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AN EXTENDED MODEL FOR DENSE GASES AND MACROMOLECULAR FLUIDS, OBTAINED WITHOUT USING TAYLOR'S EXPANSIONS

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Abstract: In this paper we consider the 14 moments model of the extended thermodynamics for dense gases and macromolecular fluids. Solutions of the restrictions imposed by the entropy principle and that of Galilean relativity for such a model were until now obtained in the literature only in an approximate manner up to a certain order with respect to thermodynamic equilibrium; for more restrictive models they were obtained up to whatever order, but by using Taylor expansions around equilibrium and without proving convergence. Here we have found an exact solution without using expansions. The idea has been to write firstly a relativistic model, for which it is easy to impose the Einsteinian relativity principle, and then taking its non relativistic limit.

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1. Introduction

The 14 moments model of the extended thermodynamics for dense gases and macromolecular fluids is firstly studied by Kremer in [11], [10] up to second order with respect to equilibrium. The balance equations to describe this model are

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$$\partial_t F + \partial_k F_k = 0\,, \quad \partial_t F_i + \partial_k G_{ki} = 0\,, \quad \partial_t F_{ij} + \partial_k G_{kij} = P_{<ij>}\,, \tag{1}$$
$$\partial_t F_{ill} + \partial_k G_{kill} = P_{ill}\,, \quad \partial_t F_{iill} + \partial_k G_{kiill} = P_{iill}\,,$$

where the independent variables are F, F_i, F_{ij}, F_{ill}, F_{iill} (that is, the densities of mass, momentum, energy, stress tensor, energy flux and trace of the flux of energy flux) and are symmetric tensors. $P_{<ij>}$, P_{ill}, P_{iill} are productions and they are symmetric tensors too. The symbol $< ij >$ in $P_{<ij>}$ denotes that this is a traceless tensor. The fluxes G_{ki}, G_{kij}, G_{kill}, G_{kiill} are constitutive functions and are symmetric over all indexes, except for k.

In ideal gases, we have also the conditions $G_{ki} = F_{ki}$, $G_{ill} = F_{ill}$, $G_{iill} = F_{iill}$ and, moreover, G_{kij} and G_{kill} are symmetric over all couples of indexes; therefore, the present case is less restrictive.

We note that, in the literature, many researchers used different set of moments to describe the hydrodynamic dense gases theory, but not all of these can be obtained in a natural way by a non relativistic limit, as proved in [3], [4]. This result can be achieved only for:

1) the 5-fields theory, for which an exact closure already exists (see [5], for example),

2) the present 14-fields theory,

3) theories with other higher number of fields, but for which representation theorems do not exist in literature.

For this reason we consider here only the 14-fields theory.

We clarify that our solution (2), (5) (6) is certainly non relativistic. However a natural question may arise: "How we have thought this complicate solution?"; the answer is that we obtained it with a mathematical tool, that is going to a relativistic formulation and, subsequently, taking its non relativistic limit.

We want that our system (1) is a symmetric hyperbolic one, with all consequent nice mathematical properties. To this end, we impose the entropy principle; in the next section it will be proved that it is equivalent to assume the existence of variables λ_A and of the functions h', ϕ'_k depending on λ_A so that the following equations hold

$$\begin{aligned}
F &= \frac{\partial h'}{\partial \lambda}, \quad F_i = \frac{\partial h'}{\partial \lambda_i}, \quad F_{il} = \frac{\partial h'}{\partial \lambda_{il}}, \\
F_{ill} &= \frac{\partial h'}{\partial \lambda_{ill}}, \quad F_{iill} = \frac{\partial h'}{\partial \lambda_{iill}}\,, \\
G_k &= \frac{\partial \phi'_k}{\partial \lambda}, \quad G_{ki} = \frac{\partial \phi'_k}{\partial \lambda_i}, \quad G_{kil} = \frac{\partial \phi'_k}{\partial \lambda_{il}},
\end{aligned} \tag{2}$$

$$G_{kill} = \frac{\partial \phi'_k}{\partial \lambda_{ill}}, \quad G_{kiill} = \frac{\partial \phi'_k}{\partial \lambda_{iill}},$$

where we have taken into account the definition $G_k = F_k$, from which the following compatibility condition $(3)_1$ holds. If we impose also the conservation law of angular momentum, we have that G_{ki} is symmetric, so that we have to impose also the condition $(3)_2$

$$\frac{\partial \phi'_k}{\partial \lambda} = \frac{\partial h'}{\partial \lambda_k}, \quad \frac{\partial \phi'_{[k}}{\partial \lambda_{i]}} = 0 . \tag{3}$$

Moreover, also in the next section, it will be proved that the Galilean relativity principle is equivalent to the following two additional conditions

$$0 = \frac{\partial h'}{\partial \lambda}\lambda_i + 2\lambda_{ij}\frac{\partial h'}{\partial \lambda_j} + \lambda_{jpp}\left(\frac{\partial h'}{\partial \lambda_{rs}}\delta_{rs}\delta_{ij} + 2\frac{\partial h'}{\partial \lambda_{ij}}\right) + 4\lambda_{ppqq}\frac{\partial h'}{\partial \lambda_{ill}}, \tag{4}$$

$$0 = \frac{\partial \phi'_k}{\partial \lambda}\lambda_i + 2\lambda_{ij}\frac{\partial \phi'_k}{\partial \lambda_j} + \lambda_{jpp}\left(\frac{\partial \phi'_k}{\partial \lambda_{rs}}\delta_{rs}\delta_{ij} + 2\frac{\partial \phi'_k}{\partial \lambda_{ij}}\right) + 4\lambda_{ppqq}\frac{\partial \phi'_k}{\partial \lambda_{ill}} + h'\delta_{ik} .$$

In other words, we have to find h' and ϕ'_k satisfying the equations (3) and (4); after that, $(2)_{1-5}$ are useful to obtain λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} as functions of our independent variables F, F_i, F_{ij}, F_{ill}, F_{iill}. Lastly, $(2)_{7-10}$ will give the constitutive functions G_{ki}, G_{kij}, G_{kill}, G_{kiill}.

Now, the restrictions (3) and (4) were imposed by Kremer in [11], [10] only up to second order with respect to equilibrium. In [3] Carrisi and Pennisi have imposed them up to whatever order (for the more restrictive model of ideal gases), but by using Taylor' s expansions around equilibrium, without worrying about convergence problems. Here we have found the exact solution without using expansions. The idea to obtain this result has been the following:

— Firstly we have assumed a relativistic model for which it is easy to impose the Einsteinian relativity principle; the results for this model can be found in Section 3;

— After that, we have shown, in Section 4, how to take the non relativistic limit of the model in Section 3; to this end we have used a methodology which is easy to find for ideal gases because in this case we have suggestions from the kinetic theory of gases; we have adopted this methodology also for our more general case. Obvious, the validity of this assumption has to be tested at the end by verifying that our results satisfy truly the equations (3) and (4). We will obtain that equations (4) are identically satisfied. Instead of this, the conditions (3) will have still to be imposed; they are the only conditions for which there is no correspondence between the relativistic case and the classical one.

— Then, in Section 5 we have taken effectively this limit and found that

h' and ϕ'_k are determined by the following equations (5) and (6), except for 4 scalar arbitrary functions H_0, H_1, H_2, H_3 which depend on the scalars (7).

There remain the further conditions (3), the requirement of convexity for the function h' and the problem of subsystems. We have exploited them, but do not report here the results, for the sake of brevity. We assure only that we have found the exact solution of $(3)_1$ without using expansions.

We close this section by reporting the results of Section 5; in this way it will be not necessary to search them throughout the paper and they will be available for the applications. They are

$$\begin{aligned} \phi'^k &= H_0V_0^k + H_1V_1^k + H_2V_2^k + H_3V_3^k\,, \\ h' &= 8H_0X_1 + H_1X_2 + H_2X_3 + H_3X_4\,, \end{aligned} \tag{5}$$

with

$$\begin{aligned} V_0^k &= -2\lambda_{kll}, \\ V_1^k &= -2\lambda_{kh}\lambda_{hll} + 4\lambda_{ppll}\lambda_k + \frac{4}{5}\lambda_{ll}\lambda_{kll}\,, \\ V_2^k &= -2\lambda_{kh}^2\lambda_{hll} + \frac{6}{5}\lambda_{ll}\lambda_{ka}\lambda_{all} + 4\lambda_{ka}\lambda_a\lambda_{ppll} \\ &- \frac{11}{25}\lambda_{ll}^2\lambda_{kll} - \lambda_{kll}\lambda_a\lambda_{all} + \lambda_k\lambda_{all}\lambda_{all} \\ &+ (tr\lambda_{ab}^2)\lambda_{kll} - \frac{12}{5}\lambda_{ppll}\lambda_{ll}\lambda_k, \\ V_3^k &= 2\lambda_{ppll}\left(2\lambda_{kh}^2\lambda_h - tr\lambda_{ab}^2\lambda_k - \frac{8}{5}\lambda_{ll}\lambda_{ka}\lambda_a + \frac{17}{25}\lambda_{ll}^2\lambda_k\right) \\ &+ (\lambda_{kh}\lambda_h)(\lambda_{all}\lambda_{all}) - \frac{4}{5}\lambda_{ll}(\lambda_{all}\lambda_{all})\lambda_k - \frac{17}{25}\lambda_{ll}^2\lambda_{ka}\lambda_{all} \\ &- (\lambda_a\lambda_{all})\lambda_{kb}\lambda_{bll} + (tr\lambda_{ab}^2)\lambda_{kc}\lambda_{cll} + \frac{4}{5}\lambda_{ll}(\lambda_a\lambda_{all})\lambda_{kll} \\ &+ \frac{8}{5}\lambda_{ll}\lambda_{kh}^2\lambda_{hll} + \frac{74}{375}\lambda_{ll}^3\lambda_{kll} - \frac{4}{5}\lambda_{ll}(tr\lambda_{ab}^2)\lambda_{kll} + (\lambda_{ab}\lambda_{all}\lambda_{bll})\lambda_k \\ &- (\lambda_{ab}\lambda_a\lambda_{bll})\lambda_{kll} + \frac{2}{3}(tr\lambda_{ab}^3)\lambda_{kll} - 2\lambda_{kh}^3\lambda_{hll}\,, \end{aligned} \tag{6}$$

$$\begin{aligned} X_1 &= \lambda_{ppll}\,, \\ X_2 &= 2\lambda_{all}\lambda_{all} - \frac{16}{5}\lambda_{ppll}\lambda_{ll}\,, \\ X_3 &= 8\lambda_{ppll}\left(\frac{11}{50}\lambda_{ll}^2 - \frac{1}{2}tr\lambda_{ab}^2\right) + 2\lambda_{ab}\lambda_{all}\lambda_{bll} - \frac{6}{5}\lambda_{ll}\lambda_{all}\lambda_{all}\,, \\ X_4 &= 2\lambda_{ab}^2\lambda_{all}\lambda_{bll} - tr\lambda_{ab}^2\lambda_{cll}\lambda_{cll} - \frac{8}{5}\lambda_{ll}\lambda_{ab}\lambda_{all}\lambda_{bll} \end{aligned} \tag{7}$$

