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ON THE SOLUTION OF NONLINEAR $\circledast^k$ OPERATOR

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Abstract: In this paper, we study the solution of nonlinear equation $\circledast^k u(x) = f(x, \triangle^{k-1}.L^k u(x))$, where $\circledast^k$ is the otimes operator iterated k times and is defined by

$$\circledast^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k,$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, $f(x)$ is the given function, k is a positive integer and $u(x)$ is an unknown function.

It is found that the existence of the solution $u(x)$ of such equation depends on the conditions of f and $\triangle^{k-1}L^k u(x)$.

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1. Introduction

The operator $\diamondsuit^k$ has been first by A. Kananthai (see [4]) and is named as the diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \quad p+q=n \tag{1.1}$$

is the dimension of the space $\mathbb{R}^n$, for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ and k is a

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nonnegative integer.

The operator $\diamondsuit$ can be expressed in the form $\diamondsuit = \triangle\Box = \Box\triangle$, where $\triangle$ is the Laplacian defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \tag{1.2}$$

and $\Box$ is the ultra-hyperbolic operator defined by

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}. \tag{1.3}$$

The linear equation $\diamondsuit^k = f(x)$, see [3], has been already studied and the convolution $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ has been obtained as a solution of such an equation, where $K_{2k,2k} = R^H_{2k}(x) * R^e_{2k}(x)$. The functions $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (2.2) and (2.5) respectively, with $\alpha = \beta = 2k$.

Next, W. Satsanit first introduced $(\circledast)^k$ operator which is defined by

$$\begin{aligned}
\circledast^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\right. \\
&\quad \left.\cdot\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k = \triangle^k\left(\triangle^2 - \frac{3}{4}(\triangle+\Box)(\triangle-\Box)\right)^k \\
&= \left(\frac{3}{4}\diamondsuit\Box + \frac{1}{4}\triangle^3\right)^k,
\end{aligned} \tag{1.4}$$

where $\diamondsuit$, $\triangle$ and $\Box$ are defined by (1.1) with $k = 1$, (1.2) and (1.3) respectively.

Now, the purpose of this work is to study the nonlinear equation

$$(\circledast)^k u(x) = f(x, \triangle^{k-1} L^k u(x)). \tag{1.5}$$

Here the operator $\circledast^k$ is defined by (1.4) and L^k is defined by

$$L^k = \left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^k$$

with f and continuous for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of R^n and $\partial\Omega$ denotes the boundary of Ω. We can find the solution $u(x)$ of (1.5) which is unique under the condition $|f(x, \triangle^{k-1} L^k u(x))| \leq N$, where N is a constant for all $x \in \Omega$ and the boundary condition is $\triangle^{k-1} L^k u(x) = 0$ for $x \in \partial\Omega$.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Let us denote by

$$u = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{2.1}$$

the nondegenerated quadratic form, and let $p + q = n$ be the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_+$ denote it closure. For any complex number α, define the function

$$R_\beta^H(u) = \begin{cases} \dfrac{u^{\frac{\beta-n}{2}}}{K_n(\beta)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\beta)$ is given by the formula

$$K_n(\beta) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\beta-n}{2})\Gamma(\frac{1-\beta}{2})\Gamma(\beta)}{\Gamma(\frac{2+\beta-p}{2})\Gamma(\frac{p-\beta}{2})}. \tag{2.3}$$

The function $R_\alpha^H(u)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki, see [5].

It is well known that $R_\alpha^H(u)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$ and suppose supp $R_\alpha^H(u) \subset \bar{\Gamma}_+$, that is supp $R_\alpha^H(u)$ is compact.

From S.E. Trione (see [6], p. 11), R_{2k}^H is an elementary solution of the operator $\square^k$ that is

$$\square^k R_{2k}^H(u) = \delta(x). \tag{2.4}$$

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and $v = x_1^2 + x_2^2 + ... + x_n^2$, the function $R_\alpha^e(v)$ denote the elliptic kernel of Marcel Riesz and be defined by

$$R_\alpha^e(v) = \frac{|x|^{\frac{\alpha-n}{2}}}{W_n(\alpha)}, \tag{2.5}$$

where

$$v = |x| = x_1^2 + x_2^2 + ... + x_n^2, \tag{2.6}$$

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.7}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \triangle^k \delta(x)$, where $\triangle^k$ is defined by (1.3). It follows that $R^e_0(x) = \delta(x)$, see [2], p. 118.

