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THE CRITICAL EXPONENT OF GLOBAL SOLUTION EXISTENCE TO A DEGENERATE PARABOLIC EQUATIONS

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Abstract: In this paper we discuss a degenerate parabolic equation with nonlocal source, and the critical exponent of global solutions existence is obtained.

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1. Introduction

In this paper, we consider the following degenerate parabolic equation with a nonlocal source

$$\frac{\partial u}{\partial t} = u^p\Big(\Delta u + a\int_\Omega u\mathrm{d}x\Big), \quad x \in \Omega, \quad t > 0, \tag{1}$$

with the initial and boundary condition

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{3}$$

where $a > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $u_0 \in C^1(\bar{\Omega})$, $u_0 > 0$ in Ω and $u_0 = 0$ on $\partial\Omega$. In the past several decades, many

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physical phenomena have been formulated into nonlocal mathematical models (see [1], [5], [3]). On the one hand, Souplet (see [6]) showed that $p = 1$ is the blow up critical exponent of the nonlocal semilinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + \Big(\int_\Omega |u|^q \mathrm{d}x \Big)^{p/q}, \quad q \geq 1, p > 0,$$

with homogeneous Dirichlet boundary condition. That is to say, if $p < 1$, the solutions are global for all initial data while if $p > 1$, the solutions blow up for sufficiently large initial data. On the other hand, it has been shown that positive solutions of parabolic equations of the form

$$\frac{\partial u}{\partial t} = u^p(\Delta u + u), \quad p > 0,$$

with homogeneous Dirichlet boundary condition, blow up on finite time if and only if $\lambda_1 < 1$, see [4], [2], [9]. Here λ_1 is the first eigenvalue of the Laplacian on Ω with zero Ditichlet data on $\partial\Omega$. But, for the problem (1)–(3), it seems that λ_1 no longer plays a crucial role by these results. In this paper, we will establish new critical exponent for global existence and nonexistence of solutions of problem (1)–(3).

Then, let us state the definition and the main results.

Definition 1. A positive solution of problem (1)-(3) is a function $u(x,t) \in C(\bar{\Omega}\times(0,T^*))\cap C^{2,1}(\Omega\times(0,T^*))$, $u(x,t) > 0$ for $(x,t) \in \Omega\times(0,T^*)$ and satisfying (1)-(3). If $T^* = +\infty$, we say the solution $u(x,t)$ is global.

Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem

$$-\Delta\varphi(x) = 1, x \in \Omega; \quad \varphi(x) = 0, x \in \partial\Omega.$$

Denote

$$\mu = \int_\Omega \varphi(x)\mathrm{d}x.$$

Theorem 1. *If $\mu > 1/a$ and $p > 0$, then there exists no global positive solution of the problem (1)-(3).*

Theorem 2. *If $\mu \leq 1/a$ or $p \leq 0$, then there exists a global solution of the problem (1)-(3).*

2. Local Existence

Let $Q_T = \Omega \times (0,T)$ and $S_T = \partial\Omega \times (0,T)$ for $0 < T < +\infty$. We first give a maximum principle (see [7]), which will be used frequently in this paper.

Lemma 1. *Suppose that $\omega(x,t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and satisfies*

$$\begin{aligned} &\frac{\partial \omega}{\partial t} - d(x,t)\Delta\omega \geq c_1(x,t)\omega + c_2(x,t)\int_\Omega c_3(x,t)\omega \mathrm{d}x, && (x,t) \in Q_T,\\ &\omega(x,t) \geq 0, && (x,t) \in S_T,\\ &\omega(x,t) \geq 0, && x \in \Omega, \end{aligned}$$

where c_1, c_2, c_3 are bounded functions and $c_2, c_3, d \geq 0$ in Q_T. Then $\omega(x,t) \geq 0$ on $\bar{Q}_T$.

