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THE SOLVABILITY OF A NEW HIGHER-ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATION

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Abstract: This paper studies the solvability of a new higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn}\cdots\Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n+b_nx_{n-d}))\big)\Big) + f\big(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \cdots, x_{n-r_{sn}}\big) = 0, \quad n \geq n_0,$$

where $n_0 \geq 0, n \geq 0, d > 0, k > 0, s > 0$ are integers, $\{a_{in}\}_{n\geq n_0} (i = 1, 2, \cdots, k)$ and $\{b_n\}_{n\geq n_0}$ are real sequences, $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\geq n_0} \subseteq \mathbb{Z}$ and $f : \{n : n \geq n_0\} \times \mathbb{R}^s \to \mathbb{R}$ is a mapping. Some sufficient conditions for existence of nonoscillatory solutions of this equation are established and expatiated through five theorems according to the range of value of the sequence b_n.

AMS Subject Classification: 34K15, 34C10
Key Words: nonoscillatory solution, higher-order neutral delay difference equation, contraction mapping

1. Introduction and Preliminaries

Recently, the interest in the study of qualitative analysis of difference equations has been increasing (see [1]-[16] and references cited therein). Some authors have payed their attention to various difference equations. For example,

$$\Delta(a_n\Delta x_n) + p_nx_{g(n)} = 0, \quad n \geq 0, \quad \text{see [13]}, \tag{1.1}$$

$$\Delta(a_n\Delta x_n) = q_nx_{n+1}, \quad \Delta(a_n\Delta x_n) = q_nf(x_{n+1}), \quad n \geq 0, \quad \text{see [10]}, \tag{1.2}$$

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$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0, \quad \text{see [5]}, \tag{1.3}$$

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1, \quad \text{see [9]}, \tag{1.4}$$

$$\Delta^2(x_n - px_{n-\tau}) = \sum_{i=1}^{m} q_i f_i(x_{n-\sigma_i}), \quad n \geq n_0, \quad \text{see [8]}, \tag{1.5}$$

$$\Delta(a_n \Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \cdots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \quad \text{see [7]}, \tag{1.6}$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} = 0, \quad n \geq n_0, \quad \text{see [14]}, \tag{1.7}$$

$$\Delta^m(x_n + c_n x_{n-k}) + p_n f(x_{n-r}) = 0, \quad n \geq n_0, \quad \text{see [3], [4], [12]}, \tag{1.8}$$

$$\Delta^m(x_n + cx_{n-k}) + \sum_{s=1}^{u} p_n^s f_s(x_{n-r_s}) = q_n, \quad n \geq n_0, \quad \text{see [15]}, \tag{1.9}$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \quad n \geq n_0. \quad \text{see [16]}. \tag{1.10}$$

The purpose of this paper is to investigate the following higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn} \cdots \Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n + b_n x_{n-d}))\big)\Big) + f\big(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \cdots, x_{n-r_{sn}}\big) = 0, \quad n \geq n_0, \tag{1.11}$$

where $n_0 \geq 0, n \geq 0, d > 0, k > 0, s > 0$ are integers, $\{a_{in}\}_{n\geq n_0} (i = 1, 2, \cdots, k)$ and $\{b_n\}_{n\geq n_0}$ are real sequences, $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\geq n_0} \subseteq \mathbb{Z}$ and $f : \{n : n \geq n_0\} \times \mathbb{R}^s \to \mathbb{R}$ is a mapping. Clearly, difference equations (1.1)-(1.10) are special cases of equation (1.11). By using Banach contraction principle, the existence of nonoscillatory solutions of equation (1.11) is established.

The forward difference Δ is defined as usual, i.e., $\Delta x_n = x_{n+1} - x_n$. The higher-order difference for a positive integer m is defined as $\Delta^m x_n = \Delta(\Delta^{m-1}x_n), \Delta^0 x_n = x_n$. Throughout this paper, assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ stand for the sets of all positive integers and integers, respectively, $\alpha = \inf\{n - r_{jn} : 1 \leq j \leq s, n \geq n_0\}$, $\beta = \min\{n_0 - d, \alpha\}$, $\lim_{n\to\infty}(n - r_{jn}) = +\infty, 1 \leq j \leq s$, l_β^∞ denotes the set of real sequences defined on the set of positive integers larger than β where any individual sequence is bounded with respect to the usual supremum norm $||x|| = \sup_{n\geq\beta}|x_n|$ for $x = \{x_n\}_{n\geq\beta} \in l_\beta^\infty$. It is well known that l_β^∞ is a Banach space under the supremum norm. Let

$$A(M, N) = \Big\{x = \{x_n\}_{n\geq\beta} \in l_\beta^\infty : M \leq x_n \leq N, n \geq \beta\Big\} \quad \text{for} \quad N > M > 0.$$