Moreover, we obtain that $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$, that is

$$\triangle^k((-1)^k R^e_{2k}(x) = \delta(x), \tag{2.8}$$

see [4], Lemma 2.4, p. 31. By (2.2) and (2.3) with $q = 0$, then $u^{\frac{\alpha-n}{2}}$ reduces to $v^{\frac{\alpha-p}{2}}$ where $v = x_1^2 + x_2^2 + ... + x_p^2$ and $K_n(\alpha)$ reduces to $K_p(\alpha) = \frac{\pi^{\frac{p-1}{2}}\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{p-\alpha}{2})}$.

By using the formulae

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$$

and

$$\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi \sec(\pi z),$$

we obtain

$$K_p(\alpha) = \frac{1}{2}\sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha),$$

where W_p is defined by (2.7) with $n = p$.

Thus for $q = 0$

$$R^H_\alpha(u) = \frac{v^{\frac{\alpha-p}{2}}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right)\frac{v^{\frac{\alpha-p}{2}}}{W_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) R^e_\alpha(v),$$

where $v = x_1^2 + x_2^2 + ... + x_p^2$.

Thus, if $\alpha = 2k$, then $R^H_{2k}(u) = 2(-1)^k R^e_{2k}(v)$ for $q = 0$ and $v = x_1^2 + x_2^2 + ... + x_p^2$.

Lemma 2.1. *Given P is a hyper-function, then*

$$P\delta^k(p) + k\delta^{(k-1)}(p) = 0,$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k derivatives.

Proof. See [7], p. 233. □

Lemma 2.2. *Given the equation*

$$\triangle^k u(x) = 0, \tag{2.9}$$

where $\triangle^k$ is defined by (1.2) and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then $u(x) = (R^e_{2(k-1)}(v))^{(m)}$ is a solution of (2.9) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension and is defined by Definition 2.1. The function $(R^e_{2(k-1)}(v))^{(m)}$ is defined by (2.5) with m-derivatives and $\alpha = 2(k-1)$.

Proof. We first to show that the generalized function $u(x) = \delta^{(m)}(r^2)$, where $r^2 = |x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ of

$$\triangle u(x) = 0. \tag{2.10}$$

Here $\triangle = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2})$ is a Laplace operator. Now

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2) = 2x_i\delta^{(m+1)}(r^2),$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2) = 2\delta^{(m+1)}r^2 + 4x_i^2\delta^{(m+2)}(r^2).$$

Thus

$$\begin{aligned}\triangle\delta^{(m)}(r^2) &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2)\\ &= 2n\delta^{(m+1)}r^2 + 4r^2\delta^{(m+2)}(r^2) = 2n\delta^{(m+1)}r^2 - 4(m+2)\delta^{(m+1)}(r^2).\end{aligned}$$

By Lemma 2.1 with $P = r^2$ we have

$$\triangle\delta^{(m)}(r^2) = (2n - 4(m+2))\delta^{(m+1)}(r^2) = 0 \quad \text{if} \quad 2n - 4(m+2) = 0,$$

or $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Thus $\delta^{(m)}(r^2)$ is a solution of (2.8) with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even. Now $\triangle^k u(x) = \triangle(\triangle^{k-1}u(x)) = 0$ then from the above proof $\triangle^{k-1}u(x) = \delta^{(m)}(r^2)$ with $m = \frac{n-4}{2}$, $n \geq 4$ and n is even.

Convolving both sides of the above equation by the function $(-1)^{k-1}$ $R^e_{2(k-1)}(x)$ we obtain

$$(-1)^{k-1}R^e_{2(k-1)} * \triangle^{k-1}u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2),$$
$$\triangle^{k-1}((-1)^{k-1}R^e_{2(k-1)}) * u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2),$$
$$\delta * u(x) = u(x) = (-1)^{k-1}R^e_{2(k-1)} * \delta^{(m)}(r^2). \tag{2.11}$$