To show the local existence of a positive solution of the problem (1)-(3), we consider the following regularized problem

$$\frac{\partial u_\varepsilon}{\partial t} = u_\varepsilon^p\Big(\Delta u_\varepsilon + a\int_\Omega u_\varepsilon \mathrm{d}x\Big), \qquad x \in \Omega, t > 0, \tag{4}$$

$$u_\varepsilon(x,t) = \varepsilon, \qquad x \in \partial\Omega, t > 0, \tag{5}$$

$$u_\varepsilon(x,0) = u_0(x) + \varepsilon, \qquad x \in \Omega, \tag{6}$$

where $0 < \varepsilon < 1$. By a similar discussion as that of Theorem $A.1$-$A.4$ (see [6]), we know that problem (4)-(6) has a unique solution $u_\varepsilon(x,t) > \varepsilon$, defined on $\bar{\Omega} \times (0,T^*)$, where T^* is the maximal existence time of the solution.

According to Lemma 1, we give a comparison principle for the problem (4)-(6).

Lemma 2. *Assume that $\omega(x,t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a nonnegative subsolution (or supersolution) of the problem (4)-(6), then $\omega(x,t) \leq (\geq) u_\varepsilon(x,t)$ on $\bar{Q}_T$.*

Using Lemma 2, we have the following

Lemma 3. *If $1 > \varepsilon_1 > \varepsilon_2 > 0$, then $u_{\varepsilon 1} \geq u_{\varepsilon 2}$ on $(0, T^*_{\varepsilon 1})$ and $T^*_{\varepsilon 1} \leq T^*_{\varepsilon 2}$.*

Then from Lemma 3, it follows that u_ε are monotone with respect to ε, so the limit $T^* = \lim_{\varepsilon\to 0} T^*_\varepsilon$ exists, and as well the point-wise limit

$$u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t) \tag{7}$$

exists for $(x,t) \in \bar{\Omega} \times (0,T^*)$. To prove $u(x,t)$ defined by (7) is a positive solution of the problem (1)-(3), we require the following regularity property.

Lemma 4. (see [8]) *For any $\varepsilon \in (0,1)$ and $T_1 < T_\varepsilon^*$, we have*

$$\int_0^{T_1}\int_\Omega \Big(\frac{\partial u_\varepsilon}{\partial t}\Big)^2 u_\varepsilon^{-p}\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(x,T_1)|^2\mathrm{d}x \le C,$$

where C is a constant independent of ε.

Denote by $\lambda_1 > 0$ and $\phi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\phi(x) = \lambda\phi(x), x \in \Omega; \quad \phi(x) = 0, x \in \partial\Omega.$$

It is well known that $\phi(x)$ may be normalized as $\phi(x) > 0$ in Ω and $\max_\Omega \phi(x) = 1$. Thus, from Lemma 2, we have the following

Lemma 5. *Let $h(x,t) = ke^{-\rho t}\phi(x)$, where $k > 0$ is sufficiently small that $k\phi(x) \le u_0(x)$ and $\rho = \lambda_1\|u\|^p_{L^\infty(0,k)}$. Then $u_\varepsilon(x,t) \ge h(x,t)$ on $\bar\Omega \times (0, T_\varepsilon^*)$.*

Then, by standard arguments (see [8]), it follows from Lemmas 2-5 that $u_\varepsilon \to u$ uniformly with the second derivatives in compact subsets of Ω and u is a solution of the problem (1)-(3) on $\Omega \times (0, T^*)$, where T^* is the maximal existence time of u. Similarly, we can show that $u(x,t)$ is continues at any point $(x,t) \in \partial\Omega$ and $u(x,t) = 0$. Thus, we have the following.