Obviously, $A(M,N)$ is a bounded closed and convex subset of l^∞_β. Put

$$\overline{b} = \limsup_{n\to\infty} b_n \quad \text{and} \quad \underline{b} = \liminf_{n\to\infty} b_n.$$

By a solution of equation (1.11), we mean a sequence $\{x_n\}_{n\geq\beta}$ with a positive integer $N_0 \geq n_0+d+|\alpha|$ such that equation (1.11) is satisfied for all $n \geq N_0$. As is customary, a solution of equation (1.11) is said to be oscillatory about zero, or simply oscillatory if the terms x_n of the sequence $\{x_n\}_{n\geq\beta}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. Existence of Nonoscillatory Solutions

In this section, a few sufficient conditions of the existence of nonoscillatory solutions of equation (1.11) are given.

Theorem 1. *Assume that there exist constants b, M and N with $N > M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1\leq i\leq k), \{b_n\}_{n\geq n_0}, \{h_n\}_{n\geq n_0}, \{q_n\}_{n\geq n_0}$ such that for $n \geq n_0$*

$$|b_n| \leq b < \frac{N-M}{2N}, \text{ eventually}, \tag{2.1}$$

$$\begin{aligned} &\big|f(n,u_1,u_2,\cdots,u_s) - f(n,v_1,v_2,\cdots,v_s)\big| \\ &\quad \leq h_n \max\big\{|u_i - v_i| : u_i, v_i \in [M,N], 1\leq i\leq s\big\}, \end{aligned} \tag{2.2}$$

$$\big|f(n,u_1,u_2,\cdots,u_s)\big| \leq q_n,\ u_i \in [M,N], 1\leq i\leq s, \tag{2.3}$$

$$\sum_{t=n_0}^{\infty} \max\Big\{\frac{1}{|a_{it}|}, h_t, q_t : 1\leq i\leq k\Big\} < +\infty. \tag{2.4}$$

Then equation (1.11) has a nonoscillatory solution in $A(M,N)$.

Proof. Choose $L \in (M+bN, N-bN)$. By (2.1) and (2.4), an integer $N_0 > n_0+d+|\alpha|$ can be chosen such that

$$|b_n| \leq b < \frac{N-M}{2N}, \quad \forall n \geq N_0, \tag{2.5}$$

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\qquad\qquad \leq \min\{L-bN-M, N-bN-L\}, \end{aligned} \tag{2.6}$$

and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|}<1-b. \tag{2.7}$$

Define a mapping $T : A(M,N) \to X$ by

$$(Tx)_n=\begin{cases} L-b_nx_{n-d}+(-1)^k\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots \\ \quad\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{f(t,x_{t-r_{1t}},x_{t-r_{2t}},\cdots,x_{t-r_{st}})}{\prod_{i=1}^{k}a_{it_i}}, & n\ge N_0, \\ (Tx)_{N_0}, & \beta\le n<N_0, \end{cases} \tag{2.8}$$

for all $x \in A(M,N)$.

For every $x \in A(M,N)$ and $n \ge N_0$, it follows from (2.3), (2.5) and (2.6) that

$$\begin{aligned}(Tx)_n &\ge L-bN-\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\left|f(t,x_{t-r_{1t}},x_{t-r_{2t}},\cdots,x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\\ &\ge L-bN-\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{q_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\ge M,\end{aligned}$$

and

$$(Tx)_n\le L+bN+\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{q_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\le N.$$