Now from (2.1)

$$R^e_{2(k-1)}(x) = \frac{|x|^{2(k-1)-n}}{W_n(\alpha)} = \frac{(|x|^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} = \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)},$$

where $r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Hence

$$\begin{aligned}R^e_{2(k-1)}(x) * \delta^{(m)}(r^2) &= \frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)} * \delta^{(m)}(r^2)\\ &= \left(\frac{(r^2)^{\frac{2(k-1)-n}{2}}}{W_n(\alpha)}\right)^{(m)} = (R^e_{2(k-1)}(x))^{(m)}.\end{aligned}$$

It follows that $u(x) = (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)}$ is a solution of (2.9) with $m =$

$\frac{n-4}{2}$, $n \geq 4$ and n is even dimension of R^n. □

Lemma 2.3. *Given the equation*

$$\square^k u(x) = 0. \tag{2.12}$$

Here $\square^k$ *is defined by (1.3) and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, *then* $u(x) = (R^H_{2(k-1)}(u))^{(m)}$ *is a solution of (2.12) with* $m = \frac{n-4}{2}$, $n \geq 4$ *and* n *is even dimension and* u *is defined by* (2.1). *The function* $(R^H_{2(k-1)}(u))^{(m)}$ *is defined by (2.2) with* m*-derivatives and* $\beta = 2(k-1)$.

Proof. We first to show that the generalized function $\delta^{(m)}(r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \ldots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2$, $p + q = n$ is a solution of the equation

$$\square u(x) = 0, \tag{2.13}$$

where $\square$ is defined by (1.2) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2 - s^2) = 2x_i\delta^{(m+1)}(r^2 - s^2),$$
$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) = 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2\delta^{(m+2)}(r^2 - s^2),$$

$$\begin{aligned}\square\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)\\ &= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2)\\ &= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2)\\ &+ 4s^2\delta^{(m+2)}(r^2 - s^2)\\ &= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m+2)\delta^{(m+1)}(r^2 - s^2)\\ &+ 4s^2\delta^{(m+2)}(r^2 - s^2)\\ &= (2p - 4(m+2))\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2).\end{aligned}$$

By Lemma 2.1 with $P = r^2 - s^2$. Similarly,

$$\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2 - s^2) = (-2q + 4(m+2))\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2).$$

Thus

$$\begin{aligned}\square\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)\\ &= (2(p+q) - 8(m+2))\delta^{(m+1)}(r^2 - s^2)\end{aligned}$$

$$
\begin{aligned}
&- 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) \\
&= (2n - 8(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4(m + 2)\delta^{(m+1)}(r^2 - s^2) \\
&= (2n - 4(m + 2))\delta^{(m+1)}(r^2 - s^2).
\end{aligned}
$$

If $2n - 4(m + 2) = 0$, we have $\Box\delta^{(m)}(r^2 - s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (2.12) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Now $\Box^k u(x) = \Box(\Box^{k-1}u(x)) = 0$.

From the above proof we have $\Box^{k-1}u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(u)$, we obtain

$$R^H_{2(k-1)}(u) * \Box^{k-1}u(x) = R^H_{2(k-1)}(u) * \delta^{(m)}(r^2 - s^2)$$

$$\Box^{k-1}(R^H_{2(k-1)}(u)) * u(x) = (R^H_{2(k-1)}(u))^{(m)}, \quad \text{where } v = (r^2 - s^2)$$

$$\delta * u(x) = u(x) = (R^H_{2(k-1)}(u))^{(m)}$$

by (2.3) and $v = r^2 - s^2$ is defined by Definition 2.1.

Thus $u(x) = (R^H_{2(k-1)}(u))^{(m)}$ is a solution of (2.12) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. $\Box$

Lemma 2.4. *Let L be the operator defined by*

$$L = (\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k, \tag{2.14}$$

where $\triangle$ and $\Box$ is defined by (1.2) and (1.3) respectively. Then we obtain $H(x)$, where

$$H(x) = \left(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{2.15}$$

and

$$S(x) = \frac{3}{4}R^e_4(v) + \frac{1}{4}(-1)^2 R^H_4(u) \tag{2.16}$$

is an elementary solution of the operator (2.14) iterated $k-$times, $S^{*k}(x)$ denotes the convolution of S it self $k-$times, $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. Moreover $H(x)$ is a tempered distribution.

