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UNIT GROUPS OF $k+1$ INDEX RADICAL ZERO COMMUTATIVE FINITE RINGS

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Abstract: In this paper, we determine the structures of the unit groups of commutative finite rings R of characteristic p^k where p is any prime integer and k is any positive integer such that if J is the Jacobson radical of R, then $J^{k+1} = (0)$, $J^k \neq (0)$ and $R/J \cong GF(p^r)$, the finite field of p^r elements for any prime p and any positive integer r.

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1. Introduction

Throughout this paper, all the rings are finite and commutative with identities, denoted by $1 \neq 0$ and ring homomorphisms preserve identity. In [7], Raghavendran proved that any finite ring with identity will contain at least one Galois ring as a subset. In [1] and [2], Chikunji determined the structure of the group of units of the ring $R = R_0 \oplus U \oplus V$ where $R_0 = GR(p^{kr}, p^k)$ is the Galois subring of R, while U and V are finitely generated R_0-modules. The author further determined the generators of R^*. Upon consideration of s, t and λ to be the number of elements in the generating sets for U, V and W respectively,

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Chikunji in [3] determined in general the structure of $1+W$ of the unit group of $R = R_0 \oplus U \oplus V \oplus W$ and the structure of the R^* of R when $s = 3$, $t = 1$, $\lambda \geq 1$ and the $\mathrm{char}R = p$. Furthermore the author generalized the solution of the cases when $s = 2$, $t = 1$; $t = s(s+1)/2$ for a fixed s, and $p \leq \mathrm{char}R \leq p^3$; and when $s = 2$, $t = 2$ and $\mathrm{char}R = p$ to the case when the annihilator, $\mathrm{ann}(J) = J^2 + W$, so that $\lambda \geq 1$.

Recall that a completely primary finite ring is a ring R with identity $1 \neq 0$ whose subset of all its zero divisors forms a unique maximal ideal. Also recall that a Galois ring is a finite ring with identity $1 \neq 0$ such that the set of all its zero divisors with 0 included forms a principal ideal. We construct commutative finite rings with unique maximal ideal J such that $J^{k+1} = (0)$ and $J^k \neq (0)$ for the cases when $\mathrm{char}R = p$, $\mathrm{char}R = p^2$ and $\mathrm{char}R = p^k$ where $k \geq 3$. Then we determine the structures of the unit groups of the rings constructed in the different cases.

The following results are fundamental to the study of unit groups of the rings to be considered in this paper.

1.1. Let R be a finite ring. Then every left unit is a right unit and every left zero divisor is a right zero divisor. Furthermore, every element of R is either a zero divisor or a unit (see [5]).

1.2. If a ring R has $n \geq 2$ left zero divisors (including zero), then R is a finite ring, and $|R| \leq n^2$ (see [6]).

1.3. Let R be a finite ring with identity $1 \neq 0$. Then, every non-trivial ideal of R consists entirely of zero divisors.

1.4. If G is a cyclic group of order n, then $G \cong \mathbf{Z}_n$.

2. $k+1$ Index Radical Zero Commutative Finite Rings

We study the unit groups R^* of a commutative finite ring R with unique maximal ideal J such that $R/J \simeq GF(p^r)$, $J^{k+1} = (0)$ and $J^k \neq (0)$, so that the characteristic of R is p^k for some prime integer p and positive integers k and r. Let R_0 be the Galois ring of the form $GR(p^{kr}, p^k)$ and let $u_i \in J$, where $1 \leq i \leq h$ so that $R = R_0 \oplus R_0u_1 \oplus ... \oplus R_0u_h$ is an additive Abelian group. On the additive Abelian group, define multiplication by the following relations: $u_iu_j = 0$ $(1 \leq i, j \leq h)$; $r_0u_i = u_ir_0$, $r_0 \in R_0$; $p^{k-1}u_i \neq 0$. From the given definition of multiplication in R, we see clearly that if $(r_0, r_1, ..., r_h)$ and

$(s_0, s_1, ..., s_h)$ are any two elements in R, then

$$(r_0, r_1, ..., r_h)(s_0, s_1, ..., s_h) = (r_0 s_0, r_0 s_1 + r_1 s_0, ..., r_0 s_h + r_h s_0).$$

It is easy to verify that the given multiplication turns the additive group into a commutative ring with identity $(1, 0, ..., 0)$.