$$
\begin{aligned}
& +\frac{17}{25}\lambda_{ll}^2\lambda_{all}\lambda_{all}+8\lambda_{ppll}\left(-\frac{37}{375}\lambda_{ll}^3+\frac{2}{5}\lambda_{ll}(tr\lambda_{ab}^2)-\frac{1}{3}tr\lambda_{ab}^3\right),\\
X_5 &= -\frac{2}{5}\lambda_{ll}^2+16\lambda_{ppll}\lambda-4\lambda_a\lambda_{all}+2tr\lambda_{ab}^2\,,\\
X_6 &= 4\lambda\lambda_{all}\lambda_{all}+8\lambda_{ppll}\left(-\frac{4}{5}\lambda\lambda_{ll}+\frac{1}{2}\lambda_a\lambda_a\right)\\
& +\frac{8}{5}\lambda_{ll}\lambda_{all}\lambda_a-\frac{4}{5}\lambda_{ll}tr\lambda_{ab}^2+\frac{8}{75}\lambda_{ll}^3-4\lambda_{ab}\lambda_a\lambda_{bll}+\frac{4}{3}tr\lambda_{ab}^3\,,\\
X_7 &= \frac{8}{15}(tr\lambda_{ab}^3)\lambda_{ll}-\frac{14}{25}\lambda_{ll}^2tr\lambda_{ab}^2+\frac{46}{375}\lambda_{ll}^4+4\lambda\lambda_{ab}\lambda_{all}\lambda_{bll}\\
& +2(tr\lambda_{ab}^2)\lambda_c\lambda_{cll}-(\lambda_a\lambda_{all})^2-\frac{12}{5}\lambda\lambda_{ll}\lambda_{all}\lambda_{all}\\
& +(\lambda_a\lambda_a)(\lambda_{bll}\lambda_{bll})-4\lambda_{ab}^2\lambda_{all}\lambda_b\\
& -8\lambda_{ppll}\left(\lambda tr\lambda_{ab}^2-\frac{1}{2}\lambda_{ab}\lambda_a\lambda_b-\frac{11}{25}\lambda\lambda_{ll}^2+\frac{3}{10}\lambda_{ll}\lambda_a\lambda_a\right)\\
& +\frac{12}{5}\lambda_{ll}\lambda_{ab}\lambda_a\lambda_{bll}-\frac{22}{25}\lambda_{ll}^2\lambda_a\lambda_{all}\,,\\
X_8 &= -\frac{34}{25}\lambda_{ll}^2\lambda_{ab}\lambda_a\lambda_{bll}+2(tr\lambda_{ab}^2)\lambda_{cd}\lambda_c\lambda_{dll}+\frac{16}{5}\lambda_{ll}\lambda_{ab}^2\lambda_a\lambda_{bll}\\
& +\frac{148}{375}\lambda_{ll}^3\lambda_a\lambda_{all}-\frac{8}{5}\lambda_{ll}(tr\lambda_{ab}^2)\lambda_c\lambda_{cll}+\frac{4}{3}(tr\lambda_{ab}^3)\lambda_c\lambda_{cll}-4\lambda_{ab}^3\lambda_a\lambda_{bll}\\
& +2\lambda_{ppll}\left(2\lambda_{ab}^2\lambda_a\lambda_b-(tr\lambda_{cd}^2)\lambda_a\lambda_a-\frac{8}{5}\lambda_{ll}\lambda_{ab}\lambda_a\lambda_b+\frac{17}{25}\lambda_{ll}^2\lambda_a\lambda_a\right)\\
& +(\lambda_{ab}\lambda_a\lambda_b)(\lambda_{cll}\lambda_{cll})-\frac{4}{5}\lambda_{ll}(\lambda_a\lambda_a)(\lambda_{bll}\lambda_{bll})-2(\lambda_a\lambda_{all})(\lambda_{bc}\lambda_b\lambda_{cll})\\
& +\frac{4}{5}\lambda_{ll}(\lambda_a\lambda_{all})^2+(\lambda_a\lambda_a)(\lambda_{bc}\lambda_{bll}\lambda_{cll})\\
& +4\lambda\lambda_{ab}^2\lambda_{all}\lambda_{bll}-2\lambda tr\lambda_{ab}^2\lambda_{cll}\lambda_{cll}-\frac{16}{5}\lambda\lambda_{ll}\lambda_{ab}\lambda_{all}\lambda_{bll}\\
& +\frac{34}{25}\lambda\lambda_{ll}^2\lambda_{all}\lambda_{all}+16\lambda\lambda_{ppll}\left(-\frac{37}{375}\lambda_{ll}^3+\frac{2}{5}\lambda_{ll}(tr\lambda_{ab}^2)-\frac{1}{3}tr\lambda_{ab}^3\right)\\
& +\frac{4}{75}\lambda_{ll}^2(tr\lambda_{ab}^3)-\frac{8}{125}\lambda_{ll}^3(tr\lambda_{ab}^2)+\frac{4}{15}\cdot\frac{37}{625}\lambda_{ll}^5\,.
\end{aligned}
$$

It is interesting that ϕ'^k has been determined except for 4 scalar functions H_0, H_1, H_2, H_3. Instead of this, if we use only the representation theorems without imposing the entropy principle and the Galilean invariance, we obtain that ϕ'^k depends on 6 arbitrary scalar functions, being a linear combination of λ_{kll}, $\lambda_{kh}\lambda_{hll}$, $\lambda_{ka}^2\lambda_{all}$, λ_k, $\lambda_{ka}\lambda_a$, $\lambda_{ka}^2\lambda_a$. We note that also h' is determined in terms of H_0,H_1, H_2, H_3. These are arbitrary functions of the 8 scalars X_1-X_8.

Instead of this, if we use only the representation theorems without imposing the entropy principle and the Galilean invariance, we obtain that all the scalar functions are arbitrary functions of the following 14 scalars λ_{ll}, $tr\lambda^2_{ab}$, $tr\lambda^3_{ab}$, $\lambda_{all}\lambda_{all}$, $\lambda_{all}\lambda_a$, $\lambda_a\lambda_a$, $\lambda_{ab}\lambda_{all}\lambda_{bll}$, $\lambda_{ab}\lambda_a\lambda_{bll}$, $\lambda_{ab}\lambda_a\lambda_b$, $\lambda^2_{ab}\lambda_{all}\lambda_{bll}$, $\lambda^2_{ab}\lambda_a\lambda_{bll}$, $\lambda^2_{ab}\lambda_a\lambda_b$, λ_{ppll}, λ.

It is useful now to verify these results: To this end we can substitute equations (5), (6) and (7) into (4) and obtain that they are identically satisfied. The corresponding calculations are long, so it will be useful to subdivide them with the following steps.

— Firstly we can verify that $(4)_1$ is satisfied with X_i instead of h', for $i = 1, \cdots, 8$; consequently, for the theorem on derivation of composite functions, it will be satisfied by whatever function of X_i, as h' is. But we have to note that, if we simplify X_8 through the Hamilton-Cayley Theorem

$$\lambda^3_{ab} = \lambda_{ll}\lambda^2_{ab} + \frac{1}{2}\left(tr\lambda^2_{cd} - \lambda^2_{ll}\right)\lambda_{ab} + \left(\frac{1}{3}tr\lambda^3_{cd} - \frac{1}{2}\lambda_{ll}tr\lambda^2_{cd} + \frac{1}{6}\lambda^3_{ll}\right)\delta_{ab}$$

it will become more complicated to verify that X_8 is a solution, because it will be necessary also to use the identity

$$\begin{aligned} 0 \;=\; & \delta_{ij}\left(-\lambda^2_{ab}\lambda_{all}\lambda_{bll} + \lambda_{ll}\lambda_{ab}\lambda_{all}\lambda_{bll} + \frac{1}{2}\lambda_{all}\lambda_{all}tr\lambda^2_{cd} - \frac{1}{2}\lambda_{all}\lambda_{all}\lambda^2_{ll}\right) \\ & +\lambda_{ij}\left(-\lambda_{ab}\lambda_{all}\lambda_{bll} + \lambda_{ll}\lambda_{all}\lambda_{all}\right) - \lambda^2_{ij}\lambda_{all}\lambda_{all} + \lambda_{ill}\lambda_{jll}\frac{1}{2}(\lambda^2_{ll} - tr\lambda^2_{cd}) \\ & -2\lambda_{ll}\lambda_{ll(i}\lambda_{j)b}\lambda_{bll} + \lambda_{ia}\lambda_{all}\lambda_{jb}\lambda_{bll} + 2\lambda_{ll(i}\lambda^2_{j)b}\lambda_{bll} \end{aligned}$$

which can be easily proved in the reference frame where λ_{ab} has the diagonal form.

— The second step is to verify that $(4)_2$ is satisfied in the case $H_0 = 1$, $H_1 = 0$, $H_2 = 0$, $H_3 = 0$. Similarly for the case $H_0 = 0$, $H_1 = 1$, $H_2 = 0$, $H_3 = 0$; then for the case $H_0 = 0$, $H_1 = 0$, $H_2 = 1$, $H_3 = 0$ and, lastly, for the case $H_0 = 0$, $H_1 = 0$, $H_2 = 0$, $H_3 = 1$. Consequently, it will be satisfied for all constant values of H_0, H_1, H_2, H_3. After that it is satisfied also in the general case: In fact, for the property on derivation of a product, the terms where H_0, H_1, H_2, H_3 are not differentiated will simplify, for the above reasons, and it remains

$$\sum_{j=0}^{3} V^k_j \cdot \sum_{i=1}^{8} \frac{\partial H_j}{\partial X_i}$$

for the right hand side of $(4)_1$ written with X_i instead of h'; the result is zero which was already verified in the first step.

We note that our exact closure (2) is expressed in terms of the parameters λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} (called Lagrange multipliers). So equations (1) give these parameters as functions of position and time; after that, equations (2) give the moments as functions of position and time. Another possibility is that to use equations $(2)_{1-5}$ to express the Lagrange multipliers in terms of the moments on their left hand sides; in this way equations (1), by using also equations $(2)_{6-10}$, give directly the moments as functions of position and time.

However, the inversion of relations $(2)_{1-5}$ can be done only through an expansion procedure; in fact, substitution of (5)-(7) into $(2)_{1-5}$ gives equations which are very difficult to invert.

This fact shows again the appropriateness of the Lagrange multipliers as independent variables: they are not only the variables through which the system (1) takes the symmetric hyperbolic form, but also allow an exact closure without use of expansions, at least for the case considered in the present paper.

For the sake of completeness, let us recall how the inversion of $(2)_{1-5}$ operates through an expansion procedure: Let us write equations $(2)_{1-5}$ at order i with respect to thermodynamical equilibrium, so it becomes a linear system whose unknowns are the Lagrange multipliers at order i and whose known terms are functions of the moments at order i and of the Lagrange multipliers at orders from 0 to $i-1$; the matrix of coefficients does not depend on i, so it is the same of the order $i=0$ and the unknowns can be obtained, without imposing further conditions.

