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APPLICATIONS OF COSINE FAMILIES FOR CONTROL THEORY

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Abstract: Some equivalent conditions for exact, null-exact and approximate controllability of "controlled" abstract second-order Cauchy problems, with A as a generator of strongly cosine continuous family have bee given. It has been also proved that the mixed wave problem is exactly controllable in the space $H_0^1[0,1]$ but it is only approximately controllable in the space $L_2[0,1]$.

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1. Introduction

This paper consists of three parts. In the first one, some properties of a cosine family have been presented. In the second, a cosine family has been used to prove the existence and uniqueness of a mild (weak) solution of the abstract second-order Cauchy problem. Usually, [1], the C_0 semigroups have been applied to an investigation of controllability of the abstract second-order Cauchy problem. Here, the families of cosine and sine families have been used. In the third part, using these families, some equivalent conditions for exact, null-exact and approximate controllability have been given. Finally, the mixed wave problem in the space $L_2[0,1]$ has been considered. This problem is approximately

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controllable in the space $L_2[0,1]$, and it is exactly controllable in the space $H_0^1[0,1]$.

Let X be a separable Banach space.

Definition 1. A one parameter family of operators $\{C(t)\} \subset \mathcal{L}(X)$, $t \in \mathbb{R}$ is called strongly continuous cosine family if:

$$C(0) = I, \tag{1}$$

$$C(s+t) + C(s-t) = 2C(s)C(t) \quad \text{for all} \quad s, t \in \mathbb{R}, \tag{2}$$

$$C(t)x \quad \text{is continuous in } t \text{ on } \mathbb{R} \text{ for each fixed } x \in X. \tag{3}$$

Definition 2. If $C(t)$, $t \in \mathbb{R}$, is a strongly continuous cosine family in X, then $S(t)$, denote by $S(t)x = \int_0^t C(t)x\, ds$, $x \in X$, $t \in \mathbb{R}$, is the one parameter family operators. It is called sine family.

Proposition 1. *Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X.*

The following statements are true:

(i) $S(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

(ii) There exist constants $M \geq 1$ and $\omega \geq 0$ such that

$\|C(t)\|_{L(X)} \leq Me^{\omega|t|}$ and $\|S(t)\|_{L(X)} \leq Me^{\omega|t|}$ *for all $t \in \mathbb{R}$,* see[5].

Definition 3. The operator $A : D(A) \subset X :\to X$ is a generator of a strongly continuous cosine family, if for some $a \geq 0$, $(a, \infty) \subset \rho(A)$, there exists strongly continuous and exponentially bounded function $C : [0, \infty) \to \mathcal{L}(X)$ such that $\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} C(t) dt$ for all λ sufficiently large; $\rho(A)$ is the resolvent set of the operator A and $R(\lambda^2, A)$ its resolvent.

Proposition 2. *Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with a generator of A. The following are true:*

(i) $D(A)$ is dense in X.

(ii) A is closed operator.

(iii) If $x \in X$ and $r, t \in \mathbb{R}$, then

$$\int_r^t S(s)xds \in D(A) \quad \text{and} \quad A\int_r^t S(s)xds = C(t)x - C(r)x.$$

(iv) If $x \in X$ and $t \in \mathbb{R}$, then $S(t)x \in E$, where $E = \{X : C(t)x$ is an once continuously differentiable function of $t\}$, see [5].

Proposition 3. *Let $f \in L_p([0,T], X)$, $p \geq 1$, then $\int_0^t S(t-s)f(s)ds \in E$, $t \in (0,T]$.*

Proof. By the properties of $C(t)$ and $S(t)$ we have the following

$$C(p)\int_0^t S(t-s)f(s)ds = \frac{1}{2}\int_0^t [S(t-s+p)+S(t-s-p)]f(s)ds.$$

The first derivative of the right-hand side is equal

$$\frac{1}{2}\int_0^t [C(t-s+p)-C(t-s-p)]f(s)ds,$$

so there exists the first derivative of the left-hand side and

$$\int_0^t S(t-s)f(s)ds \in E.$$ □

2. The Weak Solutions of Abstract Second-Order Cauchy Problem

Consider the inhomogeneous Cauchy problem

$$\frac{d^2w(t)}{dt^2} = Aw(t)+f(t),\ t\in\mathbb{R},\ w(0)=w_1,\ \frac{dw(t)}{dt}|_{t=0}=w_2,\ w_1,w_2\in X. \quad (4)$$

Definition 4. Let $f:[0,T]\to X$ be Bochner integrable and $w_1, w_2 \in X$.

$$w(t) = C(t)w_1 + S(t)w_2 + \int_0^t S(t-s)f(s)ds, \quad t \geq 0,$$

is called a mild solution of (4).