Proposition 1. *The ring R is completely primary of characteristic p^k, and:*

(i) $J = pR_0 \oplus R_0 u_1 \oplus ... \oplus R_0 u_h$,

(ii) $J^2 = p^2 R_0 \oplus pR_0 u_1 \oplus ... \oplus pR_0 u_h$,

(iii) $J^{k-1} = p^{k-1} R_0 \oplus p^{k-2} R_0 u_1 \oplus ... \oplus p^{k-2} R_0 u_h$,

(iv) $J^k = p^k R_0 \oplus p^{k-1} R_0 u_1 \oplus ... \oplus p^{k-1} R_0 u_h$,

(v) $J^{k+1} = (0)$.

Proof. First, we show that char $R = p^k$, for some prime p and positive integer k.

Since char $R_0 = p^k$, then for every $y \in R_0$, $p^k y = 0$. But

$$R = \{y_0 + y_1 u_1 + ... + y_h u_h \mid y_0, y_i \in R_0,\ u_i u_j = 0, (1 \le i, j \le h)\}.$$

Now, suppose $p^s \in R$ where $s < k$. Then, by the distributive property in R,

$$p^s (y_0 + y_1 u_1 + ... + y_h u_h) = p^s y_0 + p^s y_1 u_1 + ... + p^s y_h u_h$$

which is not identically zero. The same argument holds for any other positive integer less than p^k. So char $R = p^k$.

With the obvious identifications, we can think of R_0 as a subset of R. Now, it follows immediately from the way multiplication has been defined that if

$$J = pR_0 \oplus R_0 u_1 \oplus ... \oplus R_0 u_h,$$

then

$$J^2 = p^2 R_0 \oplus pR_0 u_1 \oplus ... \oplus pR_0 u_h,$$
$$J^{k-1} = p^{k-1} R_0 \oplus p^{k-2} R_0 u_1 \oplus ... \oplus p^{k-2} R_0 u_h,$$
$$J^k = p^k R_0 \oplus p^{k-1} R_0 u_1 \oplus ... \oplus p^{k-1} R_0 u_h,$$

and that

$$J(p^k R_0 \oplus p^{k-1} R_0 u_1 \oplus ... \oplus p^{k-1} R_0 u_h) = (p^k R_0 \oplus p^{k-1} R_0 u_1 \oplus ... \oplus p^{k-1} R_0 u_h) J = (0).$$

Hence

$$J^{k+1} = (0).$$

Also from the definition of multiplication, it follows that $RJ = JR \subseteq J$ so that J is an ideal. Suppose there is an ideal $K \supseteq J$, then by 1.1, K contains a unit $z \in R$ such that $zz^{-1} = z^{-1}z = 1$. So $K = R$. Therefore J is the unique maximal ideal in R since any maximal ideal distinct from J contains a unit.

We now show that J is indeed $pR_0 \oplus R_0u_1 \oplus ... \oplus R_0u_h$.

Let $\alpha \in R_0$ with α not a member of pR_0 and $s \in J$. We have

$$\begin{aligned}(\alpha + s)^{p^r} &= \alpha^{p^r} + t \text{ (with } t \in J) \\ &= \alpha + v \text{ (with } v \in J).\end{aligned}$$

But then $(\alpha + v)^{p-1} = 1 + q$ (with $q \in J$) and $(1 + q)^{p^{k-1}} = 1$. Hence $\alpha + s$ is invertible. Since $| J |= p^{(k(h+1)-1)r}$ and $| (R_0/pR_0)^* + J |= (p^r - 1)p^{(k(h+1)-1)r}$, it follows that $(R_0/pR_0)^* + J = R - J$ and hence all the elements outside J are invertible. Therefore R is completely primary and satisfies the given properties. □

Let R be the completely primary finite ring, with maximal ideal J such that $J^{k+1} = (0)$, $J^k \neq (0)$. Then R is of order $p^{k(h+1)r}$ and the residue field R/J is the finite field $GF(p^r)$, for some prime integer p and positive integers k, h and r. A concrete model of R_0 is the quotient $\mathbf{Z}_{p^k}[x]/(f)$ where $f \in \mathbf{Z}_{p^k}[x]$ is a monic polynomial of degree r irreducible modulo p. Then it can be deduced from the main theorem in [4] that R has a coefficient subring R_0 of the form $GR(p^{kr}, p^k)$ which is clearly a maximal Galois subring of R. A trivial case is $GR(p^k, p^k) = \mathbf{Z}_{p^k}$. Notice that since R is of order $p^{k(h+1)r}$ and $R^* = R - J$, then $| R^* |= p^{(k(h+1)-1)r}(p^r - 1)$ and $| 1 + J |= p^{(k(h+1)-1)r}$ is an Abelian p-group. Thus $R^* \cong$ (Abelian p-group)$\times$(cyclic group of order $| R/J | -1$). In the sequel, the following result due to Chikunji [3] will be useful.