Theorem 3. *The function $u(x,t)$ defined by (7) is a positive solution of the problem (1)-(3). Moreover, if $T^* < +\infty$, then*

$$\lim_{t\to T^*}\sup \max_{x\in\Omega} u(x,t) = +\infty.$$

3. Proof of Main Results

Lemma 6. *If $\mu > 1/a$, then the positive solution $u(x,t)$ of the problem (1)-(3) satisfies $u(x,t) \ge k\varphi(x)$ for $(x,t) \in \bar\Omega\times(0,T^*)$, where $k > 0$ is sufficiently small that $u_0(x) \ge k\varphi(x)$.*

Proof. Let $\omega(x,t) = u(x,t) - k\varphi(x)$, then from (1), we have

$$\frac{\partial\omega}{\partial t} = u^p\Big(\Delta\omega + a\int_\Omega \omega\mathrm{d}x\Big) + ku^p(a\mu - 1) \ge u^p(\Delta\omega + a\int_\Omega \omega\mathrm{d}x).$$

Since u is bounded before its maximal existence time T^*, we have $u(x,t) \ge k\varphi(x)$ according to Lemma 1.

Denote by $\varphi_1(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_1(x) = 1, x \in \Omega_1; \quad \varphi_1(x) = 0, x \in \partial\Omega_1.$$

Here $\Omega_1 \subset \Omega$. It is obvious that $\varphi_1(x)$ depends on Ω_1 continuously. By the comparison principle for an elliptic equation, we have $\varphi_1(x) < \varphi(x)$ on Ω_1. Let $\mu_1 = \int_{\Omega_1} \varphi_1(x)\mathrm{d}x$, then

$$\mu_1 = \int_{\Omega_1} \varphi_1(x)\mathrm{d}x < \int_{\Omega_1} \varphi(x)\mathrm{d}x < \int_{\Omega} \varphi(x)\mathrm{d}x = \mu.$$

Proof of Theorem 1. From Lemma 6 and $\mu > 1/a$, it follows that there exist $\Omega_1 \subset \Omega$ and $c_0 > 0$, such that

$$\mu_1 > \frac{1}{a}, \quad u \geq c_0 > 0, \quad \text{for } x \in \Omega_1, t \in (0, T^*). \tag{8}$$

Let

$$\Phi(s) = -\int_{c_0}^{s} x^{-p}\mathrm{d}x, \quad s \geq c_0. \tag{9}$$

We see that Φ is strictly decreasing and convex on $(c_0, +\infty)$ since $\Phi''(s) = \frac{p}{s^{p+1}} \geq 0$. Hence, the inverse function Φ^{-1} exists and is also strictly decreasing with

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi^{-1}(s) = \frac{1}{\Phi'(\Phi^{-1}(s))}. \tag{10}$$

Let $\theta(x) = \frac{\varphi_1(x)}{\mu_1}$ and define $y : (0, T^*) \to R$ by

$$y(t) = \int_{\Omega_1} \Phi(u(x,t))\theta(x)\mathrm{d}x. \tag{11}$$

Taking the derivative of $y(t)$ with respect to t, we obtain

$$\begin{aligned} y'(t) &= \int_{\Omega_1} \Phi'(u)\frac{\partial u}{\partial t}\theta(x)\mathrm{d}x = -\int_{\Omega_1} \frac{\partial u}{\partial t}u^{-p}\theta(x)\mathrm{d}x \leq \Big(\frac{1}{\mu_1} - a\Big)\int_{\Omega_1} u\mathrm{d}x \\ &\leq \frac{(1/\mu_1 - a)\int_{\Omega_1} u(x,t)\theta(x)\mathrm{d}x}{M}, \end{aligned} \tag{12}$$

where $M = \max_{x\in\Omega_1}\{\theta(x)\}$. By the convexity of Φ and Jensen's inequality, we have

$$\Phi\Big(\int_{\Omega_1} u(x,t)\theta(x)\mathrm{d}x\Big) \leq \int_{\Omega_1} \Phi(u)\theta(x)\mathrm{d}x,$$

which implies, as Φ^{-1} decreases

$$\int_{\Omega_1} u(x,t)\theta(x)\mathrm{d}x \geq \Phi^{-1}\Big(\int_{\Omega_1} \Phi(u)\theta(x)\mathrm{d}x\Big) = \Phi^{-1}(y),$$