Theorem 1. *Let $A : D(A) :\to X$ be a linear operator. The following assertions are equivalent:*

(i) A is the generator of cosine family on the Banach space X,

(ii) $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ generates an once integrated semigroups on the Banach space $X \times X$, see [3].

Proposition 4. *If A generates cosine family on X, $f \in L_p([0,T],X)$, $p > 1$, $T > 0$ and $w_1, w_2 \in X$, then the mild solution is continuous on $[0,T]$.*

Proof. Since $C(t)$, $S(t)$ are continuous on X, we can assume without loss of generality that $w_1 = 0$, $w_2 = 0$. Using the properties $C(t)$ and $S(t)$ we can write $w(t+\delta)-w(t)$ in the equivalent form

$$\begin{aligned} w(t+\delta)-w(t) &= (C(\delta)-I)\int_0^t S(t-s)f(s)ds + S(\delta)\int_0^t C(t-s)f(s)ds \\ &\quad + \int_t^{t+\delta} S(t+\delta-s)f(s)ds \end{aligned}$$

for $\delta \neq 0$. So

$$\|w(t+\delta)-w(t)\|_X \leq \|(C(\delta)-I)w(t)\|_X + \|S(\delta)v(t)\|_X + \Big(\int_t^{t+\delta}\|S(t+\delta-s)\|^q ds\Big)^{1/q}\Big(\int_t^{t+\delta}\|f(s)\|^p ds\Big)^{1/p},$$

$$\frac{1}{p}+\frac{1}{q}=1, \qquad v(t)=\int_0^t C(t-s)f(s)ds.$$

The above inequality implies the continuity of w. □

Remark 1. w is a weak solution in the following sense

$$\begin{cases} \dfrac{d^2}{dt^2}\langle w(t),w^*\rangle = \langle w(t),A^*w^*\rangle + \langle f(t),w^*\rangle \text{ a.e. on } [0,T], \\ \dfrac{d}{dt}\langle w(t),w^*\rangle|_{t=0} = \langle w_2,w^*\rangle, \\ w(0)=w_1 \quad \text{for any } w^*\in D(A^*). \end{cases}$$

Remark 2. w is also a continuous solution of the integral equation

$$w(t)=w_1+tw_2+A\int_0^t (t-r)w(r)dr+\int_0^t (t-r)f(r)dr.$$

3. The Application of Cosine Family for a Control Theory

Let Y be a subspace of X. Both Y and X are Banach spaces. Let $C(t): X\to X$, $S(t): X\to Y$ and $C(t): Y\to Y$, $t\in\mathbb{R}$.

Now, we will study "controlled" abstract Cauchy problems of the form

$$\begin{aligned} &\frac{d^2w(t)}{dt^2} = Aw(t)+Bu(t),\ t\geq 0,\ w(0)=w_1, \\ &\frac{dw(t)}{dt}|_{t=0} = w_2,\ w_1,w_2\in X. \end{aligned} \tag{5}$$

We assume that the operator A generates a strongly continuous cosine family $C(t)$, $t\in\mathbb{R}$; B is a bounded control operator from a control Banach space U to X; $u:[0,\infty)\to U$ is a locally integrable control function. By Proposition 3, the mild solution of (5) is given by the formula

$$w(t)=C(t)w_1+S(t)w_2+\int_0^t S(t-s)Bu(s)ds, \quad t\geq 0.$$

Definition 5. For some fixed $p\geq 1$ and $u(.)\in L_p([0,T],U)$ we define the controllability map

$$B_T \in \mathcal{L}(L_p([0,T],U),Y) \quad \text{by} \quad B_T[u(.)] = \int_0^T S(T-r)Bu(r)dr$$

for the control system (5), with a state Y and a control space U.