Proposition 2. *Let R be a completely primary finite ring (not necessarily commutative). Then the group of units R^* of the ring R contains a cyclic subgroup $< b >$ of order $p^r - 1$, and R^* is a semi direct product of $1 + J$ and $< b >$.*

Since the ring of consideration in our case is commutative, we notice from the above proposition that if A is a cyclic group of order $| R/J | -1$, then

$$R^* = A.(1 + J) \cong A \times (1 + J)$$

is a direct product.

We now determine the structures of the groups of units of R for any positive integer r.

3. Units of Rings of Characteristic p

Let $R_0 = \mathbf{F}_q = GF(p^r)$, the Galois field of $q = p^r$ elements. Then

$$R = \mathbf{F}_q \oplus \mathbf{F}_q u_1 \oplus ... \oplus \mathbf{F}_q u_h,$$

the Jacobson radical

$$J = \mathbf{F}_q u_1 \oplus ... \oplus \mathbf{F}_q u_h$$

and

$$J^2 = (0).$$

Since R^* is a direct product of the cyclic group A of order $p^r - 1$ and the group $1+J$ of order p^{hr}, it suffices to determine the structure of $1+J$. In this case

$$1 + J = 1 + \mathbf{F}_q u_1 \oplus ... \oplus \mathbf{F}_q u_h.$$

Proposition 3. *If* $charR = p$ *and* $h \geq 1$, *then*

$$1 + J \cong \underbrace{\boldsymbol{Z}_p^r \times ... \times \boldsymbol{Z}_p^r}_{h \text{ copies}}.$$

Proof. Let $\tau_1, ..., \tau_r \in \mathbf{F}_q$ with $\tau_1 = 1$ such that $\overline{\tau_1}, ..., \overline{\tau_r} \in \mathbf{F}_q$ form a basis for $\mathbf{F}_q$ regarded as a vector space over its prime subfield $\mathbf{F}_p$, where $q = p^r$ for any prime p and positive integer r. We note that for every $l = 1, ..., r$ and $1 \leq i \leq h$, $1 + \tau_l u_i \in 1 + J$, $(1 + \tau_l u_1)^p = 1$, $(1 + \tau_l u_1 + \tau_l u_2)^p = 1$,...,$(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_{h-1})^p = 1$, $y^p = 1$, $\forall y \in 1 + J$. For positive integers a_{1l}, a_{2l},...,a_{hl} with $a_{1l} \leq p$, $a_{2l} \leq p$,...,$a_{hl} \leq p$, we notice that

$$\prod_{l=1}^{r}\{(1+\tau_l u_1)^{a_{1l}}\}.\prod_{l=1}^{r}\{(1+\tau_l u_1 + \tau_l u_2)^{a_{2l}}\}$$

$$...\prod_{l=1}^{r}\{(1+\tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^{a_{hl}}\} = 1$$

will imply $a_{il} = p$ for every $l = 1, ..., r$ and $1 \leq i \leq h$. If we set

$$S_{1l} = \{(1 + \tau_l u_1)^{a_1} \mid a_1 = 1, ..., p\},$$

$$S_{2l} = \{(1 + \tau_l u_1 + \tau_l u_2)^{a_2} \mid a_2 = 1, ..., p\},$$

$$\vdots$$

$$S_{hl} = \{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^{a_h} \mid a_h = 1, ..., p\},$$

we see that $S_{1l}, S_{2l}, ..., S_{hl}$ are all cyclic subgroups of the group $1+J$ and they are each of order p. We also notice that as each element in $1+J$ raiseds to the power p aquals 1, then $1+J$ is an elementary Abelian group. Now, since

$$\prod_{l=1}^{r} |< 1+\tau_l u_1 >| \,.\, \prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2 >|$$

$$\ldots \prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2+\ldots+\tau_l u_h >| = p^{hr}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the hr subgroups S_{1l}, S_{2l},...,S_{hl} is direct. So their product exhausts the group $1+J$. □

