p dy\}^{\frac{1}{p}}\{\int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx\}^{\frac{1}{q}}; \end{aligned}$$

$$0 \quad < \quad \int_{\frac{1}{n}}^{n} x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx = \widetilde{J}_n > (\frac{r}{1-\lambda})^p \int_0^\infty y^{p(1+\frac{1-\lambda}{r})-1} f^p(y)dy.$$

Hence $0 < \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g_n^q(x)dx < \infty$, for $n \to \infty$, both the above inequalities still preserve the strict sign-inequalities by (11). And one has (15).

By the reverse Hölder's inequality, one has

$$\widetilde{I} = \int_0^\infty [x^{\frac{1-\lambda}{r}-\frac{1}{p}} \int_x^\infty f(y)dy][x^{\frac{1}{p}-\frac{1-\lambda}{r}} g(x)]dx$$
$$\geq \widetilde{J}^{\frac{1}{p}} \{ \int_0^\infty x^{q(1-\frac{1-\lambda}{r})-1} g^q(x)dx \}^{\frac{1}{q}}. \quad (16)$$

On the other hand, suppose that (15) is valid. Then by (16), one has (11), which is equivalent to (15). One conforms that the constant factor in (15) is the best possible. Otherwise, one can cause a contradiction by (16) the the constant factor in (11) is the best possible. This proves the theorem.

Note. Combining Theorems 4-5, one may obtain Theorem 1. For $r = p$ in (5), one has (4). Hence (5) is a best generalization of (4).

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