That is, $T(A(M,N)) \subseteq A(M,N)$. It is claimed that T is a contraction mapping on $A(M,N)$. In fact, (2.2), (2.5) and (2.7) guarantee that for any $x, y \in A(M,N)$ and $n \ge N_0$

$$\begin{aligned}\left|(Tx)_n-(Ty)_n\right| &\le |b_n||x_{n-d}-y_{n-d}|+\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\\ &\quad\frac{\left|f(t,x_{t-r_{1t}},x_{t-r_{2t}},\cdots,x_{t-r_{st}})-f(t,y_{t-r_{1t}},y_{t-r_{2t}},\cdots,y_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\\ &\le b\|x-y\|+\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t\max\{|x_{t-r_{jt}}-y_{t-r_{jt}}|:1\le j\le s\}}{\left|\prod_{i=1}^{k}a_{it_i}\right|}\\ &\le b\|x-y\|+\|x-y\|\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|}=k\|x-y\|,\end{aligned}$$

where $k = b+\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|}<1$. This implies that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in A(M, N),$$

that is, T is a contraction mapping on $A(M, N)$. Consequently T has a unique fixed point $x \in A(M, N)$, which is a bounded nonoscillatory solution of equation (1.11). This completes the proof. □

Theorem 2. *Assume that there exist constants M and N with $N > \frac{2-\underline{b}}{1-\overline{b}}M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k), \{b_n\}_{n\geq n_0}, \{h_n\}_{n\geq n_0}, \{q_n\}_{n\geq n_0}$, satisfying (2.2)-(2.4) and*

$$b_n \geq 0, \text{ eventually, and } 0 \leq \underline{b} \leq \overline{b} < 1. \tag{2.9}$$

Then equation (1.11) has a nonoscillatory solution in $A(M, N)$.

Proof. Choose $L \in (M + \frac{1+\overline{b}}{2}N, N + \frac{\underline{b}}{2}M)$. By (2.9) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b}}{2} \leq b_n \leq \frac{1+\overline{b}}{2}, \ \forall n \geq N_0, \tag{2.10}$$

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \leq \min\Big\{L - M - \frac{1+\overline{b}}{2}N, N - L + \frac{\underline{b}}{2}M\Big\}, \tag{2.11}$$

and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t}{|\prod_{i=1}^{k} a_{it_i}|} < \frac{1-\overline{b}}{2}. \tag{2.12}$$

Define a mapping $T : A(M, N) \to X$ as (2.8). The rest of the proof is similar to that in Theorem 1. This completes the proof. □

Theorem 3. *Assume that there exist constants M and N with $N > \frac{2+\overline{b}}{1+\underline{b}}M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k), \{b_n\}_{n\geq n_0}, \{h_n\}_{n\geq n_0}, \{q_n\}_{n\geq n_0}$, satisfying (2.2)-(2.4) and*

$$b_n \leq 0, \text{ eventually, and } -1 < \underline{b} \leq \overline{b} \leq 0. \tag{2.13}$$

Then equation (1.11) has a nonoscillatory solution in $A(M, N)$.

Proof. Choose $L \in (\frac{2+\overline{b}}{2}M, \frac{1+\underline{b}}{2}N)$. By (2.13) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\frac{\underline{b}-1}{2} \leq b_n \leq \frac{\overline{b}}{2}, \ \forall n \geq N_0, \tag{2.14}$$

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{q_t}{|\prod_{i=1}^{k}a_{it_i}|}\le \min\Big\{L-\frac{2+\overline{b}}{2}M,\frac{1+\underline{b}}{2}N-L\Big\}, \tag{2.15}$$

and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{|\prod_{i=1}^{k}a_{it_i}|}<\frac{1+\underline{b}}{2}. \tag{2.16}$$

Define a mapping $T:A(M,N)\to X$ as (2.8). The rest of the proof is similar to that in Theorem 1. This completes the proof. □

Theorem 4. *Assume that there exist constants M and N with $N > \frac{\underline{b}(\overline{b}^2-\underline{b})}{\overline{b}(\underline{b}^2-\overline{b})}M>0$ and sequences $\{a_{in}\}_{n\ge n_0}(1\le i\le k),\{b_n\}_{n\ge n_0},\{h_n\}_{n\ge n_0},\{q_n\}_{n\ge n_0}$, satisfying (2.2)-(2.4) and*

$$b_n>1,\ \textit{eventually},\ 1<\underline{b}\ \textit{and}\ \overline{b}<\underline{b}^2<+\infty. \tag{2.17}$$

Then equation (1.11) has a nonoscillatory solution in $A(M,N)$.