For $1 \leq p < \infty$, the system (5) is called:

(j) exactly p-controllable on $[0,T]$ if $rgB_T = Y$,

(jj) approximately p-controllable on $[0,T]$ if $\overline{rgB_T} = Y$,

(jjj) exactly p-null controllable on $[0,T]$ if $rg(C(T) \oplus S(T)) \subset rgB_T$, where $rg(C(T) \oplus S(T))$

$$= \{z \in Y : z = z_1 + z_2,\ z_1 = C(T)y,\ y \in Y,\ z_2 = S(T)x,\ x \in X\}.$$

Lemma 1. *Let V, W, Z be Banach spaces and $P \in \mathcal{L}(V,Z)$ and $T \in \mathcal{L}(W,Z)$. The following conditions are equivalent:*

(i) (a) $\overline{rgP} \subset \overline{rgT}$. (b) $\ker T^* \subset \ker P^*$.

If, additionally, the spaces, V, W, and, Z are reflexive, then

(ii) (a) $rgP \subset rgT$. (b) *There exists $\gamma > 0$ such $||P^*z^*||_{Y^*} \leq \gamma ||T^*z^*||$ for all $z^* \in Z^*$, see [2].*

Theorem 2. *Let be a control system of (5) with a state reflexive space Y and a reflexive control space U, $1 < p < \infty$.*

The following conditions are equivalent:

(j) (a) (5) is exactly p-controllable on $[0,T]$.

(b) There exists $\gamma > 0$ such $||y^*||_{Y^*} \leq \gamma ||B^*S(.)^*y^*||_{L^q([0,T],U^*)}$, for all $y^* \in Y^*$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(jj) (a) (5) is approximately p-controllable on $[0,T]$.

(b) $\ker B^*S(s)^* = \{0\}$ a.e. for $s \in [0,T]$.

(jjj) (a) is exactly p-null controllable on $[0,T]$.

(b) There exists $\gamma > 0$ such

$$\gamma \|B^*S(.)^*y^*\|_{L^q([0,T],U^*)} \geq \|C(T)^*y^*\|_{Y^*} + \|S(T)^*y^*\|_{Y^*},$$

for all $y^* \in Y^*$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The proof of the above theorem is based on Lemma 1 and on the following remark.

Remark 3. In the case (jjj), an equivalent form of $w(t)$ has been used

$$w(t) = \left[\begin{array}{cc} C(t), & S(t) \end{array}\right] \left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] + \int_0^t S(t-s)Bu(s)ds$$

in order to find the adjoint operator to the operator $[C(t), S(t)] \left[\begin{array}{c} w_1 \\ w_2 \end{array}\right]$.

This adjoint equals to $\left[\begin{array}{c} C^*(t) \\ S^*(t) \end{array}\right] y^*$, $y^* \in Y^*$.

Theorem 3. *If $Y = X$, then the system* (5) *is neither the exactly controllable on $[0, T]$, nor the null-exactly controllable on $[0, T]$. If B^* is a surjective and U, X are reflexive spaces the system* (5) *is approximately controllable on $[0, T]$.*

Proof. A range of B_T is the proper subset of X ($rgB_T \subset E$), so the system (5) is not the exactly controllable on $[0, T]$. The system (5) is not the null-exactly controllable on $[0, T]$ because $rgB_T \subset E$ but $rg(C(T) \oplus S(T)) \subset X$.

By the Theorem 2(jj) we have $B^*S(T-r)^*x^* = 0$ a.e. for $r \in [0, T]$. B^* is a surjective, so $S(T-r)^*x^* = 0$. $\frac{dS(T-r)^*x^*}{dr} = -C(T-r)^*x^*$ and $C(T-r)^*x^* = 0$. So for $r = T$, $C(0)^*x^* = 0 \Leftrightarrow I^*x^* = 0$. It implies that $x^* = 0$. It means that the system (5) is approximately controllable on $[0, T]$. □

4. Example

Consider the hyperbolic mixed problem with a control function $z(., t) \in L_2[0, 1]$

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} = \dfrac{\partial^2 w(t,x)}{\partial x^2} + z(t,x) \text{ for } x \in [0,1] \text{ and } ; 0 < t \leq T, \\ w(0,x) = w_1(x), \ \dfrac{\partial w(t,x)}{\partial t}|_{t=0} = w_2(x) \text{ for } x \in [0,1], \\ w(t,0) = 0, \ w(t,l) = 0 \text{ for } t \geq 0. \end{cases} \tag{6}$$

This problem can be formulated as an abstract Cauchy problem in the Hilbert space $L_2[0, 1]$:

$$\begin{aligned} &\frac{d^2 w(t)}{dt^2} = Aw(t) + z(t), \quad t \in \mathbb{R}, \quad w(0) = w_1, \\ &\frac{dw(t)}{dt}|_{t=0} = w_2, \quad w_1, w_2 \in L_2[0,1], \end{aligned} \tag{7}$$

where $A = \frac{\partial^2}{\partial x^2}$ with the domain $D(A) = H^2[0, 1] \cap H_0^1[0, 1]$.