We conclude this section saying that the present one is a macroscopic approach, i.e., based on the entropy principle. Another possible formulation of extended thermodynamics for dense gases is based on the kinetic approach; in order to permit a comparison between the two approaches, we cite now some references on the kinetic approach. In [7] Enskog introduced a kinetic theory for dense gases which yields a very good approximate of the behavior of gases. Later, hydrodynamical like equations have been derived from the kinetic equation; see, for example, the Chapman-Enskog method [8].

In 1988, Kremer and Rosa [9] obtained hydrodynamic equations from the local equilibrium distribution function as kernel linearizing the collision integral in Enskog's equation; in this way, they were able to derive sound dispersion relations for monoatomic gases by using normal mode analysis. Basing on this last paper, in 1991 Marques and Kremer [10] obtained linearized hydrodynamic equations involving the second order terms of the collision integral; in this way they improved the results previously known in literature and, furthermore, they obtained linearized Burnett equations for monoatomic gases.

In [11] Ugawa and Cordero obtained extended hydrodynamic equations derived from Enskog's equation by using Grad's moment expansion method in the bi-dimensional case; among other results, they discussed the nature of a simple one-dimensional heat conduction problem and were able to show that, not too far from equilibrium, the nonequilibrium pressure in this case depends on the density, temperature and heat flux vector.

We want also to say that, with the present work, we do not aim to exhaust everything that can be said about dense gases and macromolecular fluids; see, for example, [12] for other aspects whose relation with the present results can be investigated in the future. We aim simply to give more light to usual procedures previously used in extended thermodynamics, such as Taylor expansions, hyperbolicity, and the problem of subsystems. We hope also to stimulate further discussions on this subject, in the future.

2. The Principles of Entropy and of Galilean Relativity

We want that our system (1) is symmetric and hyperbolic, with all the consequent nice mathematical properties. To this end we impose that all the solutions of equations (1) satisfy also the entropy inequality

$$\partial_t h + \partial_k \phi_k = \sigma \geq 0.$$

For Liu' s Theorem [13] this is equivalent to assume the existence of Lagrange multipliers λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} such that

$$\begin{aligned} dh &= \lambda dF + \lambda_i dF_i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill} + \lambda_{iill} dF^{iill}\,, \\ d\phi_k &= \lambda dF_k + \lambda_i dG_{ik} + \lambda_{ij} dG_{ijk} + \lambda_{ill} dG_{illk} + \lambda_{iill} dG_{iillk}, \end{aligned} \tag{8}$$

besides a residual inequality which we leave for the sake of brevity. The Lagrange multipliers are also called "main field" (see [14]). Let us now impose the Galilean relativity principle by considering the following change of independent variables

$$F_{i_1 i_2 \ldots i_n} = \sum_{k=0}^{n} \binom{n}{k} m_{(i_1 i_2 \ldots i_k} v_{i_{k+1} \ldots i_n)} \tag{9}$$

which can be found in [14], [21] and that, applied to our case, becomes

$$\begin{aligned} F &= m, \qquad (10) \\ F_i &= m v_i + m_i, \\ F_{ij} &= m v_i v_j + m_{ij} + 2 m_{(i} v_{j)}\,, \\ F_{ill} &= m_{ill} + m_{ll} v_i + 2 m_{il} v_l + m v^2 v_i + m_i v^2 + 2 m_l v_i v_l\,, \end{aligned}$$

$$F_{iill} = m_{iill} + mv^4 + 4m_i v_i v^2 + 2m_{ii} v^2 + 4m_{il} v_i v_l + 4m_{iil} v_l.$$

Also in [14], [21] we can find how change the constitutive functions $G_{ki_1\cdots i_n}$ when the reference frame changes, i.e.,

$$H_{ki_1 i_2\cdots i_n} = \sum_{j=0}^{n} \binom{n}{j} M_{k(i_1\cdots i_j} v_{i_{j+1}\cdots i_n)}, \tag{11}$$

where the functions $H_{ki_1\cdots i_n}$ are defined by

$$G_{ki_1\cdots i_n} = v_k F_{i_1\cdots i_n} + H_{ki_1\cdots i_n}. \tag{12}$$

It is interesting that (11) looks like (9), except that they do not act on the index k. In our particular case, equations (11) become

$$\begin{aligned}
H_k &= M_k\,, \\
H_{ki} &= M_k v_i + M_{ki}\,, \\
H_{kij} &= M_k v_i v_j + 2M_{k(i} v_{j)} + M_{kij}\,, \\
H_{kill} &= M_k v_i v^2 + M_{ki} v^2 + 2M_{kl} v_i v_l + 2M_{kil} v_l + M_{kll} v_i + M_{kill}\,, \\
H_{kiill} &= M_k v^4 + 4M_{ki} v_i v^2 + 2M_{kii} v^2 + 4M_{kil} v_l v_i + 4M_{kiil} v_l \\
&+ M_{kiill}.
\end{aligned} \tag{13}$$

Consequently, the functions $G_{ki_1\cdots i_n}$ transform as follows

$$\begin{aligned}
G_{ki} &= m v_i v_k + m_i v_k + M_k v_i + M_{ki}\,, \\
G_{kij} &= m v_i v_j v_k + m_{ij} v_k + 2m_{(i} v_{j)} v_k + M_k v_i v_j + 2M_{k(i} v_{j)} + M_{kij}\,, \\
G_{kill} &= m_{ill} v_k + m_{ll} v_i v_k + 2m_{il} v_l v_k + mv^2 v_i v_k + m_i v^2 v_k \\
&+ 2m_l v_i v_l v_k + M_{ki} v^2 + 2M_{kl} v_i v_l + M_{kll} v_i + 2M_{kil} v_l \\
&+ M_k v_i v^2 + M_{kill}, \\
G_{kiillkk} &= m_{iill} v_k + mv^4 v_k + 4m_i v_i v^2 v_k + 2m_{ii} v^2 v_k + 4m_{il} v_i v_l v_k \\
&+ 4m_{iil} v_l v_k + M_k v^4 + 4M_{ki} v_i v^2 + 2M_{kii} v^2 + 4M_{kil} v_i v_l \\
&+ 4M_{kiil} v_l + M_{kiill}.
\end{aligned} \tag{14}$$

We note that from (12) and (11), for $n = 0$, and from $G_k = F_k$ it follows $M_k = F_k - Fv_k$. This and $(10)_{1,2}$ yield $M_k = m_k$. The new variables m, m_i, m_{ij}, m_{ill}, m_{iill} and M_i, M_{ij}, M_{kij}, M_{kill}, M_{kiill} have the same symmetries of $F_{i_1\ldots i_n}$ and $G_{ki_1\ldots i_n}$.

Let us now substitute into equations (8) the expressions which we have above found for the variables and the constitutive functions. In this way equations (8) become

$$\begin{aligned}
dh &= \lambda^I dm + \lambda^I_i dm_i + \lambda^I_{ij} dm_{ij} + \lambda^I_{ill} dm_{ill} \\
&+ \lambda^I_{iill} dm_{iill} + (\lambda^I_i m + 2\lambda^I_{ij} m_j + \lambda^I_{jll} m_{ll} \delta_{ij}
\end{aligned}$$

$$
\begin{aligned}
& + \quad 2\lambda^I_{jll} m_{ij} + 4\lambda^I_{ppqq} m_{ill}) dv_i \qquad (15) \\
d(\phi_k) &= (\lambda^I dm + \lambda^I_i dm_i + \lambda^I_{ij} dm_{ij} + \lambda^I_{ill} dm_{ill} \\
&+ \lambda^I_{iill} dm_{iill}) v_k + \lambda^I dM_k + \lambda^I_i dM_{ki} + \lambda^I_{il} dM_{kil} \\
&+ \lambda^I_{ill} dM_{kill} + \lambda^I_{iill} dM_{kiill} + (\lambda^I m + \lambda^I_i mi + \lambda^I_{ij} m_{ij} \\
&+ \lambda^I_{ill} m_{ill} + \lambda^I_{ppqq} m_{iill}) dv_k + (\lambda^I_i M_k + 2\lambda^I_{ij} M_{kj} \\
&+ \lambda^I_{ipp} M_{kll} + 2\lambda^I_{lpp} M_{kil} + 4\lambda^I_{ppqq} M_{kill}) dv_i, \qquad (16)
\end{aligned}
$$

with

$$
\begin{aligned}
\lambda^I &= \lambda + \lambda_i v_i + \lambda_{ij} v_i v_j + \lambda_{ill} v_i v^2 + \lambda_{ppqq} v^4 \,, \\
\lambda^I_i &= \lambda_i + 2\lambda_{ij} v_j + \lambda_{ill} v^2 + 2\lambda_{jpp} v_i v_j + 4\lambda_{ppqq} v_i v^2 \,, \\
\lambda^I_{ij} &= \lambda_{ij} + \lambda_{hpp} v_h \delta_{ij} + 2\lambda_{pp(i} v_{j)} + 4\lambda_{ppqq} v_j v_i + 2\lambda_{hhpp} v^2 \delta_{ij} \,, \\
\lambda_{ill} &= \lambda_{ipp} + 4\lambda_{hhpp} v_i \\
\lambda^I_{iill} &= \lambda_{ppqq}.
\end{aligned}
$$

The Galilean relativity principle imposes that h, $\phi_k - hv_k$, M_{ki}, M_{kij}, M_{kill}, M_i do not depend on v_i. For this condition on h and $\phi_k - hv_k$ we obtain

$$
\begin{aligned}
\frac{\partial h}{\partial v_i} = 0 &= m\lambda^I_i + 2\lambda^I_{ij} m_j + \lambda^I_{jpp}(m_{ll}\delta_{ij} + 2m_{ij}) \\
&+ 4\lambda^I_{ppqq} m_{ill}, \\
\frac{\partial(\phi^k - hv_k)}{\partial v_i} = 0 &= M_k \lambda^I_i + 2M_{kj}\lambda^I_{ij} + M_{kll}\lambda^I_{ipp} + 2M_{kil}\lambda^I_{lpp} \\
&+ 4\lambda^I_{ppqq} M_{kill} + h'\delta_{ik} \,, \qquad (17)
\end{aligned}
$$

where h' is defined by

$$
h' = \lambda^I m + \lambda^I_i m_i + \lambda^I_{il} m_{il} + \lambda^I_{ill} m_{ill} + \lambda^I_{iill} m_{iill} - h;
$$