Proof. From (3.1), we have

$$\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^k H(x) = \delta(x),$$

or we can write

$$\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^{k-1} H(x) = \delta(x).$$

Convolving both sides of the above equation by $R_4^H(u) * (-1)^2 R_4^e(v)$,

$$\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)\left(R_4^H(u) * (-1)^2 R_4^e(v)\right)\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^{k-1} H(x) = \delta(x) * R_4^H(u) * (-1)^2 R_4^e(v),$$

$$\left(\frac{3}{4}\Box^2 (R_4^H(u)) * (-1)^2 R_4^e(v)) + \frac{1}{4}\triangle^2((-1)^2 R_4^e(v)) * (R_4^H(u))\right) \left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^{k-1} H(x) = \delta(x) * R_4^H(u) * (-1)^2 R_4^e(v).$$

By (2.4) and (2.8), we obtain

$$\left(\frac{3}{4}\delta * (-1)^2 R_4^e(v) + \frac{1}{4}\delta * R_4^H(u)\right) \cdot \left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^{k-1} H(x) = R_4^H(u) * (-1)^2 R_4^e(v),$$

or

$$\left(\frac{3}{4}(-1)^2 R_4^e(v) + \frac{1}{4} R_4^H(u)\right)\left(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2\right)^{k-1} H(x) = R_4^H(u) * (-1)^2 R_4^e(v).$$

Keeping on convolving both sides of the above equation by $R_4^H(u) * (-1)^2 R_4^e(v)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * H(x) = \left(R_4^H(u) * (-1)^2 R_4^e(v)\right)^{*k}.$$

The symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(u)$, we have

$$\left(R_4^H(u) * (-1)^2 R_4^e(v)\right)^{*k} = R_{4k}^H(u) * (-1)^{2k} R_{4k}^e(v),$$

$$S^{*k}(x) * H(x) = \left(R_{4k}^H(u) * (-1)^{2k} R_{4k}^e(v)\right),$$

$$H(x) = \left(R_{4k}^H(u) * (-1)^2 R_{4k}^e(v)\right) * (S^{*k})^{*-1}$$

is an elementary solution of (2.14). □

Lemma 2.5. *Given the equation*

$$\triangle u(x) = f(x, u(x)), \tag{2.17}$$

where f is defined and has continuous first deivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ is the boundary of Ω. Assume that f is bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (2.17)

Proof. We can prove the existence of the solution $u(x)$ of (2.10) by the method of iterations and the Schuder's estimates. The details of the proof are given by Courant and Hilbert, see [1], pp. 369-372. □

3. Main Results

Theorem 3.1. *Consider the nonlinear equation*

$$\circledast^k u(x) = f(x, \triangle^{k-1} L^k u(x)), \tag{3.1}$$

where $\circledast^k$ is the operator iterated k times defined

$$\begin{aligned}
\circledast^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\cdot\right. \\
&\quad \left.\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k = \triangle^k\left(\triangle^2 - \frac{3}{4}(\triangle + \Box)(\triangle - \Box)\right)^k \\
&= \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k,
\end{aligned}$$

L is the operator iterated k times defined by

$$Ł^k = (\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k, \tag{3.2}$$

and $\triangle^{k-1}$ is the Laplacian operator iterated $k-1$ times, defined by (1.2). Let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let f be a bounded function, that is

$$|f(x, \triangle^{k-1} L^k u(x)| \leq N, \quad x \in \Omega, \tag{3.3}$$

and the boundary condition

$$\triangle^{k-1} L^k u(x) = 0, \quad x \in \partial\Omega, \tag{3.4}$$

then we obtain

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(v) * (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(u)) * (S^{*k})^{*-1} * w(x) \tag{3.5}$$

as a solution of (3.1) with the boundary condition

$$u(x) = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (R^H_{2(k-2)}(u))^{(m)} \tag{3.6}$$

for $x \in \partial\Omega$, $m = \frac{n-4}{2}$, $k = 2, 3, 4, 5, ...$ and $w(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$ and u defined by (2.1) and v defined by (2.6), $R^H_{2(k-2)}(u)$ and R^H_{4k} are given by (2.2) and $R^e_{4k}(v)$ is given by (2.5).

