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ON THE GREEN FUNCTION OF THE OPERATOR $(\circledcirc + m^4)^k$ RELATED TO THE KLEIN-GORDON OPERATOR AND HELMHOLTZ OPERATOR

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Abstract: In this paper, we study the Green function of the operator $(\circledcirc + m^4)^k$ which is iterated $k-$times and m is positive real number. At first we find the Green function of the operator $(\circledcirc + m^4)^k$ and after that we apply such a Green function to solve the solution of the equation $(\circledcirc + m^4)^k G(x) = f(x)$, where f is a generalized function and $G(x)$ is an unknown for $x \in \mathbb{R}^n$.

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1. Introduction

Consider the ultra-hyperbolic operator iterated k times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k .$$

S.E. Trione [7] has shown that the generalized function $R_{2k}^H(x)$ defined by (2.2) is the unique elementary solution of the operator $\Box^k$, that is $\Box^k R_{2k}^H(x) = \delta$,

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where $x \in R^n$ the n dimensional Euclidian space. Also M. Aguirre Tellez (see [4], pp. 147-149) has proved that $R_{2k}(x)$ exists only if n is an odd with p odd and q even, or only n is an even with p odd and q odd.

Furthermore, we also know that the function $R^e_{2k}(x)$ defined by (2.4) is an elementary solution of the Laplace operator

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k .$$

Next, the operator $\circledcirc^k$ has been first introduced by W. Satsanit and is defined by

$$\begin{aligned} \circledcirc^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial {x_j}^2}\right)^2\right]^k \\ &= \left[\left(\frac{\triangle + \square}{2}\right)^2 + \left(\frac{\triangle - \square}{2}\right)^2\right]^k \\ &= \left(\frac{\triangle^2 + \square^2}{2}\right)^k, \end{aligned} \tag{1.1}$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}, \tag{1.2}$$

$\square$ is the ultra-hyperbolic operator defined by

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}, \tag{1.3}$$

and

$$\begin{aligned} \left(\circledcirc + m^4\right)^k &= \left(\frac{\triangle^2 + \square^2}{2} + m^4\right)^k \\ &= \left(\frac{1}{2}\left(\triangle^2 + m^4\right) + \frac{1}{2}\left(\square^2 + m^4\right)\right)^k \\ &= \left(\frac{1}{2}((\square + m^2)^2 - m^2(\square + \triangle) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)^k . \end{aligned} \tag{1.4}$$

Here $\square + m^2$ is the ultra-hyperbolic Klein-Gordon operator and $\triangle + m^2$ is the Helmholtz operator and they are defined by

$$\square + m^2 = \sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial {x_j}^2} + m^2 \tag{1.5}$$

and

$$\triangle + m^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} + m^2. \tag{1.6}$$

The purpose of this work is to find the Green function of the operator $\left(\circledcirc + m^4\right)^k$, that is

$$\left(\circledcirc + m^4\right)^k G(x) = \delta(x), \tag{1.7}$$

where $G(x)$ is the Green function, δ is the Dirac-delta function, k is a nonnegative integer and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Before finding the Green function of (1.7), the following definitions and concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denoted by

$$u = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2. \tag{2.1}$$

The nondegenerated quadratic form and $p+q = n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and $\overline{\Gamma}_+$ denote it closure. For any complex number α, define the function

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}. \tag{2.3}$$

The function $R_\alpha^H(u)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [3].

It is well known that $R_\alpha^H(u)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$ and suppose supp $R_\alpha^H(u) \subset \bar{\Gamma}_+$, that is supp $R_\alpha^H(u)$ is compact.

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and the function $R_\alpha^e(v)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R_\alpha^e(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \tag{2.4}$$

where

$$v = x_1^2 + x_2^2 + ... + x_n^2, \tag{2.5}$$

$$H_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.6}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

By (2.2) and (2.3) with $q = 0$, $u^{\frac{\alpha-n}{2}}$ reduces to $v^{\frac{\alpha-p}{2}}$ where $v = x_1^2 + x_2^2 + ... + x_p^2$ and $K_n(\alpha)$ reduces to $K_p(\alpha) = \frac{\pi^{\frac{p-1}{2}}\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{p-\alpha}{2})}$.

By using the formulae

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)$$

and

$$\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi \sec(\pi z),$$

we obtain

$$K_p(\alpha) = \frac{1}{2}\sec\left(\frac{\pi\alpha}{2}\right) H_p(\alpha),$$

where H_p is defined by (2.6) with $n = p$.