In this way equations (15) and (16) become respectively

$$
\begin{aligned}
dh^I &= \lambda^I dm + \lambda^I_i dm_i + \lambda^I_{il} dm_{il} + \lambda^I_{ill} dm_{ill} + \lambda^I_{iill} dm_{iill}, \qquad (18) \\
d\phi^I_k &= \lambda^I dM_k + \lambda^I_i dM_{ki} + \lambda^I_{il} dM_{kil} + \lambda^I_{ill} dM_{kill} + \lambda^I_{ppqq} dM_{kiill},
\end{aligned}
$$

with $\phi^I_k = \phi_k - hv_k$. Let us also define

$$
\phi'_k = \lambda^I M_k + \lambda^I_i M_{ki} + \lambda^I_{il} M_{kil} + \lambda^I_{ill} M_{kill} + \lambda^I_{ppqq} M_{kiill} - \phi^I_k \,;
$$

so that equations (18) become

$$
\begin{aligned}
dh' &= md\lambda^I + m_i d\lambda^I_i + m_{il} d\lambda^I_{il} + m_{ill} d\lambda^I_{ill} + m_{iill} d\lambda^I_{iill}, \\
d\phi'_k &= M_k d\lambda^I + M_{ki} d\lambda^I_i + M_{kil} d\lambda^I_{il} + M_{kill} d\lambda^I_{ill} + M_{kiill} d\lambda^I_{ppqq} \,, \qquad (19)
\end{aligned}
$$

from which, by taking λ^I, λ^I_i, λ^I_{ij}, λ^I_{ill}, λ^I_{iill} as independent variables and, by taking the derivatives with respect to the various components of the main field, it follows

$$\begin{aligned} m &= \frac{\partial h'}{\partial \lambda^I}, \quad m_i = \frac{\partial h'}{\partial \lambda^I_i}, \quad m_{il} = \frac{\partial h'}{\partial \lambda^I_{il}}, \\ m_{ill} &= \frac{\partial h'}{\partial \lambda^I_{ill}}, \quad m_{iill} = \frac{\partial h'}{\partial \lambda^I_{iill}}, \\ M_k &= \frac{\partial \phi'_k}{\partial \lambda^I}, \quad M_{ki} = \frac{\partial \phi'_k}{\partial \lambda^I_i}, \quad M_{kil} = \frac{\partial \phi'_k}{\partial \lambda^I_{il}}, \\ M_{kill} &= \frac{\partial \phi'_k}{\partial \lambda^I_{ill}}, \quad M_{kiill} = \frac{\partial \phi'_k}{\partial \lambda^I_{iill}}. \end{aligned} \tag{20}$$

These last equations are nothing more than (2), but in the new reference frame. By substituting in (17) from (20) we obtain equations whose expression, in the previous reference frame, are (4). Consequently, (2) and (4) are equivalent to the principles of entropy and of Galilean relativity. Other interesting aspects can be found in [19], which are adapted for the present case in [4] and [5]. In order to find the general solution of (4), let us write a relativistic counterpart of our equations.

3. Relativistic Extended Thermodynamics for Dense Gases and Macromolecular Fluids

According to the idea explained in the introduction, let us firstly consider a relativistic model for which it is easy to impose the Einsteinian relativity principle and which will have equations (1) as non relativistic limit. This result can be achieved with the balance equations

$$\partial_\alpha T^{\alpha\beta} = 0 \quad , \quad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma} \, , \tag{21}$$

where $T^{\alpha\beta}$ is not symmetric, while $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$ are symmetric only with respect to the indexes $\beta\gamma$. This system is the generalization of that introduced in [15]. The first of these is the conservation law of momentum-energy, while the trace of the second one is the conservation law of mass, so that

$$I^{\beta\gamma} g_{\beta\gamma} = 0 \, . \tag{22}$$

Now let us briefly resume the usual procedure of extended thermodynamics. The entropy principle is expressed in terms of h^α, called (entropy density – entropy flux density) tensor, such that

$$\partial_\alpha h^\alpha = \sigma \geq 0 \tag{23}$$

for every solution of equations (21).

For Liu's Theorem [13] equation (23) is equivalent to assuming the existence of the Lagrange multipliers $\lambda_\beta,\ \lambda_{\beta\gamma}$ such that

$$\partial_\alpha h^\alpha - \sigma - \lambda_\beta \partial_\alpha T^{\alpha\beta} - \lambda_{\beta\gamma}(\partial_\alpha A^{\alpha\beta\gamma} - I^{\beta\gamma}) = 0 \tag{24}$$

for every value of the independent variables. By differentiating it becomes

$$dh^\alpha = \lambda_\beta dT^{\alpha\beta} + \lambda_{\beta\gamma} dA^{\alpha\beta\gamma}\,; \quad -\sigma + \lambda_{\beta\gamma} I^{\beta\gamma} = 0. \tag{25}$$

If we define h'^α by

$$h^\alpha = -h'^\alpha + \lambda_\beta T^{\alpha\beta} + \lambda_{\beta\gamma} A^{\alpha\beta\gamma}, \tag{26}$$

then equation (25) can be rewritten as

$$dh'^\alpha = \ T^{\alpha\beta} d\lambda_\beta + \ A^{\alpha\beta\gamma} d\lambda_{\beta\gamma}. \tag{27}$$

This last equation, by taking $\lambda_\beta,\ \lambda_{\beta\gamma}$ as independent variables, becomes

$$T^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \lambda_\beta}, \quad A^{\alpha\beta\gamma} = \frac{\partial h'^\alpha}{\partial \lambda_{\beta\gamma}}. \tag{28}$$

It follows that from the knowledge of h'^α we obtain $T^{\alpha\beta}$ and $A^{\alpha\beta\gamma}$; we have only a condition on h'^α: it has to satisfy the Einsteinian relativity principle. For well known representation theorems as [18] and [20], we have that

$$h'^\alpha = \ h_0 \lambda^\alpha + h_1 \lambda^{\alpha\gamma}\lambda_\gamma + h_2 \overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma + \ h_3 \overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma, \tag{29}$$

where the following definitions have been used

$$\overset{2}{\lambda}{}^{\alpha\gamma} = \lambda^{\alpha\beta}\lambda_{\beta\delta} g^{\delta\gamma}, \quad \overset{3}{\lambda}{}^{\alpha\gamma} = \lambda^{\alpha\beta}\lambda_{\beta\delta}\lambda^{\delta\gamma}$$

with h_i scalar functions depending on

$$Q_1 = \lambda^\beta_\beta, \quad Q_2 = \overset{2}{\lambda}{}^{\alpha\gamma} g_{\alpha\gamma}, \quad Q_3 = \overset{3}{\lambda}{}^{\alpha\gamma} g_{\alpha\gamma}, \quad Q_4 = \overset{4}{\lambda}{}^{\alpha\gamma} g_{\alpha\gamma}, \tag{30}$$

$$P_0 = \lambda_\beta \lambda^\beta, \quad P_1 = \lambda_\beta \lambda_\gamma \lambda^{\beta\gamma}, \quad P_2 = \lambda_\beta \lambda_\gamma \overset{2}{\lambda}{}^{\beta\gamma}, \quad P_3 = \lambda_\beta \lambda_\gamma \overset{3}{\lambda}{}^{\beta\gamma}.$$

4. The Non Relativistic Limit of the Previous Model

In order to take this limit, let us consider a modified procedure of that used in [1], [2] for ideal gases. We assume that this procedure holds also for macromolecular gases; this assumption does not lead to wrong results because we have already verified, at the end of Section 1, that these results are correct. The procedure is subdivided in two parts. In the first of these, we transform the system (21) in another one which has the 3-dimensional form. It is shown how, with this transformation, the entropy principle (23) transforms into the entropy principle for the new system; also the expression of h'^α and of the Lagrange multipliers λ_β and $\lambda_{\beta\gamma}$ are obtained in terms of the corresponding ones for the new system. The pertinent calculations are reported in the first two parts of Appendix A.

However, the non relativistic limit of this new system does not yield an independent set of equations; so we have to consider a linear combination of the equations constituting the new system, before taking the non relativistic limit. This is a second transformation in 3-dimensional form; the coefficients of the linear combination are suggested by the kinetic theory. The details are reported in Parts 3 and 4 of Appendix A. There it is shown how, with this second transformation, the entropy principle (23) transforms into the entropy principle for the last system; also the expression of h'^α and of the Lagrange multipliers λ_β and $\lambda_{\beta\gamma}$ are obtained in terms of the corresponding ones for the last system. The results are equations (31) of the next section. The non relativistic limit of this last system now yields the independent field equations (72); our starting equations (1) are coincident with these if and only if $G^k = F^k$, so that also this condition has to be imposed and this will be done in a future work.

The functions h' and ϕ'^k, which are the corresponding expressions of h'^α in the last system, are also called "potentials" because of relations (2). They will be found in the next section.

5. Determination of the Potentials

With equations (81) and (82) we have found that the functions h' and ϕ'^k for the new system (72) are exactly the same obtained, through the above mentioned passages, from the 4-potential h'^α for the system (21). Consequently, we may obtain them from the result (29) and (30) of the relativistic system. To this end it is useful the relation, obtained in Appendix A, between the relativistic Lagrange multipliers in terms of those for the system (72). They are

$$\lambda^\beta{}_\gamma = \frac{c^2}{m_0^2}\left[\begin{pmatrix} -8\lambda_{ppll} & 0_j \\ 0_i & -4\lambda_{ppll}\delta_{ij}\end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 & -\lambda_{jll} \\ \lambda_{ill} & 0_{ij}\end{pmatrix}\right. \tag{31}$$
$$\left. + \frac{1}{c^2}\begin{pmatrix} \frac{2}{3}\lambda_{ll} & 0_j \\ 0_i & \lambda_{ij} + \frac{1}{3}\lambda_{ll}\delta_{ij}\end{pmatrix} + \frac{1}{c^4}\begin{pmatrix} -\lambda & 0_j \\ 0_i & -\lambda\delta_{ij}\end{pmatrix}\right],$$

$$\lambda^\beta = \frac{c^3}{m_0}\left[\begin{pmatrix} 8\lambda_{ppll} \\ \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ \\ -2\lambda_{ill} \end{pmatrix} + \frac{1}{c^2}\begin{pmatrix} -\frac{2}{3}\lambda_{ll} \\ \\ 0_i \end{pmatrix} + \frac{1}{c^3}\begin{pmatrix} 0 \\ \\ \lambda_i \end{pmatrix}\right] .$$

We evaluate now the 4-vectors occurring in (29) and the scalars (30). The effective calculations are reported in Appendix B, so that we point here only to the adopted strategy. By using the expressions (31) we see that the 4-vectors $\overset{i}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ occurring in equation (29) have not a finite limit for $c \to \infty$. But, in equation (29), the scalar coefficients h_i are arbitrary, so that every invertible linear combination of $\overset{i}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ can replace $\overset{i}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ in the expression (29), without loss of generality. We have only to find linear combinations whose limits are finite and independent. The results are equations (6).

The scalar coefficients H_i are arbitrary functions of $Q_1 - Q_4$ and $P_0 - P_3$; this fact allows us to consider an invertible transformation of $Q_1 - Q_4$, $P_0 - P_3$ into new scalars whose limits are finite and independent. The results are equations (7).