Proof. Take $\varepsilon\in(0,\underline{b}-1)$ sufficiently small satisfying

$$1<\underline{b}-\varepsilon<\overline{b}+\varepsilon<(\underline{b}-\varepsilon)^2 \tag{2.18}$$

and

$$\big((\overline{b}+\varepsilon)(\underline{b}-\varepsilon)^2-(\overline{b}+\varepsilon)^2\big)N>\big((\overline{b}+\varepsilon)^2(\underline{b}-\varepsilon)-(\underline{b}-\varepsilon)^2\big)M. \tag{2.19}$$

Choose $L\in\big((\overline{b}+\varepsilon)M+\frac{\overline{b}+\varepsilon}{\underline{b}-\varepsilon}N,(\underline{b}-\varepsilon)N+\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M\big)$. By (2.18) and (2.4), an integer $N_0>n_0+d+|\alpha|$ can be chosen such that

$$\underline{b}-\varepsilon<b_n<\overline{b}+\varepsilon,\quad \forall b\ge N_0, \tag{2.20}$$

$$\begin{aligned}&\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{q_t}{|\prod_{i=1}^{k}a_{it_i}|}\\&\le\min\Big\{\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}L-(\underline{b}-\varepsilon)M-N,\frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M+(\underline{b}-\varepsilon)N-L\Big\},\end{aligned} \tag{2.21}$$

and

$$\sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{|\prod_{i=1}^{k}a_{it_i}|}<\underline{b}-\varepsilon-1. \tag{2.22}$$

Define a mapping $T:A(M,N)\to X$ by

$$(Tx)_n=\begin{cases}\frac{L}{b_{n+d}}-\frac{x_{n+d}}{b_{n+d}}+\frac{(-1)^k}{b_{n+d}}\sum_{t_1=n}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\\ \quad\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{f(t,x_{t-r_{1t}},x_{t-r_{2t}},\cdots,x_{t-r_{st}})}{\prod_{i=1}^{k}a_{it_i}}, & n\ge N_0,\\ (Tx)_{N_0}, & \beta\le n<N_0,\end{cases} \tag{2.23}$$

for all $x \in A(M, N)$. The rest of the proof is similar to that in Theorem 1. This completes the proof. ☐

Theorem 5. *Assume that there exist constants M and N with $N > \frac{1+\underline{b}}{1+\overline{b}} M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k), \{b_n\}_{n\geq n_0}, \{h_n\}_{n\geq n_0}, \{q_n\}_{n\geq n_0}$, satisfying (2.2)-(2.4) and*

$$b_n < -1, \text{ eventually, } -\infty < \underline{b} \text{ and } \overline{b} < -1. \tag{2.24}$$

Then equation (1.11) has a nonoscillatory solution in $A(M, N)$.

Proof. Take $\epsilon \in \big(0, -(1+\overline{b})\big)$ sufficiently small satisfying

$$\underline{b} - \epsilon < \overline{b} + \epsilon < -1 \tag{2.25}$$

and

$$(1 + \overline{b} + \epsilon)N < (1 + \underline{b} - \epsilon)M. \tag{2.26}$$

Choose $L \in \big((1+\overline{b}+\epsilon)N, (1+\underline{b}-\epsilon)M\big)$. By (2.25) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\underline{b} - \epsilon < b_n < \overline{b} + \epsilon, \quad \forall n \geq N_0, \tag{2.27}$$

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\leq \min\Big\{\Big(\overline{b} + \epsilon + \frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}\Big)M - \frac{\overline{b}+\epsilon}{\underline{b}-\epsilon}L, L - (1+\overline{b}+\epsilon)N\Big\}, \end{aligned} \tag{2.28}$$

and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t}{|\prod_{i=1}^{k} a_{it_i}|} < -\overline{b} - \epsilon - 1. \tag{2.29}$$

Define a mapping $T : A(M, N) \to X$ as (2.23). The rest of the proof is similar to that in Theorem 1. This completes the proof. ☐

Remark 1. Theorems 1-5 extend and improve Theorem 1 of Cheng [5], Theorems 2.3-2.7 of Liu, Xu and Kang [7] and corresponding theorems in [3], [4], [8]-[16].

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