Thus for $q = 0$

$$R_\alpha^H(u) = \frac{v^{\frac{\alpha-p}{2}}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right)\frac{v^{\frac{\alpha-p}{2}}}{H_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) R_\alpha^e(v),$$

where $v = x_1^2 + x_2^2 + ... + x_p^2$.

Thus, if $\alpha = 2k$ then

$$R_{2k}^H(u) = 2(-1)^k R_{2k}^e(v) \tag{2.7}$$

for $q = 0$ and $v = x_1^2 + x_2^2 + ... + x_p^2$.

Definition 2.3. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and the function $W_\alpha^H(u, m)$ be defined by

$$W_\alpha^H(u, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\alpha}{2} + r\right)}{r!\Gamma\left(\frac{\alpha}{2}\right)} \left(m^2\right)^r R_{\alpha+2r}^H(u), \tag{2.8}$$

where $R_{\alpha+2r}^H(u)$ is defined by (2.2) and m is a nonnegative real number.

Lemma 2.1. *Given the equation*

$$\left(\Box + m^2\right)^k K(x) = \delta(x), \tag{2.9}$$

where $\left(\Box + m^2\right)^k$ *is the operator iterated* $k-$*times defined by (1.3) then* $K(x) = W_{2k}^H(u, m)$ *is an elementary solution or Green function of (2.9), where* $W_{2k}^H(u, m)$

is defined by (2.8) with $\alpha = 2k$.

Proof. See [5]. □

From (1.2) if $q = 0$ then $\left(\Box + m^2\right)^k$ reduces to the Helmholtz operator $\left(\triangle + m^2\right)^k$, where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}.$$

Thus, by (2.9), for $q = 0$ we obtain the equation

$$\left(\triangle + m^2\right)^k K(x) = \delta(x) \tag{2.10}$$

with an elementary solution $K(x) = W_{2k}^H(v, m)$, where

$$v = x_1^2 + x_2^2 + \cdots + x_p^2.$$

Now,

$$W_{2k}^H(v, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{2k}{2} + r\right)}{r! \Gamma\left(\frac{2k}{2}\right)} \left(m^2\right)^r R_{2k+2r}^H(v). \tag{2.11}$$

In general, if $p = n$

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and

$$v = x_1^2 + x_2^2 + \cdots + x_n^2,$$

we obtain

$$K(x) = W_{2k}^e(v, m) \tag{2.12}$$

as an elementary solution of (2.10).

Lemma 2.2. (The Existence of the Convolution $W_{2k}^H(u, m) * W_{2k}^e(v, m)$) *The convolution $W_{2k}^H(u, m) * W_{2k}^e(v, m)$ exists and is a tempered distribution, where $W_{2k}^H(u, m)$ and $W_{2k}^e(v, m)$ are defined by (2.8) and (2.11) with $\alpha = 2k$.*

Proof. From (2.8) and (2.11), we have

$$\begin{aligned} W_{2k}^H(u, m) * W_{2k}^e(v, m) &= \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k + r)}{r! \Gamma(k)} \left(m^2\right)^r R_{2k+2r}^H(u)\right) \\ &\quad * \left(\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k + r)}{r! \Gamma(k)} \left(m^2\right)^r R_{2k+2r}^e(v)\right) \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(k + s)}{s! \Gamma(k)} \left(m^2\right)^s \frac{(-1)^r \Gamma(k + r)}{r! \Gamma(k)} \end{aligned}$$

$$\left(m^2\right)^r \cdot 2(-1)^{k+r}\left[R^H_{2k+2s}(u) * R^e_{2k+2r}(v)\right].$$

Kananthai (see [2]) has shown $R^H_{2k+2s}(u) * R^e_{2k+2r}(v)$ exists and is a temperd distribution. It follows that $W^H_{2k}(u,m) * W^e_{2k}(v,m)$ exists and also is a tempered distribution. □