4. Units of Rings of Characteristic p^2

Let $k=2$. By the definition of multiplication in the ring R, and the properties of the Jacobson radical J, $p^2 m = 0$, $m \in J$. Therefore $p^2 \in J^2$. Consider $R_0 = GR(p^{2r}, p^2)$, the Galois ring of characteristic p^2 and order p^{2r}. Then

$$R = R_0 \oplus R_0 u_1 \oplus \ldots \oplus R_0 u_h,$$

$$J = pR_0 \oplus R_0 u_1 \oplus \ldots \oplus R_0 u_h,$$

$$J^2 = p^2 R_0 \oplus pR_0 u_1 \oplus \ldots \oplus pR_0 u_h,$$

$$J^3 = (0).$$

Since R^* is a direct product of the cyclic group, say A of order $p^r - 1$ by the group $1+J$ of order $p^{(2h+1)r}$, it suffices to determine the structure of $1+J$. We notice that

$$1+J = 1+pR_0 \oplus R_0 u_1 \oplus \ldots \oplus R_0 u_h.$$

Proposition 4. *If* $\mathrm{char} R = p^2$ *and* $h \geq 1$, *then*

$$1+J \cong \boldsymbol{Z}_p^r \times \underbrace{\boldsymbol{Z}_{p^2}^r \times \ldots \times \boldsymbol{Z}_{p^2}^r}_{h \text{ copies}}.$$

Proof. Let $\tau_1, \ldots, \tau_r \in R_0$ with $\tau_1 = 1$ such that $\overline{\tau_1}, \ldots, \overline{\tau_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbf{F}_p$. We note that for every $l = 1, \ldots, r$, $(1+p\tau_l)^p = 1$, $(1+\tau_l u_1)^{p^2} = 1$, $(1+\tau_l u_1+\tau_l u_2)^{p^2} = 1$,..., $(1+\tau_l u_1+\tau_l u_2+\ldots+\tau_l u_h)^{p^2} = 1$. For positive integers $a_l, b_{1l}, b_{2l}, \ldots, b_{hl}$ with $a_l \leq p$, $b_{1l} \leq p^2$,...,$b_{hl} \leq p^2$, we notice that

$$\prod_{l=1}^{r}\{(1+p\tau_l)^{a_l}\}.\prod_{l=1}^{r}\{(1+\tau_l u_1)^{b_{1l}}\}.\prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2)^{b_{2l}}\}$$

$$...\prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{b_{hl}}\}=1$$

will imply $a_l=p$, $b_{il}=p^2$ for every $l=1,...,r$ and $1\leq i\leq h$. If we set

$$T_l=\{(1+p\tau_l)^a \mid a=1,...,p\},$$

$$S_{1l}=\{(1+\tau_l u_1)^{b_1} \mid b_1=1,...,p^2\},$$

$$S_{2l}=\{(1+\tau_l u_1+\tau_l u_2)^{b_2} \mid b_2=1,...,p^2\},$$

$$\vdots$$

$$S_{hl}=\{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{b_h} \mid b_h=1,...,p^2\},$$

we see that T_l, S_{1l}, S_{2l},...,S_{hl} are all cyclic subgroups of the group $1+J$ and they are of the orders indicated by their definition. Since

$$\prod_{l=1}^{r}|<1+p\tau_l>|\,.\prod_{l=1}^{r}|<1+\tau_l u_1>|\,.\prod_{l=1}^{r}|<1+\tau_l u_1+\tau_l u_2>|$$

$$...\prod_{l=1}^{r}|<1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h>|=p^{(2h+1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups T_l, S_{1l}, S_{2l},...,S_{hl} is direct. So their product exhausts the group $1+J$. □

5. Units of Rings of Characteristic p^k, where $k\geq 3$

Let $k\geq 3$. By the definition of multiplication in the ring R, and the properties of the Jacobson radical J, $p^k m=0$, $m\in J$. Therefore $p^k\in J^k$. Consider $R_0=GR(p^{kr},p^k)$, the Galois ring of characteristic p^k and order p^{kr}. Then

$$R=R_0\oplus R_0u_1\oplus...\oplus R_0u_h,$$

$$J=pR_0\oplus R_0u_1\oplus...\oplus R_0u_h,$$

$$J^2=p^2R_0\oplus pR_0u_1\oplus...\oplus pR_0u_h,$$

$$\vdots$$

$$J^k=p^kR_0\oplus p^{k-1}R_0u_1\oplus...\oplus p^{k-1}R_0u_h,$$

$$J^{k+1}=(0).$$

Since R^* is a direct product of the cyclic group, say A of order $p^r - 1$ by the group $1+J$ of order $p^{(k(h+1)-1)r}$, it suffices to determine the structure of $1+J$. Again

