International Journal of Pure and Applied Mathematics
Volume 57 No. 1 2009, 23-25

A GENERALIZATION OF THE REJECTION METHOD OF SIMULATION THROUGH THE RADON-NIKODÝM DERIVATIVE

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Abstract: A generalization of the acceptance-rejection method in $\mathbb{R}^n$ is proposed for simulating outcomes of a random experiment in an arbitrary probability space (Ω, Σ, μ), comparing with the outcomes of (Ω, Σ, ν), where there is a finite constant $M \geq 1$ such that $\mu \leq M\nu$.

AMS Subject Classification: 28A35, 60A10
Key Words: acceptance-rejection method, Radon-Nikodým derivative

1. Introduction

In this paper we expose a generalization of the acceptance-rejection method for simulating n-dimensional continuous random variables (e.g. [1], Chapter II, pp. 41-42).

The classical acceptance-rejection method establishes that if we have two probability density functions f and g in $\mathbb{R}^n$, a constant $M \geq 1$ such that $Mg \geq f$, a sequence $(Y_k)_{k=1}^{\infty}$ of independent random vectors in $\mathbb{R}^n$ with density g, and other sequence $(U_k)_{k=1}^{\infty}$ of independent random variables with uniform distribution in $[0, 1]$ also independent of the first sequence, then for all Borel sets $A \subset \mathbb{R}^n$, we have

Received: August 22, 2009

$$\int_A f(x)\mathrm{d}x$$
$$= \mathrm{P}\left(\bigcup_{k=1}^{\infty}\left[Y_k \in A,\, U_k \leq \frac{f(Y_k)}{Mg(Y_k)},\ \text{and } U_j > \frac{f(Y_j)}{Mg(Y_j)},\ \text{for } j<k\right]\right).$$

Moreover, if N is the firs k such that $U_k \leq \frac{f(Y_k)}{Mg(Y_k)}$, then

$$\mathrm{E}(N) = M.$$

So, if we want to simulate the value x of a random vector X with density f, but it is easier to simulate the value of a random vector Y with density g and there is a constant M such that, $Mg \geq f$, we simulate independent random numbers u_k and independent random values y_k of Y, until the first time t in which $u_t \leq \frac{f(y_t)}{Mg(y_t)}$, and the expected numbers of iterations of this algorithm is M. Thus, in order to reduce the number of iterations, it is important that M do not be too large.

In [2], pp. 53-54 is given a similar algorithm for simulating discrete random variable by mean of the probability mass function.

The next section provides a similar technique for simulating the result of a random experiment whose sample space is an arbitrary set Ω and the probability space (Ω, Σ, μ) is such that there is other probability space (Ω, Σ, ν) and a number $M > 1$ for which $\mu \leq M\nu$. The method can be interesting when the simulation of a result in (Ω, Σ, ν) is easier than that of (Ω, Σ, μ) and we can evaluate some Radon-Nikodým derivatives.

2. The General Acceptance-Rejection Method

We are interested in simulating the results of a random experiment whose probability space is (Ω, Σ, μ).

Theorem 1. *If μ and ν are probability measures in Σ such that $\mu \leq M\nu$; ψ is a measure in Σ such that $\nu \ll \psi$; f is a Radon-Nikodým derivative of μ with respect to ψ; g is a Radon-Nikodým derivative of ν with respect to ψ; λ is the Lebesgue measure in $\mathbb{R}$; and $B = \left\{(\omega, u) \in \Omega \times [0,1] : u \leq \frac{f(\omega)}{Mg(\omega)}\right\}$; then*

$$\mu = \nu \otimes \lambda(\cdot \times [0,1]|B).$$

Proof. Let $A \in \Sigma$,

$$\begin{aligned}\nu \otimes \lambda(A \times [0,1] \cap B) &= \int_A \int_{\left[0, \frac{f(\omega)}{Mg(\omega)}\right]} \mathrm{d}\lambda(u)\mathrm{d}\nu(\omega) \\ &= \int_A \frac{f(\omega)}{Mg(\omega)}\mathrm{d}\nu(\omega) = \int_A \frac{f(\omega)}{Mg(\omega)} g(\omega)\mathrm{d}\psi(\omega) \\ &= \frac{1}{M}\mu(A).\end{aligned}$$

Now,

$$\begin{aligned}\nu \otimes \lambda(B) &= \int_\Omega \int_{\left[0, \frac{f(\omega)}{Mg(\omega)}\right]} \mathrm{d}\lambda(u)\mathrm{d}\nu(\omega) = \int_\Omega \frac{f(\omega)}{Mg(\omega)}\mathrm{d}\nu(\omega) \\ &= \int_\Omega \frac{f(\omega)g(\omega)}{Mg(\omega)}\mathrm{d}\psi(\omega) = \frac{1}{M},\end{aligned}$$

So $\nu \otimes \lambda(A \times [0,1]|B) = \frac{\frac{1}{M}\mu(A)}{\frac{1}{M}} = \mu(A)$. □

The following result is an immediate consequence of the previous theorem.

Corollary 2. *If μ and ν are probability measures in Σ such that $\mu \leq M\nu$; f is a Radon-Nikodým derivative of μ with respect to ν; λ is the Lebesgue measure in $\mathbb{R}$; and*

$$B = \left\{(\omega, u) \in \Omega \times [0,1] : u \leq \frac{f(\omega)}{M}\right\};$$

then

$$\mu = \nu \otimes \lambda(\cdot \times [0,1]|B).$$

References

[1] L. Devroye, *Non-Uniform Random Variates Generations*, Springer-Verlag, New York (1986).

[2] S.M. Ross, *Simulation*, 3-rd Edition, Academic Press, London (2002).