3. Main Results

Theorem 3.1. *Given the equation*

$$\left(\circledcirc + m^4\right)^k G(x) = \delta(x), \tag{3.1}$$

then

$$G(x) = \left(W^H_{4k}(u,m) * W^e_{4k}(v,m))\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.2}$$

is a Green function for the operator $\left(\circledcirc + m^4\right)^k$ *iterated* $k-$*times, where* $\circledcirc$ *is defined by (1.4),* m *is a nonnegative real number and*

$$S(x) = \left(W^e_4(v,m) - m^2 H(u,v) + W^H_4(u,m)\right), \tag{3.3}$$

where

$$H(u,v) = \left(W^H_4(u,m) * W^e_4(v,m) * (R^H_{-2}(u) + R^e_{-2}(v)\right),$$

$S^{*k}(x)$ *denotes the convolution of* S *itself* $k-$*times,* $\left(S^{*k}(x)\right)^{*-1}$ *denotes the inverse of* $S^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. From (1.4), we have

$$\begin{aligned}\left(\circledcirc + m^4\right)^k G(x) &= \left(\frac{1}{2}\left(\Box + m^2\right)^2 - m^2\left(\Box + \triangle\right) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)^k G(x)\\ &= \delta(x),\end{aligned}$$

or we can write

$$\begin{aligned}&\left(\frac{1}{2}\left(\Box + m^2\right)^2 - m^2\left(\Box + \triangle\right) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)\\ &\qquad \cdot\left(\frac{1}{2}\left(\Box + m^2\right)^2 - m^2\left(\Box + \triangle\right) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)^{k-1} G(x) = \delta(x).\end{aligned}$$

By Lemma 2.3 with $k = 2$, we obtain $W^H_4(u,m) * W^e_4(v,m)$ exists and is a tempered distribution.

Convolving both sides of the above equation by $W^H_4(u,m) * W^e_4(v,m)$,

$$\left(\frac{1}{2}(\Box+m^2)^2-m^2(\Box+\triangle)+\frac{1}{2}(\triangle+m^2)^2\right)\cdot W_4^H(u,m)*W_4^e(v,m)$$
$$\cdot\left(\frac{1}{2}(\Box+m^2)^2-m^2(\Box+\triangle)+\frac{1}{2}(\triangle+m^2)^2\right)^{k-1}G(x)$$
$$=\delta(x)*W_4^H(u,m)*W_4^e(v,m),\quad (3.4)$$

or

$$((\frac{1}{2}\left(\Box+m^2\right)^2W_4^H(u,m)*W_4^e(v,m)-m^2\left(\Box+\triangle\right)W_4^H(u,m)*W_4^e(v,m)$$
$$+\frac{1}{2}\left(\triangle+m^2\right)^2\cdot W_4^H(u,m)*W_4^e(v,m))\cdot((\frac{1}{2}\left(\Box+m^2\right)^2-m^2\left(\Box+\triangle\right)$$
$$+\frac{1}{2}\left(\triangle+m^2\right)^2)^{k-1}G(x)=\delta(x)*W_4^H(u,m)*W_4^e(v,m).\quad (3.5)$$

By Lemma 2.1, we have $K(x)=W_{2k}^H(u,m)$ and $I(x)=W_{2k}^e(v,m)$ are elementary solution of (2.9) and (2.10) respectively. Here $W_{2k}^H(u,m)$ is defined by (2.8) with $\alpha=2k$ and $W_{2k}^e(v,m)$ is defined by (2.11) with $\alpha=2k$. That is

$$\left(\Box+m^2\right)^kK(x)=\delta(x)$$

and

$$\left(\triangle+m^2\right)^kI(x)=\delta(x).$$

Now, by (2.8)

$$W_4^H(u,m)=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(2+r\right)}{r!\Gamma\left(2\right)}\left(m^2\right)^rR_{4+2r}^H(u)$$

and

$$\Box W_4^H(u,m)=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(2+r\right)}{r!\Gamma\left(2\right)}\left(m^2\right)^r\Box R_{4+2r}^H(u).$$

By Trione [7] and Telles (see [4]) we have

$$\Box R_{4+2r}^H(u)=R_{4+2r-2}^H(u)=R_{4+2r}^H(u)*R_{-2}^H(u).$$

Thus

$$\Box W_4^H(u,m)=W_4^H(u,m)*R_{-2}^H(u).$$

Similarly, by (2.11),

$$\triangle W_4^e(v,m)=\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma\left(2+r\right)}{r!\Gamma\left(2\right)}\left(m^2\right)^r2\triangle R_{4+2r-2}^e(v),$$