$$1 + J = 1 + pR_0 \oplus R_0u_1 \oplus ... \oplus R_0u_h.$$

Proposition 5. *If* $\mathrm{char}R = p^k$ *where* $k \geq 3$ *and* $h \geq 1$, *then*

$$1+J \cong \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_{2^{k-2}} \times \boldsymbol{Z}_{2^{k-1}}^{r-1} \times \underbrace{\boldsymbol{Z}_{2^k}^r \times ... \times \boldsymbol{Z}_{2^k}^r}_{h \text{ copies}} & \text{if } p = 2, \\ \boldsymbol{Z}_{p^{k-1}}^r \times \underbrace{\boldsymbol{Z}_{p^k}^r \times ... \times \boldsymbol{Z}_{p^k}^r}_{h \text{ copies}} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let $\tau_1, \tau_2, ..., \tau_r \in R_0$ with $\tau_1 = 1$ such that $\overline{\tau_1}, \overline{\tau_2}, ..., \overline{\tau_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbf{F}_p$. We consider the two cases separately.

Suppose $p = 2$.

If $l = 1, ..., r$ and y is an element of R_0 such that $x^2 + x + \overline{y} = \overline{0}$ over R_0/pR_0, has no solution in the field R_0/pR_0, we obtain the following results: $(-1 + 2^{k-1}\tau_1) \in 1 + pR_0$, $(-1 + 2^{k-1}\tau_1)^2 = 1$, $(1 + 4y)^{2^{k-2}} = 1$ and $w^{2^{k-1}} = 1$ for every $w \in 1 + pR_0$. We also notice that $(1 + 2\tau_l)^{2^{k-1}} = 1$ for $l = 2, ..., r$, $(1 + \tau_l u_1)^{2^k} = 1$, $(1 + \tau_l u_1 + \tau_l u_2)^{2^k} = 1$,..., $(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^{2^k} = 1$, for every $l = 1, ..., r$.

Now, for positive integers $a, b, c_l, d_{1l}..., d_{hl}$ with $a \leq 2$, $b \leq 2^{k-2}$, $c_l \leq 2^{k-1}$ for l=2,...,r and $d_{il} \leq 2^k$ for every $l = 1, ..., r$ and $1 \leq i \leq h$, we notice that

$$(-1 + 2^{k-1}\tau_1)^a.(1 + 4y)^b.\prod_{l=2}^{r}(1 + 2\tau_l)^{c_l}.\prod_{l=1}^{r}\{(1 + \tau_l u_1)^{d_{1l}}\}$$

$$.\prod_{l=1}^{r}\{(1 + \tau_l u_1 + \tau_l u_2)^{d_{2l}}\}...\prod_{l=1}^{r}\{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^{d_{hl}}\} = 1$$

will imply $a = 2$, $b = 2^{k-2}$, $c_l = 2^{k-1}$ for $l = 2, ..., r$ and $d_{il} = 2^k$ for every $l = 1, ..., r$ and $1 \leq i \leq h$.

If we set

$$H = \{(-1 + 2^{k-1}\tau_1)^a \mid a = 1, 2\},$$

$$Q = \{(1 + 4y)^b \mid b = 1, ..., 2^{k-2}\},$$

$$T_l = \{(1 + 2\tau_l)^c \mid c = 1, ..., 2^{k-1}\},$$

for $l = 2, ..., r$

$$S_{1l} = \{(1 + \tau_l u_1)^{d_1} \mid d_1 = 1, ..., 2^k\},$$

$$S_{2l} = \{(1+\tau_l u_1+\tau_l u_2)^{d_2} \mid d_2 = 1, ..., 2^k\},$$

$$\vdots$$

$$S_{hl} = \{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{d_h} \mid d_h = 1, ..., 2^k\},$$

we see that H, Q, T_l, S_{1l}, S_{2l},...,S_{hl} are all cyclic subgroups of the group $1+J$ and they are of the orders indicated by their definition. Since

$$|< -1+2^{k-1}\tau_l >| \,.\, |< 1+4y >| \,.\prod_{l=2}^{r} |< 1+2\tau_l >| \,.\prod_{l=1}^{r} |< 1+\tau_l u_1 >|$$
$$.\prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2 >| \,...\prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h >|$$
$$= 2^{(k(h+1)-1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r+1$ subgroups H, Q, T_l, S_{1l}, S_{2l},...,S_{hl} is direct. So their product exhausts the group $1+J$.