Note. We have obtained the decomposition (31) by considering a modified procedure of that introduced in [1], [2] for ideal gases; instead of this, if we use exactly the procedure of [1], [2], we have to substitute (31) with

$$\begin{aligned} \lambda^\beta{}_\gamma &= \frac{c^2}{m_0^2}\left[\begin{pmatrix} -8\lambda_{ppll} & 0_j \\ 0_i & -4\lambda_{ppll}\delta_{ij} \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 & -\lambda_{jll} \\ \lambda_{ill} & 0_{ij} \end{pmatrix}\right. \\ &+ \left.\frac{1}{c^2}\begin{pmatrix} 0 & 0_j \\ 0_i & \lambda_{ij} \end{pmatrix} + \frac{1}{c^3}\begin{pmatrix} 0 & -\frac{1}{2}\lambda_j \\ \frac{1}{2}\lambda_i & 0_{ij} \end{pmatrix} + \frac{1}{c^4}\begin{pmatrix} -\lambda & 0_j \\ 0_i & 0_{ij} \end{pmatrix}\right] , \\ \lambda^\beta &= \frac{c^3}{m_0}\left[\begin{pmatrix} 8\lambda_{ppll} \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ -2\lambda_{ill} \end{pmatrix}\right] . \end{aligned} \tag{32}$$

Another procedure is present in literature (see [7], [17]) also for ideal gases. If we want to follow it, then we have to substitute (31) with

$$\begin{aligned} \lambda^\beta{}_\gamma =& \frac{c^2}{m_0^2}\left[\begin{pmatrix} -3\lambda_{ppll} & 0_j \\ 0_i & \lambda_{ppll}\delta_{ij} \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 & -\lambda_{jll} \\ \lambda_{ill} & 0_{ij} \end{pmatrix} + \frac{1}{c^2}\begin{pmatrix} 0 & 0_j \\ 0_i & \lambda_{<ij>} \end{pmatrix}\right] , \\ \lambda^\beta =& \frac{c^3}{m_0}\left[\begin{pmatrix} 8\lambda_{ppll} \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ -2\lambda_{ill} \end{pmatrix} + \frac{1}{c^2}\begin{pmatrix} -\frac{2}{3}\lambda_{ll} \\ 0_i \end{pmatrix} + \frac{1}{c^3}\begin{pmatrix} 0 \\ \lambda_i \end{pmatrix}\right] , \\ \xi =& c^4\left(5\lambda_{ppll} - \frac{2}{3}\frac{1}{c^2}\lambda_{ll} + \frac{\lambda}{c^4}\right) . \end{aligned} \tag{33}$$

We have performed calculations also with (32) and with (33) instead of (31), but we do not report them for the sake of brevity. The interesting result is that the pertinent polynomials in $1/c$ are different between them and from those here obtained, but the limits for c going to infinity are the same with all 3 approaches!

6. Conclusions

We consider very interesting the results here obtained, because until now nobody found exact solutions in the extended thermodynamics for a 14 moments model. This allows to review the procedures previously used and which were based on Taylor' s expansions. Moreover, it opens the possibility for many other further deepening considerations; besides those already indicated throughout the paper, for example one could try to obtain the same result for the case of many moments, or to extend this procedure to the case with additional symmetry conditions. We hope that this will be a great spur also for other researchers.

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Appendix A: On Two Transformations in 3-Dimensional Form and the Non Relativistic Limit

A.1. A First Transformation in 3-Dimensional Form

Equation $(21)_1$ for $\beta = 0, i$ becomes

$$\frac{1}{c}\partial_t T^{00} + \partial_k T^{k0} = 0\,, \tag{34}$$

$$\frac{1}{c}\partial_t T^{0i} + \partial_k T^{ki} = 0\,. \tag{35}$$

Similarly, equation $(21)_2$ for $\beta\gamma = 00$, $\beta\gamma = 0i$, $\beta\gamma = ij$ becomes

$$\frac{1}{c}\partial_t A^{000} + \partial_k A^{k00} = I^{00}\,,\ \frac{1}{c}\partial_t A^{00i} + \partial_k A^{k0i} = I^{0i}\,,\ \frac{1}{c}\partial_t A^{0ij} + \partial_k A^{kij} = I^{ij}\,. \tag{36}$$

Now we adopt the following change of variables

$$T^{00} = m_0^4 c F_2,\ T^{k0} = m_0^4 G_2^k,\ T^{0i} = m_0^4 F_2^i,\ T^{ki} = \frac{m_0^4}{c} G_2^{ki}, \tag{37}$$

$$A^{000} = m_0^5 c^2 F_3,\ A^{00i} = m_0^5 c F_3^i,\ A^{0ij} = m_0^5 F_3^{ij}, \tag{38}$$

$$A^{k00} = m_0^5 c G_3^k,\ A^{k0i} = m_0^5 G_3^{ki},\ A^{kij} = m_0^5 \frac{1}{c} G_3^{kij}, \tag{39}$$

$$I^{00} = cRm_0^5,\ I^{0i} = Q^i m_0^5,\ I^{ij} = \frac{1}{c} p^{ij} m_0^5 . \tag{40}$$

In their terms equations (34), (35) and (36) become

$$\partial_t F_2 + \partial_k G_2^k = 0, \quad \partial_t F_2^i + \partial_k G_2^{ki} = 0, \quad \partial_t F_3 + \partial_k G_3^k = R, \tag{41}$$

$$\partial_t F_3^i + \partial_k G_3^{ki} = Q^i, \quad \partial_t F_3^{ij} + \partial_k G_3^{kij} = p^{ij} .$$

The last of these can be also subdivided in

$$\partial_t F_3^{ll} + \partial_k G_3^{kll} = p^{ll}, \quad \partial_t F_3^{<ij>} + \partial_k G_3^{k<ij>} = p^{<ij>} , \tag{42}$$

while equation (23) can be subdivided in

$$\frac{1}{c}\partial_t h^0 + \partial_k h^k = \sigma . \tag{43}$$

With the following changes of names

$$h^0 = m_0^3 h, \qquad h^k = \frac{m_0^3}{c}\phi^\kappa, \qquad \sigma = \frac{m_0^3}{c}\sigma^\star, \tag{44}$$

equation (43) becomes

$$\partial_t h + \partial_k \phi^k = \sigma^\star . \tag{45}$$

A.2. A First Transformation of the Lagrange Multipliers

Equation $(25)_1$ for $\alpha = 0$ becomes

$$dh^0 = \lambda_0 dT^{00} + \lambda_i dT^{0i} + \lambda_{00} dA^{000} + 2\lambda_{0i} dA^{00i} + \lambda_{ij} dA^{0ij} . \tag{46}$$

By using equation $(44)_1$ and (37)-(39), it transforms into

$$\begin{aligned} d(m_0^3 h) &= \lambda_0 d(m_0^4 c F_2) + \lambda_i d(m_0^4 F_2^i) + \lambda_{00} d(m_0^5 c^2 F_3) \\ &+ 2\lambda_{0i} d(m_0^5 c F_3^i) + \lambda_{ij} d(m_0^5 F_3^{ij}) \end{aligned} \tag{47}$$

from which

$$\begin{aligned} dh &= \lambda_0 m_0 c d(F_2) + \lambda_i m_0 d(F_2^i) + \lambda_{00} m_0^2 c^2 d(F_3) \\ &+ 2\lambda_{0i} m_0^2 c d(F_3^i) + \lambda_{ij} m_0^2 d(F_3^{ij}) \\ &= \ell dF_2 + \ell_i dF_2^i + \eta dF_3 + \mu_i dF_3^i + \mu_{ij} dF_3^{ij} , \end{aligned} \tag{48}$$

with

$$\ell = \lambda_0 m_0 c\,,\ \ell_i = \lambda_i m_0\,,\ \eta = \lambda_{00} m_0^2 c^2\,,\ \mu_i = 2\lambda_{0i} m_0^2 c\,,\ \mu_{ij} = \lambda_{ij} m_0^2\,. \tag{49}$$

Similarly, equation $(25)_1$ for $\alpha = k$ becomes

$$dh^k = \lambda_0 dT^{k0} + \lambda_i dT^{ki} + \lambda_{00} dA^{k00} + 2\lambda_{0i} dA^{k0i} + \lambda_{ij} dA^{kij}. \tag{50}$$

By using equation (44) and (37), (39), it transforms into

$$\begin{aligned} d(\frac{m_0^3}{c}\phi^k) &= \lambda_0 d(m_0^4 G_2^k) + \lambda_i d(\frac{m_0^4}{c} G_2^{ki}) + \lambda_{00} d(m_0^5 c G_3^k) \\ &+ 2\lambda_{0i} d(m_0^5 G_3^{ki}) + \lambda_{ij} d(m_0^5 \frac{1}{c} G_3^{kij}) \end{aligned}$$

which can be rewritten also as

$$\begin{aligned} d\phi^k &= \lambda_0 c m_0 d(G_2^k) + \lambda_i m_0 d(G_2^{ki}) + \lambda_{00} c^2 m_0^2 d(G_3^k) \\ &+ 2\lambda_{0i} c m_0^2 d(G_3^{ki}) + \lambda_{ij} m_0^2 d(G_3^{kij}) \\ &= \ell dG_2^k + \ell_i dG_2^{ki} + \eta dG_3^k + \mu_i dG_3^{ki} + \mu_{ij} dG_3^{kij}\,, \end{aligned} \tag{51}$$

where we have used (49). The equations (48) and (51) represent the entropy principle for the system (41); moreover, equation (49) gives the transformation of the Lagrange multipliers.

Let us now find the transformation of h'^α; equation (26) for $\alpha = 0$ is

$$h^0 = -h'^0 + \lambda_0 T^{00} + \lambda_i T^{0i} + \lambda_{00} A^{000} + 2\lambda_{i0} A^{0i0} + \lambda_{ij} A^{0ij}\,. \tag{52}$$

By using equations $(44)_1$, (49) and (37)-(39), equation (52) becomes

$$m_0^3 h = -h'^0 + \ell m_0^3 F_2 + \ell_i m_0^3 F_2^i + \eta m_0^3 F_3 + \mu_i m_0^3 F_3^i + \mu_{ij} m_0^3 F_3^{ij}. \tag{53}$$

This can be rewritten as

$$h = \frac{-h'^0}{m_0^3} + \ell F_2 + \ell_i F_2^i + \eta F_3 + \mu_i F_3^i + \mu_{ij} F_3^{ij}, \tag{54}$$

or

$$h = -h' + \ell F_2 + \ell_i F_2^i + \eta F_3 + \mu_i F_3^i + \mu_{ij} F_3^{ij}, \tag{55}$$

where we have defined

$$h'^0 = m_0^3 h'\,. \tag{56}$$

In this way we have found the counterpart of $(44)_1$ for h'^0. Let us now find the transformation of h'^α for $\alpha = k$; from equation (26) we find

$$h^k = -h'^k + \lambda_0 T^{k0} + \lambda_i T^{ki} + \lambda_{00} A^{k00} + 2\lambda_{i0} A^{ki0} + \lambda_{ij} A^{kij}. \tag{57}$$