and we have

$$\triangle R_{4+2r}^e(v)=R_{4+2r-2}^e(v)=R_{4+2r}^e(v)*R_{-2}^e(v),$$

$$\triangle W_4^e(v,m) = W_4^e(v,m) * R_{-2}^e(v).$$

So equation (3.5) becomes

$$\Big(W_4^e(v,m) - m^2 H(u,v) + W_4^H(u,m)\Big) \cdot \left(\frac{1}{2}\left(\Box + m^2\right)^2 - m^2\left(\triangle + \Box\right) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)^{k-1} \cdot G(x) = W_4^H(u,m) * W_4^e(v,m),$$

where

$$H(u,v) = \left(W_4^H(u,m) * W_4^e(v,m) * (R_{-2}^H(u) + R_{-2}^e(v)\right).$$

So, we can write by Lemma 2.4 and (3.2)

$$S(x) * \left(\frac{1}{2}\left(\Box + m^2\right)^2 - m^2\left(\triangle + \Box\right) + \frac{1}{2}\left(\triangle + m^2\right)^2\right)^{k-1} \cdot G(x) = W_4^H(u,m) * W_4^e(v,m).$$

Keeping on convolving both sides of the above equation by $W_4^H(u,m) * W_4^e(v,m)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * G(x) = \left(W_4^H(u,m) * W_4^e(v,m)\right)^{*k}. \tag{3.6}$$

The symbol $*k$ denotes the convolution of itself $k-$times. By Telles [4], we have

$$\left(W_4^H(u,m) * W_4^e(v,m)\right)^{*k} = W_{4k}^H(u,m) * W_{4k}^e(v,m).$$

Thus,

$$S^{*k}(x) * G(x) = W_{4k}^H(u,m) * W_{4k}^e(v,m).$$

We consider the function $S^{*k}(x)$, since $W_4^H(u,m) * W_4^e(v,m)$ by Lemma 2.2 and also $\delta(x)$ is a tempered distribution. By Kananthai (1997) (see [2]) $R_{-2}^H(u) + R_{-2}^e(v)$ is a tempered distribution with compact support. Thus $S(x)$ defined by (3.2) is a tempered distribution and by Lemma 2.2 again, we obtain $S^{*k}(x)$ is a tempered distribution.

Now, $W_{4k}^H(u,m) * W_{4k}^e(v,m) \in \mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution. Thus $W_{4k}^H(u,m) * W_{4k}^e(v,m) \in \mathcal{D}'_{\mathcal{R}}$. It follows that $W_{4k}^H(u,m) * W_{4k}^e(v,m)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence, by Zemanian (see [6], pp. 150-151), the equation (3.6) has a unique solution

$$G(x) = \left(W_{4k}^H(u,m) * W_{4k}^e(v,m)\right) * \Big(S^{*k}(x)\Big)^{*-1},$$

where $\left(S^{*k}(x)\right)^{*-1}$ is an inverse of S^{*k} in the convolution algebra, $G(x)$ is called the Green function of the operator $\left(\circledcirc + m^4\right)^k$.

Since $W^H_{4k}(u,m) * W^e_{4k}(v,m)$ and $\left(S^{*k}(x)\right)^{*-1}$ are tempered distribution, then by Donoghue (see [1], p. 152) $\left(W^H_{4k}(u,m) * W^e_{4k}(v,m)\right) * \left(S^{*k}(x)\right)^{*-1}$ is tempered distribution. It follows that $G(x)$ is a tempered distribution. □

Theorem 3.2. (An Application of Green Function) *Given the equation*

$$\left(\circledcirc + m^4\right)^k K(x) = f(x), \tag{3.7}$$

where $f(x)$ is a generalized function, $K(x)$ is an unknown function and $x \in \mathbb{R}^n$. Then

$$K(x) = G(x) * f(x)$$

is a unique solution of the equation (3.7) where $G(x)$ is a Green function for $\left(\circledcirc + m^4\right)^k$.

Proof. By convolving both sides of (3.7) by $G(x)$, where $G(x)$ is a Green function for $\left(\circledcirc + m^4\right)^k$ in Theorem 3.1, we have

$$G(x) * \left(\circledcirc + m^4\right)^k K(x) = G(x) * f(x),$$

$$\left(\circledcirc + m^4\right)^k G(x) * K(x) = G(x) * f(x),$$

$$\delta(x) * K(x) = G(x) * f(x).$$

Thus,

$$K(x) = G(x) * f(x).$$

Sine $G(x)$ is unique, by Theorem 3.1. It follows that $K(x) = G(x) * f(x)$ is unique. □

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