International Journal of Pure and Applied Mathematics
Volume 57 No. 3 2009, 435-445

WAVELET SETS FROM THREE-INTERVAL SCALING SETS AND CONTRACTIBILITY

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Abstract: In this paper, we determine wavelet sets arising from three-interval scaling sets and study contractibility of such families of wavelet sets.

AMS Subject Classification: 42C15, 42C40
Key Words: scaling set, wavelet set, contractibility

1. Introduction

A function $\psi \in L^2(\mathbb{R})$ is called an *orthonormal wavelet* (or simply a *wavelet*) if $\{2^{\frac{n}{2}}\psi(2^n \cdot +k)\}_{n,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$. It is known that the Lebesgue measure $|supp\,\widehat{\psi}|$ of the support of $\widehat{\psi}$ of a wavelet ψ is at least 2π. A wavelet ψ for which $|supp\,\widehat{\psi}|$ is 2π, is said to be a *Minimally Supported Frequency* (MSF) wavelet [2], [5]. For an MSF wavelet ψ, there is a measurable set W of measure 2π such that $\widehat{\psi} = \chi_W$. The set W is called a *wavelet set* [2], [5], [6]. In case ψ is an MRA wavelet, W is said to be a *wavelet set associated with an* MRA.

Received: November 17, 2009

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Ha, Kang, Lee and Seo [4] characterized wavelet sets in $\mathbb{R}$ which are unions of three disjoint intervals. These are precisely

$$W(j,p) \equiv \Big[-2(1-\frac{2p+1}{2^{j+1}-1})\pi,\ -(1-\frac{2p+1}{2^{j+1}-1})\pi\Big] \cup \Big[\frac{2(p+1)\pi}{2^{j+1}-1}, \frac{2(2p+1)\pi}{2^{j+1}-1}\Big] \cup \Big[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}, \frac{2^{j+2}(p+1)\pi}{2^{j+1}-1}\Big], \quad \text{(A)}$$

for natural numbers j and p satisfying $j \geq 2$ and $1 \leq p \leq 2^j - 2$ together with $-W(j,p)$. They obtain that $W(j,p)$ for an odd p is a non-MRA wavelet set. Determination of wavelet sets of $\mathbb{R}$ which are unions of pairwise disjoint intervals attracted several workers who made significant contribution towards this end [2, 4].

A measurable set S of $\mathbb{R}$ containing a neighborhood of zero and contained in $2S$ is a *scaling set* if each element of S uniquely corresponds with an element of $[a, a+2\pi)$, $a \in \mathbb{R}$, by a 2π-integral translate and vice versa [1, 3]. It is known that if a wavelet set W arises from a scaling set S, that is to say that $W = 2S \backslash S$, then $S = \bigcup_{j\in\mathbb{N}} 2^{-j}W$ [1]. Such a wavelet set is associated with an MRA [1, 3]. By an *n-interval wavelet set* (*scaling set*) we mean a wavelet set (scaling set) consisting of n intervals of $\mathbb{R}$ which are mutually separated.

In an earlier paper, we characterized scaling sets having three-intervals [8]. In this article, we determine all wavelet sets which arise from these scaling sets. Such wavelet sets lie in $\mathcal{W}_n^s$ for $n = 3, 4, 5$ or 6, where $\mathcal{W}_n^s$ denotes the collection of all n-interval wavelet sets arising from scaling sets. We provide examples of wavelet sets lying in $\mathcal{W}_n^s$ for $n = 3, 4, 5$ or 6 which do not arise from these scaling sets. Furthermore, it is obtained that several wavelet sets provided by Dai and Larson [2] are determined by three-interval scaling sets. We find that there are seven kinds of four-interval wavelet sets arising from three-interval scaling sets, the collection of which are denoted by $\mathcal{W}_{4,i}^s$, $i = 1, 2, ..., 7$. The corresponding sets of wavelets are denoted by $\mathcal{W}_{4,i}$. Also, three-interval scaling sets give rise to eight kinds of five-interval wavelet sets $\mathcal{W}_{5,j}^s$, $j = 1, 2, ..., 8$, and three kinds of six-interval wavelet sets $\mathcal{W}_{6,k}^s$, $k = 1, 2, 3$. We denote corresponding families of wavelets by $\mathcal{W}_{5,j}$ and $\mathcal{W}_{6,k}$.