This, by using again equations $(44)_2$, (49) and (37)-(39), can be written as

$$\frac{m_0^3}{c}\phi^k = -h'^k + \frac{\ell}{m_0 c} m_0^4 G_2^k + \frac{\ell_i}{m_0}\frac{m_0^4}{c} G_2^{ki}$$

$$+ \quad \frac{\eta}{m_0^2 c^2} m_0^5 c G_3^k + \frac{\mu_i}{m_0^2 c} m_0^5 G_3^{ki} + \frac{\mu_{ij}}{m_0^2} \frac{m_0^5}{c} G_3^{kij},$$

or

$$\phi^k = -\phi'^k + \ell G_2^k + \ell_i G_2^{ki} + \eta G_3^k + \mu_i G_3^{ki} + \mu_{ij} G_3^{kij}, \tag{58}$$

with

$$h'^k = \frac{m_0^3}{c} \phi'^k. \tag{59}$$

Equations (56) and (59) are the counterparts of $(44)_{1,2}$ for h'^0 and h'^k. Equations (55) and (58) are the counterpart of equation (26) for the system (41). Let us finish by considering the counterpart of mass conservation (22), that is $-I^{00} + I^{ij}\delta_{ij} = 0$ or, by use of (40), $-\, cRm_0^5 + \frac{1}{c} p^{ij} m_0^5 \delta_{ij} = 0$. In other words,

$$R = \frac{1}{c^2} p^{ij} \delta_{ij} \,. \tag{60}$$

A.3. The Second Transformation in 3-Dimensional Form

A.3.1. Suggestions from the Kinetic Theory for Ideal Gases

For ideal gases the variables $F_2, F_2^i, F_3, F_3^i, F_3^{ij}$ have counterparts in statistical mechanics, where they are defined as moments of the distribution function $\widetilde{f}$, i.e., by means of the following integrals

$$F_2 \;=\; \int \tilde{f} \gamma^6 d\mathbf{u} \,, \quad F_2^i = \int \tilde{f} \gamma^6 u^i d\mathbf{u} \,, \tag{61}$$
$$F_3 \;=\; \int \tilde{f} \gamma^7 d\mathbf{u} \,, \quad F_3^i = \int \tilde{f} \gamma^7 u^i d\mathbf{u} \,, \quad F_3^{ij} = \int \tilde{f} \gamma^7 u^i u^j d\mathbf{u} \,,$$

where γ is the Lorentz factor $(1 - \frac{u^2}{c^2})^{-\frac{1}{2}}$.

If we take the limits of these expressions for $c \to \infty$ we obtain $F_2 = F_3$, $F_2^i = F_3^i$ so that they will be no more independent variables; to avoid this problem, we take suitable invertible linear combinations of the equations before taking the limits.

In particular, as first equation we take the linear combination of $(41)_3$ and $(42)_1$ trough the coefficients 1 and $\frac{-1}{c^2}$, respectively; so we obtain the following equation $(72)_1$ which is the *conservation law of mass* with

$$F = F_3 - \frac{1}{c^2} F_3^{ll} \,, \quad G^k = G_3^k - \frac{1}{c^2} G_3^{kll} \,; \tag{62}$$

moreover, we have taken into account equation (60).

As second equation we take $(41)_2$ which can be written as the following equation $(72)_2$ which is the *conservation law of momentum* with

$$F^i = F_2^i\,, \quad G^{ki} = G_2^{ki}. \tag{63}$$

As third equation we take the linear combination of $(41)_1$, $(41)_3$, $(42)_1$ trough the coefficients $2c^2$, $-2c^2$, 2 respectively; so it becomes

$$\partial_t F^{ll} + \partial_k G^{kll} = 0 \tag{64}$$

which is the *conservation law of energy*, with

$$F^{ll} = 2c^2(F_2 - F_3 + \frac{1}{c^2}F_3^{ll})\,, \quad G^{kll} = 2c^2(G_2^k - G_3^k + \frac{1}{c^2}G_3^{kll})\,. \tag{65}$$

As fourth equation we take $(42)_2$ which we write as

$$\partial_t F^{<ij>} + \partial_k G^{k<ij>} = p^{<ij>}, \tag{66}$$

where

$$F^{<ij>} = F_3^{<ij>}\,, \quad G^{k<ij>} = G_3^{k<ij>}\,. \tag{67}$$

We transform furtherly equation (66) adding to it equation (64) multiplied by $\frac{\delta^{ij}}{3}$ and obtaining the following equation $(72)_3$ with

$$F^{ij} = F_3^{ij} + \frac{c^2\delta^{ij}}{3}(2F_2 - 2F_3 + \frac{F_3^{ll}}{c^2}), \tag{68}$$

$$G^{kij} = G_3^{kij} + \frac{c^2\delta^{ij}}{3}(2G_2^k - 2G_3^k + \frac{G_3^{kll}}{c^2}). \tag{69}$$

Equation $(72)_3$ encloses both (64) and (66). As other equation we take the linear combination of $(41)_4$ and $(41)_2$ trough the coefficients $2c^2$ and $-2c^2$, respectively; so we obtain the following equation $(72)_4$ with

$$F^{ill} = 2c^2(F_3^i - F_2^i)\,,\, G^{kill} = 2c^2(G_3^{ki} - G_2^{ki})\,,\, p^{ill} = 2c^2 Q^i\,. \tag{70}$$

Finally, as last equation we take the linear combination of $(41)_1$, $(41)_3$, $(42)_1$ trough the coefficients $-8c^4$, $+8c^4$, $-4c^2$ respectively; so we obtain the following equation $(72)_5$ with

$$\begin{aligned} F^{iill} &= -8c^4F_2 + 8c^4F_3 - 4c^2F_3^{ll}, \\ G^{kiill} &= -8c^4G_2^k + 8c^4G_3^k - 4c^2G_3^{kll}, \\ p^{iill} = 8c^4R - 4c^2p^{ll} &= 8c^2p^{ll} - 4c^2p^{ll} = 4c^2p^{ll}, \end{aligned} \tag{71}$$

where equation (60) has been used. The complete system is

$$\begin{aligned} &\partial_t F + \partial_k G^k = 0, \;\; \partial_t F^i + \partial_k G^{ki} = 0, \;\; \partial_t F^{ij} + \partial_k G^{kij} = p^{<ij>}\,, \\ &\partial_t F^{ill} + \partial_k G^{kill} = p^{ill}, \;\; \partial_t F^{iill} + \partial_k G^{kiill} = p^{iill}. \end{aligned} \tag{72}$$

Note that the only difference between this system and (1) is that in $(72)_1$ the vector G^k occurs, while in $(1)_1$ there is F^k; for this reason we will have to impose the further condition $G^k = F^k$.

A.3.2. Reasons for the Above Choice of Coefficients

The coefficients in the above linear combinations have been chosen for the following reasons: from (61), $(62)_1$, $(63)_1$, $(65)_1$, $(67)_1$, $(70)_1$, $(71)_1$ it follows

$$\begin{aligned}
&F = \int \widetilde{f}\gamma^7(1-\frac{u^2}{c^2})d, \quad F^i = \int \widetilde{f}\gamma^6 u^i d\mathbf{u}u = \int \widetilde{f}\gamma^5 d\mathbf{u},\\
&F^{ll} = 2c^2 \int \widetilde{f}\gamma^7(\frac{1}{\gamma}-1+\frac{u^2}{c^2})d\mathbf{u},\\
&F^{<ij>} = \int \widetilde{f}\gamma^7(u^iu^j - \frac{1}{3}u^2\delta^{ij})d\mathbf{u}, \quad F^{ill} = 2c^2\int \widetilde{f}\gamma^7 u^i(1-\frac{1}{\gamma})d\mathbf{u},\\
&F^{iill} = \int \widetilde{f}\gamma^7(-8c^4\frac{1}{\gamma}+8c^4-4c^2u^2)d\mathbf{u}\,,
\end{aligned}$$

which have limits $\int \widetilde{f}d\mathbf{u}$, $\int \widetilde{f}u^i d\mathbf{u}$, $\int \widetilde{f}u^2 d\mathbf{u}$, $\int \widetilde{f}u^{<i}u^{j>}d\mathbf{u}$, $\int \widetilde{f}u^iu^2 d\mathbf{u}$, $\int \widetilde{f}u^4 d\mathbf{u}$, where we have taken into account also that

$$\frac{1}{\gamma} = \sqrt{1-\frac{u^2}{c^2}} = 1 - \frac{1}{2}\frac{u^2}{c^2} - \frac{1}{8}\frac{u^4}{c^4} + \frac{u^6}{c^6}(...)\,. \tag{73}$$

From the previous expressions it follows that F^{ij} has limit $\int \widetilde{f}u^iu^j d\mathbf{u}$. Obviously, the previous properties hold in the case of ideal gases; we assume that the corresponding change of equations is appropriate also for dense gases and for macromolecular fluids. In this way generality is not lost because the Galilean relativity principle has the same form for both cases.

A.4. Second Transformation of the Lagrange Multipliers

The change of equations in the previous section induces another one on the Lagrange multipliers and we want now to determine it.

We observe that the equations $(62)_1$, $(63)_1$, (68), $(70)_1$, $(71)_1$ give F, F^i, F^{ij}, F^{ill}, F^{iill} in terms of F_2, F_2^i, F_3, F_3^i, F_3^{ij}. Let us now take the inverse of these relations, which are

$$\begin{aligned}
&F_2 = \frac{F^{ll}}{2c^2} + F, \quad F_2^i = F^i,\\
&F_3 = F + \frac{F^{ppll}}{4c^4} + \frac{F^{ll}}{c^2}, \quad F_3^i = \frac{F^{ill}}{2c^2} + F^i, \quad F_3^{ij} = F^{ij} + \frac{\delta^{ij}}{3}\frac{F^{ppll}}{4c^2}
\end{aligned} \tag{74}$$

and these we now substitute in equation (48); so it becomes

$$dh = ld(\frac{F^{ll}}{2c^2}+F) + l_i d(F^i) + \eta d(F + \frac{F^{ppll}}{4c^4} + \frac{F^{ll}}{c^2})$$

$$\begin{aligned} &+ \quad \mu_i d(\frac{F^{ill}}{2c^2} + F^i) + \mu_{ij} d(F^{ij} + \frac{1}{3}\delta^{ij}\frac{F^{ppll}}{4c^2}) \\ &= \quad \lambda dF + \lambda_i dF^i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill} \lambda_{ppll} dF^{ppll} \end{aligned} \tag{75}$$

with

$$\lambda = l + \eta, \quad \lambda_i = l_i + \mu_i, \quad \lambda_{ij} = \mu_{ij} + (\frac{l}{2c^2} + \frac{\eta}{c^2})\delta_{ij}, \tag{76}$$
$$\lambda_{ill} = \frac{\mu_i}{2c^2}, \quad \lambda_{ppll} = \frac{\eta}{4c^4} + \frac{\mu_{ll}}{12c^2}\,.$$

In this way we have found the first part of the entropy principle for the new system.