Proof.

$$\begin{aligned}(\circledast)^k u(x) &= \triangle^k(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) \\ &= \triangle\triangle^{k-1}(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) \\ &= f(x, \triangle^{k-1}(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x)) \\ &= f(x, \triangle^{k-1} L^k u(x)).\end{aligned}$$

Since $u(x)$ has continuous derivatives up to order $6k$ for $k = 1, 2, 3, ...$ we can assume

$$\triangle^{k-1}(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) = w(x), \quad \forall x \in \Omega. \tag{3.7}$$

Thus, (3.1) can be written in the form

$$\circledast^k u(x) = \Box w(x) = f(x, w(x)) \tag{3.8}$$

by (3.3)

$$|f(x, w)| \leq N, \quad \forall x \in \Omega, \tag{3.9}$$

and by (3.4), $w(x) = 0$ or $\triangle^{k-1}(\frac{3}{4}\Box^2 + \frac{1}{4}\triangle^2)^k u(x) = 0$ for $x \in \partial\Omega$. Thus by Lemma 2.3 there exists a unique solution $w(x)$ of (3.8) which satisfies (3.9).

Now consider the equation (3.7) we have $\Box^{k-1} R^H_{2(k-1)}(x) = \delta$, where δ is the Dirac-delta function, that is $(-1)^{2k} R^e_k(x)$ and $R^H_{2(k-1)}(x)$ are the elementary solution of the operators $\triangle^k$ and $\Box^{k-1}$ respectively, see [2], p. 118 and [6], p. 11, i.e.

$$\triangle^k(-1)^k R^e_{2k}(x) = \delta(x)$$

and

$$\Box^{k-1} R^H_{2(k-1)}(u) = \delta(x).$$

The function $R^e_{2k}(x)$ and $R^H_{2(k-1)}(u)$ and defined (2.5)and (2.2) respectively.

Thus, convolving both sides of (3.7) by

$$(-1)^{k-1} R^e_{2(k-1)}(u) * \left(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\right) * (S^{*k})^{*-1},$$

we obtain

$$(-1)^{k-1} R^e_{2(k-1)}(v) * \left(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\right) * (S^{*k})^{*-1} * \triangle^{k-1} L^k u(x)$$

$$= \Big((-1)^{k-1} R^e_{2(k-1)}(v) * \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1}\Big) * w(x).$$

By the properties of convolution, we obtain

$$\begin{aligned}
&\triangle^{k-1}(-1)^{k-1}(R^e_{2(k-1)}(v)) * Ł^k \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1} * w(x) \\
&\quad = \Big((-1)^{k-1} R^H_{2(k-1)}(u) * \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1}\Big) * w(x),
\end{aligned}$$

or

$$\delta*\delta*u(x) = \Big((-1)^{k-1} R^e_{2(k-1)}(u) * \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) * (S^{*k})^{*-1}\Big)*w(x).$$

Thus

$$u(x) = \Big((-1)^{k-1} R^e_{2(k-1)}(v) * \Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)\Big) *(S^{*k})^{*-1}\Big) * w(x) \quad (3.10)$$

as a solution of (3.1)

Next, consider the equaion

$$\triangle^{k-1} L^k u(x) = 0 \quad (3.11)$$

for $x \in \partial\Omega$. By Lemma 2.2, we have

$$L^k u(x) = (-1)^{k-2}(R^e_{2(k-2)}(v))^{(m)}.$$

Convolving both sides of the above equation by $(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1}$, we obtain

$$\begin{aligned}
&(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} L^k u(x) \\
&\qquad = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (R^e_{2(k-2)}(v))^{(m)})
\end{aligned}$$

or

$$\begin{aligned}
&L^k(\Big(R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1}\Big) * u(x) \\
&\qquad = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (-1)^{k-2}(R^e_{2(k-2)}(v))^{(m)}).
\end{aligned}$$

By Lemma 2.4, we obtain

$$\begin{aligned}
&\delta * u(x) = u(x) \\
&\quad = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (-1)^{k-2}(R^e_{2(k-2)}(v))^{(m)}. \quad (3.12)
\end{aligned}$$

Thus

$$u(x) = (R^H_{4k}(u) * (-1)^{2k} R^e_{4k}(v)) * (S^{*k})^{*-1} * (-1)^{k-2}(R^e_{2(k-2)}(v))^{(m)}), \quad (3.13)$$

for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \ldots$.

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