Suppose p is odd.

We notice that for every $l = 1, ..., r$, $(1+p\tau_l)^{p^{k-1}} = 1$, $(1+\tau_l u_1)^{p^k} = 1$, $(1+\tau_l u_1+\tau_l u_2)^{p^k} = 1$,...,$(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{p^k} = 1$.

Now, for positive integers $a_l, b_{1l}, b_{2l}, ..., b_{hl}$ with $a_l \leq p^{k-1}$, $b_{il} \leq p^k$ for $1 \leq i \leq h$, we notice that

$$\prod_{l=1}^{r}\{(1+p\tau_l)^{a_l}\}.\prod_{l=1}^{r}\{(1+\tau_l u_1)^{b_{1l}}\}$$
$$.\prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2)^{b_{2l}}\}...\prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{b_{hl}}\} = 1$$

will imply $a_l = p^{k-1}$, $b_{il} = p^k$, for every $l = 1, ..., r$ and $1 \leq i \leq h$. If we set

$$T_l = \{(1+p\tau_l)^a \mid a = 1, ..., p^{k-1}\},$$

$$S_{1l} = \{(1+\tau_l u_1)^{b_1} \mid b_1 = 1, ..., p^k\},$$

$$S_{2l} = \{(1+\tau_l u_1+\tau_l u_2)^{b_2} \mid b_2 = 1, ..., p^k\},$$

$$\vdots$$

$$S_{hl} = \{(1+\tau_l u_1+\tau_l u_2+...+\tau_l u_h)^{b_h} \mid b_h = 1, ..., p^k\},$$

we see that T_l, S_{1l}, S_{2l},...,S_{hl} are all cyclic subgroups of the group $1+J$ and they are of the orders indicated by their definition. Since

$$\prod_{l=1}^{r} |< 1 + p\tau_l >| \,.\, \prod_{l=1}^{r} |< 1 + \tau_l u_1 >|$$
$$.\prod_{l=1}^{r} |< 1 + \tau_l u_1 + \tau_l u_2 >| \ldots \prod_{l=1}^{r} |< 1 + \tau_l u_1 + \tau_l u_2 + \ldots + \tau_l u_h >|$$
$$= p^{(k(h+1)-1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups T_l, S_{1l}, S_{2l},...,S_{hl} is direct. So their product exhausts the group $1 + J$. □

Theorem 1. *The unit group R^* of the commutative completely primary finite ring R of characteristic p^k with maximal ideal J such that $J^{k+1} = (0)$ and $J^k \neq (0)$, with invariants p, k, r and h, where $p \in J$, is a direct product of cyclic groups as follows:*

i) If $h \geq 1$, $r \geq 1$ and char$R = p$, *then*

$$R^* \cong \boldsymbol{Z}_{p^r-1} \times \underbrace{\boldsymbol{Z}_p^r \times \ldots \times \boldsymbol{Z}_p^r}_{h \text{ copies}}.$$

ii) If $h \geq 1$, $r \geq 1$ and char$R = p^2$, *then*

$$R^* \cong \boldsymbol{Z}_{p^r-1} \times \boldsymbol{Z}_p^r \times \underbrace{\boldsymbol{Z}_{p^2}^r \times \ldots \times \boldsymbol{Z}_{p^2}^r}_{h \text{ copies}}.$$

iii) If $h \geq 1$, $r \geq 1$ and char $R = p^k; k \geq 3$, *then*

$$R^* \cong \begin{cases} \boldsymbol{Z}_{2^r-1} \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_{2^{k-2}} \times \boldsymbol{Z}_{2^{k-1}}^{r-1} \times \underbrace{\boldsymbol{Z}_{2^k}^r \times \ldots \times \boldsymbol{Z}_{2^k}^r}_{h \text{ copies}} & \text{if } p = 2, \\ \boldsymbol{Z}_{p^r-1} \times \boldsymbol{Z}_{p^{k-1}}^r \times \underbrace{\boldsymbol{Z}_{p^k}^r \times \ldots \times \boldsymbol{Z}_{p^k}^r}_{h \text{ copies}} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. It follows from the proofs of Propositions 3, 4 and 5 □

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