For the sequel it will be useful to take the inverse of equations (76). They are

$$\eta = 8c^4\lambda_{ppll} - \frac{2}{3}c^2\lambda_{ll} + \lambda, \quad \ell = -8c^4\lambda_{ppll} + \frac{2}{3}c^2\lambda_{ll}, \quad \mu_i = 2c^2\lambda_{ill}, \tag{77}$$
$$\ell_i = \lambda_i - 2c^2\lambda_{ill}, \quad \mu_{ij} = \lambda_{ij} - (4c^2\lambda_{ppll} - \frac{\lambda_{ll}}{3} + \frac{\lambda}{c^2})\delta_{ij}.$$

Similarly, equations $(62)_2$, $(63)_2$, (69), $(70)_2$, $(71)_2$ give G^k, G^{ki}, G^{kij},G^{kill}, G^{kiill}, in terms of G_2^k, G_2^{ki}, G_3^k, G_3^{ki} G_3^{kij}. The inverses of these relations are

$$G_2^k = \frac{G^{kll}}{2c^2} + G^k, \quad G_2^{ki} = G^{ki} \tag{78}$$
$$G_3^k = G^k + \frac{G^{kppll}}{4c^4} + \frac{G^{kll}}{c^2}, \quad G_3^{ki} = \frac{G^{kill}}{2c^2} + G^{ki}, \quad G_3^{kij} = G^{kij} + \frac{1}{3}\delta^{ij}\frac{G^{kppll}}{4c^2},$$

which now we substitute into equation (51) which now becomes

$$\begin{aligned} d\phi^k &= \ell d(\frac{G^{kll}}{2c^2} + G^k) + \ell_i dG^{ki} + \eta d(G^k + \frac{G^{kppll}}{4c^4} + \frac{G^{kll}}{c^2}) \\ &+ \quad \mu_i d(\frac{G^{kill}}{2c^2} + G^{ki}) + \mu_{ij} d(G^{kij} + \frac{1}{3}\delta^{ij}\frac{G^{kppll}}{4c^2}) \\ &= \lambda dG^k + \lambda_i dG^{ki} + \lambda_{ij} dG^{kij} + \lambda_{ill} dG^{kill} + \lambda_{ppll} dG^{kppll}\,, \end{aligned} \tag{79}$$

thanks to equation (76).

In this way we have obtained the second part of the entropy principle. We deduce now the transformation of h' and ϕ'^k; to this end, let us take equation (55) and substitute equations (74); so we obtain

$$\begin{aligned} h &= -h' + l(\frac{F^{ll}}{2c^2} + F) + l_i F^i + \eta(F + \frac{F^{ppll}}{4c^4} + \frac{F^{ll}}{c^2}) \\ &+ \quad \mu_i(\frac{F^{ill}}{2c^2} + F^i) + \mu_{ij}(F^{ij} + \frac{\delta^{ij}}{3}\frac{F^{ppll}}{4c^2}) \end{aligned} \tag{80}$$

which, for equation (76), can be written in the following way

$$h = -h' + \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ill} F^{ill} + \lambda_{ppll} F^{ppll}. \tag{81}$$

From equations (58), by substituting from equations (78) we find

$$\begin{aligned} \phi^k &= -\phi'^k + \ell(\frac{G^{kll}}{2c^2} + G^k) + \ell_i G^{ki} \\ &+ \eta(G^k + \frac{G^{kppll}}{4c^4} + \frac{G^{kll}}{c^2}) + \mu_i(\frac{G^{kill}}{2c^2} + G^{ki}) + \mu_{ij}(G^{kij} + \frac{\delta^{ij} G^{kppll}}{12c^2}) \\ &= -\phi'^k + \lambda G^k + \lambda_i G^{ki} + \lambda_{ij} G^{kij} + \lambda_{ill} G^{kill} + \lambda_{ppll} G^{kppll}, \end{aligned} \tag{82}$$

for equations (76).

The equations (75) and (79) gives the entropy principle for the balance equations (72). The equations (81) and (82) give the counterparts of (55) and (58) for the new system (72). We have now to obtain h' and ϕ'^k from (29) expressing them in terms of the new Lagrange multipliers and then taking the limits for $c \longrightarrow \infty$. From the present results we can also write the relativistic Lagrange multipliers in terms of those for the system (72). By deducing λ_0, λ_i, λ_{00}, λ_{0i} and λ_{ij} from (49), and by substituting in their expressions equations (77) we obtain the expression (31).

Appendix B: Limits of the 4-Vectors Occurring in Equation (29) and of the Scalars (30)

B.1. The Scalars Q_1 - Q_4 .

Let us begin with $(30)_1$. From equation $(31)_1$ we find

$$Q_1 = \frac{c^2}{m_0^2}\left(-20\lambda_{ppll} + \frac{1}{c^2}\frac{8}{3}\lambda_{ll} - \frac{4}{c^4}\lambda\right). \tag{83}$$

Consequently, if we assume that the scalar functions depend on Q_1 as composite functions through $\frac{m_0^2}{-20c^2}Q_1$, then their limits will be functions of $\lambda_{ppll} = X_1$.

Let us consider now equation $(30)_2$. From equation $(31)_1$ we find

$$Q_2 = \frac{c^4}{m_0^4}\left(112\lambda_{ppll}^2 + 0(\frac{1}{c})\right),$$

where $0(\frac{1}{c})$ has limit zero when c goes to infinity. Therefore, if we assume that the scalar functions depend on Q_2 as composite functions through $\frac{m_0^4}{c^4}Q_2$, then their limits will be functions of $112\lambda_{ppll}^2$. But this is not independent from X_1, so that this result is too much restrictive. The idea is now to find a number k

such that $\frac{m_0^4}{c^4}(Q_2+kQ_1^2)$ has zero limit for c going to infinity. We find $k=-\frac{7}{25}$. After that, we see that

$$Q_2-\frac{7}{25}Q_1^2=\frac{c^2}{m_0^4}\left(-2\lambda_{all}\lambda_{all}+\frac{16}{5}\lambda_{ppll}\lambda_{ll}+0(\frac{1}{c})\right)\,. \tag{84}$$

Therefore, if we assume that the scalar functions depend on Q_1 and Q_2 as composite functions through $\frac{m_0^2}{-20c^2}Q_1$ and $\frac{m_0^4}{c^2}(Q_2-\frac{7}{25}Q_1^2)$, then their limits will be functions of X_1 and X_2.

Going on in a similar way, we look for the numbers k_1 and k_2 such that $\frac{m_0^6}{c^4}[Q_3+k_1Q_1(Q_2-\frac{7}{25}Q_1^2)+k_2Q_1^3]$ has zero limit for c going to infinity. We find $k_1=-\frac{9}{10}$ and $k_2=-\frac{11}{125}$. After that, we obtain

$$Q_3-\frac{9}{10}Q_1(Q_2-\frac{7}{25}Q_1^2)-\frac{11}{125}Q_1^3=\frac{c^2}{m_0^6}\left(-\frac{3}{2}X_3+0(\frac{1}{c})\right)\,. \tag{85}$$

Then the limit of a scalar function depends on X_1, X_2 and X_3.

Similarly, we search the numbers k_3, k_4, k_5, k_6 such that $\frac{m_0^8}{c^4}\{Q_4+k_3Q_1[Q_3-\frac{9}{10}Q_1(Q_2-\frac{7}{25}Q_1^2)-\frac{11}{125}Q_1^3]+k_4(Q_2-\frac{7}{25}Q_1^2)^2+k_5Q_1^2(Q_2-\frac{7}{25}Q_1^2)+k_6Q_1^4]$ has zero limit for c going to infinity. We find $k_3=-\frac{16}{15}$, $k_4=-\frac{1}{2}$, $k_5=-\frac{14}{25}$, $k_6=-\frac{19}{625}$. After that, we obtain

$$Q_4-\frac{16}{15}Q_1\left[Q_3-\frac{9}{10}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{11}{125}Q_1^3\right]-\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)^2$$
$$-\frac{14}{25}Q_1^2\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{19}{625}Q_1^4=\frac{c^2}{m_0^8}\left(-2X_4+0(\frac{1}{c})\right). \tag{86}$$

Therefore, the limit of a scalar function depends on X_1, X_2, X_3 and X_4.

Before proceeding with the other scalars $(30)_{5-8}$, let us evaluate the 4-vectors of which (29) is a linear combination.

B.2. Limits of the 4-Vectors in the Expression of the Relativistic Potential

From the equations (56), (59) and $(31)_2$ it follows that the term $h_0\lambda^\alpha$ contributes to ϕ'^k the term $-2H_0\lambda_{kll}$ and to h' the term $8H_0\lambda_{ppll}$, where H_0 is the limit of $\frac{c^3}{m_0^4}h_0$ for c going to infinity.

But, from equations (59) and (31) it follows that the term $h_1\lambda^{\alpha\gamma}\lambda_\gamma$ contributes to ϕ'^k the term $\frac{c^5}{m_0^6}h_1\,16\lambda_{ppll}\lambda_{kll}$ which is parallel to the previous one. In order not to lose generality, it is better to look for a number a such that

$(\lambda^{\alpha\gamma}\lambda_\gamma + a\, Q_1\, \lambda^\alpha)\, \frac{m_0^3}{c^5}$ has zero limit. We find $a = -\frac{2}{5}$. After that, we obtain

$$\lambda^{\alpha\gamma}\lambda_\gamma - \frac{2}{5}\, Q_1\, \lambda^\alpha = \frac{c^3}{m_0^3}\left[\begin{pmatrix} X_2 \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ V_1^i \end{pmatrix} + 0\begin{pmatrix} 1 \\ c \end{pmatrix}\right]. \tag{87}$$

Now a linear combination, with arbitrary coefficients, of λ^α, $\lambda^{\alpha\gamma}\lambda_\gamma$, $\overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$, $\overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ is also a linear combination, with arbitrary coefficients, of λ^α, $\lambda^{\alpha\gamma}\lambda_\gamma - \frac{2}{5}\, Q_1\, \lambda^\alpha$, $\overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$, $\overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$, so we can suppose that in (29) there is $\lambda^{\alpha\gamma}\lambda_\gamma - \frac{2}{5}\, Q_1\, \lambda^\alpha$ instead of $\lambda^{\alpha\gamma}\lambda_\gamma$; After that, from equations (56) and (59) it follows that the term $h_1(\lambda^{\alpha\gamma}\lambda_\gamma - \frac{2}{5}\, Q_1\, \lambda^\alpha)$ contributes to ϕ'^k the term $H_1(-2\lambda_{ih}\lambda_{hll} + 4\lambda_{ppll}\lambda_i + \frac{4}{5}\lambda_{ll}\lambda_{ill})$ and to h' the term $H_1 X_2$, where H_1 is the limit of $\frac{c^3}{m_0^6} h_1$ for c going to infinity.

Proceeding furtherly in this way, we search the numbers a_1, a_2, a_3, such that $\frac{m_0^5}{c^5}\left\{\overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma + a_1 Q_1 \lambda^{\alpha\gamma}\lambda_\gamma + \left[a_2 Q_1^2 + a_3(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^\alpha\right\}$ has zero limit for c going to infinity. We find $a_1 = -\frac{3}{5}$, $a_2 = \frac{2}{25}$, $a_3 = -\frac{1}{2}$. After that, we obtain

$$\begin{aligned} &\overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma - \frac{3}{5}Q_1\lambda^{\alpha\gamma}\lambda_\gamma + \left[\frac{2}{25}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^\alpha \\ &= \frac{c^3}{m_0^5}\left[\begin{pmatrix} X_3 \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ V_2^i \end{pmatrix} + 0\begin{pmatrix} 1 \\ c \end{pmatrix}\right]. \end{aligned} \tag{88}$$

This 4-vector can replace $\overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ in (29); so we find that it, together with the factor h_2, contributes to ϕ'^k the term $H_2(-2\lambda_{kh}^2\lambda_{hll} + \frac{6}{5}\lambda_{ll}\lambda_{ka}\lambda_{all} + 4\lambda_{ka}\lambda_a\lambda_{ppll} - \frac{11}{25}\lambda_{ll}^2\lambda_{kll} - \lambda_{kll}\lambda_a\lambda_{all} + \lambda_k\lambda_{all}\lambda_{all} + (tr\lambda_{ab}^2)\lambda_{kll} - \frac{12}{5}\lambda_{ppll}\lambda_{ll}\lambda_k)$ and to h' the term $H_2 X_3$, with H_2 limit of $\frac{c^3}{m_0^8} h_2$ for c going to infinity.