which inserted into (12) gives

$$y'(t) \leq \frac{(1/\mu_1 - a)\Phi^{-1}(y(t))}{M}. \tag{13}$$

Let

$$H(t) = \Phi^{-1}(y(t)), \quad t \in (0, T^*).$$

From (10) and the smoothness of y, we know that $H(t) \in C((0, T^*)) \cap C^1((0, T^*))$ and

$$H'(t) = \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Phi^{-1}\Big)(y) \cdot y'(t) = -H(t)^p y'(t).$$

Denote $b = (a\mu_1 - 1)/(\mu_1 M)$, then (13) turns into

$$H'(t) \geq bH(t)^{p+1}, \quad t \in (0, T^*). \tag{14}$$

Furthermore,

$$H(0) = \Phi^{-1}(y(0)) \geq \Phi^{-1}(0) = c_0. \tag{15}$$

Thus, integrating (14) from 0 to T^*, we have

$$\int_{H(0)}^{H(T^*)} \frac{\mathrm{d}s}{s^{p+1}} \geq bT^*.$$

Then, from $p > 0$,

$$T^* \leq \frac{1}{b}\int_{c_0}^{+\infty} \frac{\mathrm{d}s}{s^{p+1}} < +\infty$$

which means $u(x,t)$ can exist no later than $t = T^*$, and the proof is completed. □

Proof of Theorem 2. We take the function $W(x,t) = K\varphi(x) - u(x,t)$, where K is sufficiently large that $K\varphi(x) - u_0(x) \geq 0$ and $u(x,t)$ is the positive solution of the problem (1)-(3) defined by (7). Suppose $T^* < +\infty$, then, from $\mu < 1/a$, we have

$$\frac{\partial W}{\partial t} = u^p\Big(\Delta W + a\int_\Omega W \mathrm{d}x\Big) + Ku^p(1 - a\mu) \geq u^p\Big(\Delta W + a\int_\Omega W \mathrm{d}x\Big).$$

It follows from Lemma 1 that $u(x,t) \leq K\varphi(x)$ for all $0 < t < T^*$. This contradicts Theorem 1, hence, u exists globally.

Next, we show that if $p \leq 0$, then the positive $u(x,t)$ defined by (7) is global.

Choosing $c > \|u_0\|_{L^\infty(\Omega)}$, we consider the initial problem

$$z'(t) = bz^{p+1}(t), \quad z(0) = c.$$

Clearly, it follows from $p \leq 0$ that z exists for all $0 < t < +\infty$ and $z(t) \geq c > 0$. Now, let $\omega(x,t) = z(t) - u(x,t)$, then

$$\omega(x,t) = z(t) - u(x,t) = z(t) \geq c \geq 0, \quad x \in \partial\Omega,$$
$$\omega(x,0) = z(0) - u(x,0) = c - u_0(x) \geq 0,$$

and

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{\partial z}{\partial t} + u^p\Big(\Delta\omega + a\int_\Omega (\omega - z)\mathrm{d}x\Big)\\ &= u^p\Big(\Delta\omega + a\int_\Omega \omega\mathrm{d}x\Big) + bz^{p+1} - au^p\int_\Omega z\mathrm{d}x\\ &\geq u^p\Big(\Delta\omega + a\int_\Omega \omega\mathrm{d}x\Big) + (b - a|\Omega|)(z^p - u^p)z\\ &= u^p\Big(\Delta\omega + a\int_\Omega \omega\mathrm{d}x\Big) + p(b - a|\Omega|)z\int_0^1 \Big(\tau z + (1-\tau)u\Big)^{p-1}\mathrm{d}\tau\omega.\end{aligned}$$

It follows from Lemma 1 that $\omega(x,t) \leq z(t)$ for all $0 < t < T^*$. Hence, u exists globally. This completes the proof of Theorem 2. □

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