Remark 4. In [4] it has been proved that the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}$ with the domain $D(\mathcal{A}) = H^2[0,1] \cap H_0^1[0,1] \times L_2[0,1]$ generates exponentially bounded non-degenerate once integrated semigroup on the Banach space $L_2[0,1] \times L_2[0,1]$. By Theorem 1 the operator $A = \frac{\partial^2}{\partial x^2}$ generates the cosine family on $L_2[0,1]$. In our case

$$C(t)\phi = 2\sum_{n=1}^{\infty} \langle \phi, \varphi_n \rangle \cos(n\pi t)\varphi_n \quad \text{and}$$

$$S(t)\phi = 2\sum_{n=1}^{\infty} \langle \phi, \varphi_n \rangle \frac{\sin(n\pi t)}{n\pi}\varphi_n \quad \text{for } \phi \in L_2[0,1],$$

where $\varphi_n(x) = \sin(n\pi x)$, and $\langle .,. \rangle$ denotes inner product on $L_2[0,1]$.

Remark 5. The operator $A = \frac{\partial^2}{\partial x^2}$ with the domain $D(A) = H^2[0,1] \cap H_0^1[0,1]$ is self adjoint in $L_2[0,1]$, so $S(t)^* = S(t)$.

Remark 6. By Theorem 3, the system described by (7) in the space $L_2[0,1]$, is the approximately controllable on any interval $[0,T]$, $T > 0$, by is not the null-exactly controllable and it is not exactly controllable.

Theorem 4. *The system described in* (7) *is exactly controllable on interval* $[0,T]$ *in the space* $H_0^1[0,1]$.

Proof. At first we denote we observe that in the space $H_0^1[0,1]$, $S(t)^* = -AS(t)$. So the condition (j)(b) from Theorem 2 is in the form: there exists $\gamma_1 > 0$ such that

$$\gamma_1 \int_0^T \| -AS(t)\phi\|^2_{L_2[0,1]} \geq \|\phi\|^2_{H_0^1[0,1]} \quad \text{for } \phi \in H_0^1[0,1].$$

This condition is equivalent to the following one

$$\gamma_1 \int_0^T \sum_{n=1}^{\infty} 4n^2\pi^2 \langle \phi, \sin n\pi \rangle^2 \sin^2 n\pi t dt \geq \sum_{n=1}^{\infty} n^2\pi^2 \langle \phi, \sin n\pi \rangle^2.$$

The above inequality yields the system of inequalities for any $n \in$N and any $T > 0$

$$4\gamma_1 (\tfrac{T}{2} - \tfrac{\sin 2n\pi T}{4n\pi}) \geq 1.$$

It is easy to prove that for each $n \in$N and any $T > 0$, $\frac{T}{2} - \frac{\sin 2n\pi T}{4n\pi} > 0$. $\lim_{n\to\infty}(\frac{T}{2} - \frac{\sin 2n\pi T}{4n\pi}) = \frac{T}{2}$, so there exists $m \in N$ such that for $n > m$, $\frac{T}{2} - \frac{\sin 2n\pi T}{4n\pi} > \frac{T}{4}$.

Let $\gamma = \frac{1}{\gamma_1}$. We choose γ in the following way

$$0 < \gamma < \min\left(\min_{k\in\{1,2,\dots,m\}} \gamma_k, \frac{\pi}{4}\right),$$

where $0 < \gamma_k < \frac{T}{2} - \frac{\sin 2k\pi T}{4k\pi}$, $k = 1, 2, \dots, m$.

This means that the system (6) is exactly controllable on any interval $[0, T]$, for $T > 0$. □

Remark 7. In [1], applying C_0 semigroups, it has been proved that the system (7) is exactly controllable on $[0, T]$ in the space $H_0^1[0, 1] \times L_2[0, 1]$.

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