Finally, we search the numbers a_4, a_5, a_6, a_7, a_8, a_9, such that

$$\begin{aligned} &\frac{m_0^7}{c^5}\{\ \overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma + a_4 Q_1\ \overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma \\ &+\left[a_5 Q_1^2 + a_6(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^{\alpha\gamma}\lambda_\gamma + \left[a_7 Q_3 + a_8 Q_1^3 + a_9 Q_1(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^\alpha\ \}, \end{aligned}$$

has zero limit for c going to infinity. We find $a_4 = -\frac{4}{5}$, $a_5 = \frac{1}{5}$, $a_6 = -\frac{1}{2}$, $a_7 = -\frac{1}{3}$, $a_8 = \frac{1}{75}$, $a_9 = \frac{2}{5}$. After that we have the result

$$\begin{aligned} &\overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma - \frac{4}{5}Q_1\ \overset{2}{\lambda}{}^{\alpha\gamma}\lambda_\gamma + \left[\frac{1}{5}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^{\alpha\gamma}\lambda_\gamma \\ &\left[-\frac{1}{3}Q_3 + \frac{1}{75}Q_1^3 + \frac{2}{5}Q_1(Q_2 - \frac{7}{25}Q_1^2)\right]\lambda^\alpha \end{aligned} \tag{89}$$

$$= \frac{c^3}{m_0^7}\left[\begin{pmatrix} X_4 \\ 0_i \end{pmatrix} + \frac{1}{c}\begin{pmatrix} 0 \\ V_3^i \end{pmatrix} + 0\left(\frac{1}{c}\right)\right] .$$

It may replace $\overset{3}{\lambda}{}^{\alpha\gamma}\lambda_\gamma$ in (29), so that it, together with the factor h_3, contributes to ϕ'^k the term

$$H_3\left[2\lambda_{pplI}\left(-2\lambda_{kh}^2\lambda_h - tr\lambda_{ab}^2\lambda_k - \frac{8}{5}\lambda_{ll}\lambda_{ka}\lambda_a + \frac{17}{25}\lambda_{ll}^2\lambda_k\right) + (\lambda_{kh}\lambda_h)(\lambda_{all}\lambda_{all})\right.$$
$$- \frac{4}{5}\lambda_{ll}(\lambda_{all}\lambda_{all})\lambda_k - \frac{17}{25}\lambda_{ll}^2\lambda_{ka}\lambda_{all} - (\lambda_{all}\lambda_{all})\lambda_{kb}\lambda_{bll} + (tr\lambda_{ab}^2)\lambda_{kc}\lambda_{cll}$$
$$+ \frac{4}{5}\lambda_{ll}(\lambda_a\lambda_{all})\lambda_{kll} + \frac{8}{5}\lambda_{ll}\lambda_{kh}^2\lambda_{hll} + \frac{74}{375}\lambda_{ll}^3\lambda_{kll} - \frac{4}{5}\lambda_{ll}(tr\lambda_{ab}^2)\lambda_{kll}$$
$$\left. + (\lambda_{ab}\lambda_{all}\lambda_{bll})\lambda_k - (\lambda_{ab}\lambda_a\lambda_{bll})\lambda_{kll} + \frac{2}{3}(tr\lambda_{ab}^3)\lambda_{kll} - 2\lambda_{kh}^3\lambda_{hll}\right]$$

and to h' the term H_3X_4, with H_3 limit of $\frac{c^3}{m_0^8}h_3$ for c going to infinity.

B.3. The Scalars $\mathbf{P}_0$ - $\mathbf{P}_3$

From equation $(31)_2$ we find $P_0 = \frac{c^6}{m_0^2}\left(64\lambda_{ppll}^2 + 0(\frac{1}{c})\right)$ so that the limit of $\frac{m_0^2}{c^6}P_0$ is a function of X_1. In order to obtain a less restrictive result, we may substitute P_0 with $P_0 + \frac{4}{25}Q_1^2m_0^2c^2$, because this will eliminate the term $\frac{c^6}{m_0^2}64\lambda_{ppll}^2$ of P_0; in fact we obtain

$$P_0 + \frac{4}{25}Q_1^2m_0^2c^2 = \frac{c^4}{m_0^2}\left(2X_2 + 0(\frac{1}{c})\right) .$$

In this way we have a better result, even if it is still not enough because it is not independent from X_2; to this end let us substitute $P_0 + \frac{4}{25}Q_1^2m_0^2c^2$ with $P_0 + \frac{4}{25}Q_1^2m_0^2c^2 + 2(Q_2 - \frac{7}{25}Q_1^2)m_0^2c^2$ and the result is satisfactory because is equal to

$$\frac{c^2}{m_0^2}\left(-\frac{2}{5}\lambda_{ll}^2 + 16\lambda_{ppll}\lambda - 4\lambda_a\lambda_{all} + 2tr\lambda_{ab}^2 + 0(\frac{1}{c})\right) .$$

In this way we have found the new scalar X_5.

Let us now consider the scalar P_1 in $(30)_6$. It is obvious, for equation (87) that it is better to replace it with $P_1 - \frac{2}{5}Q_1P_0$; but this has limit a scalar which is a function of X_1, X_2, X_3. Briefly, let us look for the numbers b_1 and b_2 such that

$$\frac{m_0^4}{c^4}\left\{P_1 - \frac{2}{5}Q_1P_0 + b_1Q_1\left(Q_2 - \frac{7}{25}Q_1^2\right)m_0^2c^2\right.$$

$$+b_2\left[Q_3-\frac{9}{10}Q_1(Q_2-\frac{7}{25}Q_1^2)-\frac{11}{125}Q_1^3\right]m_0^2c^2\Bigg\}$$

has zero limit.

We find $b_1=\frac{2}{5}$, $b_2=\frac{4}{3}$. After that we have

$$P_1-\frac{2}{5}Q_1P_0+\frac{2}{5}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)m_0^2c^2 \\ +\frac{4}{3}\left[Q_3-\frac{9}{10}Q_1(Q_2-\frac{7}{25}Q_1^2)-\frac{11}{125}Q_1^3\right]m_0^2c^2=\frac{c^2}{m_0^4}\left(X_6+0(\frac{1}{c})\right),$$

from which the new scalar X_6.

Let s consider now the scalar P_2 in $(30)_7$. It is obvious, for equation (88) that it is better replace it with $P_2-\frac{3}{5}Q_1P_1+\left[\frac{2}{25}Q_1^2-\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)\right]P_0$; but also this is not enough; then we search the numbers b_3 and b_4 such that

$$\frac{m_0^6}{c^4}\Bigg\{P_2-\frac{3}{5}Q_1P_1+\left[\frac{2}{25}Q_1^2-\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)\right]P_0 \\ +b_3Q_1\left[Q_3-\frac{9}{10}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{11}{125}Q_1^3\right]m_0^2c^2\Bigg\} \\ +b_4\frac{m_0^6}{c^4}\Bigg\{Q_4-\frac{16}{15}Q_1\left[Q_3-\frac{9}{10}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{11}{125}Q_1^3\right] \\ -\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)^2-\frac{14}{25}Q_1^2\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{19}{625}Q_1^4\Bigg\}m_0^2c^2$$

has zero limit.

We find $b_3=\frac{4}{15}$, $b_4=1$. After that it follows

$$P_2-\frac{3}{5}Q_1P_1+\left[\frac{2}{25}Q_1^2-\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)\right]P_0 \\ +\frac{4}{15}Q_1\left[Q_3-\frac{9}{10}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{11}{125}Q_1^3\right]m_0^2c^2 \\ +\Bigg\{Q_4-\frac{16}{15}Q_1\left[Q_3-\frac{9}{10}Q_1\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{11}{125}Q_1^3\right] \\ -\frac{1}{2}\left(Q_2-\frac{7}{25}Q_1^2\right)^2-\frac{14}{25}Q_1^2\left(Q_2-\frac{7}{25}Q_1^2\right)-\frac{19}{625}Q_1^4\Bigg\}m_0^2c^2 \\ =\frac{c^2}{m_0^6}\left(X_7+0(\frac{1}{c})\right),$$

from which the new scalar X_7.

Finally, let us consider the scalar P_3 in $(30)_8$. It is obvious, for equation

(89) that it is better replace it with $P_3 - \frac{4}{5}Q_1P_2 + \left[\frac{1}{5}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2)\right] P_1 + \left[-\frac{1}{3}Q_3 + \frac{1}{75}Q_1^3 + \frac{2}{5}Q_1(Q_2 - \frac{7}{25}Q_1^2)\right] P_0$; but this also is not enough; then we look for a number b_5 such that

$$\frac{m_0^8}{c^6}\left\{P_3 - \frac{4}{5}Q_1P_2 + \left[\frac{1}{5}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2)\right] P_1 + \left[-\frac{1}{3}Q_3 + \frac{1}{75}Q_1^3 + \frac{2}{5}Q_1(Q_2 - \frac{7}{25}Q_1^2)\right] P_0\right\} + b_5 Q_1 \frac{m_0^8}{c^6}\left\{Q_4 - \frac{16}{15}Q_1\left[Q_3 - \frac{9}{10}Q_1\left(Q_2 - \frac{7}{25}Q_1^2\right) - \frac{11}{125}Q_1^3\right] - \frac{1}{2}\left(Q_2 - \frac{7}{25}Q_1^2\right)^2 - \frac{14}{25}Q_1^2\left(Q_2 - \frac{7}{25}Q_1^2\right) - \frac{19}{625}Q_1^4\right\} m_0^2c^2$$

has zero limit.

We find $b_5 = \frac{1}{5}$. Then we can evaluate

$$P_3 - \frac{4}{5}Q_1P_2 + \left[\frac{1}{5}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2)\right] P_1 + \left[-\frac{1}{3}Q_3 + \frac{1}{75}Q_1^3 + \frac{2}{5}Q_1(Q_2 - \frac{7}{25}Q_1^2)\right] P_0 + \frac{1}{5}Q_1\left\{Q_4 - \frac{16}{15}Q_1\left[Q_3 - \frac{9}{10}Q_1\left(Q_2 - \frac{7}{25}Q_1^2\right) - \frac{11}{125}Q_1^3\right] - \frac{1}{2}\left(Q_2 - \frac{7}{25}Q_1^2\right)^2 - \frac{14}{25}Q_1^2\left(Q_2 - \frac{7}{25}Q_1^2\right) - \frac{19}{625}Q_1^4\right\} m_0^2c^2 = \frac{c^2}{m_0^8}\left(X_8 + 0(\frac{1}{c})\right),$$

from which the last new scalar X_8.