In Section 2, we describe the characterization of a three-interval scaling sets obtained by us [8] together with a sketch of proof for completion. Section 3 is devoted to the determination of all wavelet sets arising from three-interval scaling sets, while in Section 4, we obtain that families $\mathcal{W}_{4,i}^s$, $i = 1, 2, ..., 7$, $\mathcal{W}_{5,j}^s$, $j = 1, 2, ..., 8$, $\mathcal{W}_{6,k}^s$, $k = 1, 2, 3$ and the corresponding families of wavelets are contractible. It is worth to mention that considering the set $\mathcal{W}$ of all orthonormal wavelets as a subspace formed by the induced metric of $L^2(\mathbb{R})$, the completeness property and the topological properties like connectedness and pathconnectedness for $\mathcal{W}$ and certain of its subsets have drawn attention of many contributors in the field of wavelets during the past one decade [7, 9].

2. A Characterization of Three-Interval Scaling Sets

Recall that a measurable set S of $\mathbb{R}$ containing a neighborhood of zero is a scaling set iff $S \subset 2S$, and S is 2π-translation congruent to $[\delta,\, \delta + 2\pi)$, where $\delta \in \mathbb{R}$ [1, 3]. That S is 2π-translation congruent to $[\delta,\, \delta + 2\pi)$ means that each element of S uniquely corresponds with an element of $[\delta,\, \delta + 2\pi)$, where $\delta \in \mathbb{R}$, by a 2π-integral translate and vice versa. A scaling set having three intervals is called a *three-interval scaling set*, the collection of which is denoted by S_3. Below we quote a characterization of three-interval scaling sets [8].

Result 2.1. *There are precisely three kinds of three-interval scaling sets described as follows:*

(i) $S_3^1 = \{[\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma) \cup [\alpha + 2\pi,\, \beta + 2\pi) : 2\alpha \geq \gamma,\, \alpha + 2\pi \geq 2\beta,\, 2\gamma \geq \beta + 2\pi\}$,

(ii) $S_3^2 = \{[\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma) : 2\alpha \geq \gamma,\, \alpha + 2\pi \geq 2\gamma\}$, *and*

(iii) $S_3^3 = \{[\beta - 4\pi,\, \gamma - 4\pi) \cup [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) : 2\gamma \leq \alpha + 2\pi, \gamma \leq 2\beta, 2\alpha \leq \beta\}$,

where $0 < \alpha < \beta < \gamma < 2\pi$.

Sketch of the Proof. For $\alpha, \beta, \gamma \in (0, 2\pi)$ such that $\alpha < \beta < \gamma$, we consider the set $[\alpha,\, \beta) \cup [\beta,\, \gamma) \cup [\gamma,\, \alpha + 2\pi)$ in $\mathbb{R}$. We obtain the three-interval scaling sets of $\mathbb{R}$ by translating $[\alpha,\, \beta)$, $[\beta,\, \gamma)$ and $[\gamma,\, \alpha + 2\pi)$ by integral multiples of 2π. Since a scaling set of $\mathbb{R}$ has to contain a neighborhood of zero, we have to translate the interval $[\gamma,\, \alpha + 2\pi)$ by -2π. Translate $[\alpha,\, \beta)$ by $2n\pi$ and $[\beta,\, \gamma)$ by $2m\pi$ such that the three-intervals $[\gamma - 2\pi,\, \alpha)$, $[\alpha + 2n\pi,\, \beta + 2n\pi)$ and $[\beta + 2m\pi,\, \gamma + 2m\pi)$ are mutually separated. Write

$$S = [\gamma - 2\pi,\, \alpha) \cup [\alpha + 2n\pi,\, \beta + 2n\pi) \cup [\beta + 2m\pi,\, \gamma + 2m\pi).$$

By construction, S is 2π-translation congruent to $[\alpha,\, \alpha + 2\pi)$. Therefore, in order that S be a scaling set, due to the condition that $S \subset 2S$, we are left with following choices:

Choice I. $m = 0$ and $n = 1$,

Choice II. $m = 0$ and $n = -1$, and

Choice III. $m = -2$ and $n = -1$,

which provide, respectively, scaling sets of the kind in S_3^1, S_3^2 or S_3^3.

Conversely, suppose $S = I_1 \cup I_2 \cup I_3$, where $I_1 = [a, b)$, $I_2 = [c, d)$, and $I_3 = [e, f)$ of $\mathbb{R}$, is a three-interval scaling set of $\mathbb{R}$. Assume that I_1 contains a neighborhood of zero.

(i) In case $a < b < c < d < e < f$, we obtain scaling sets of the kind in S_3^1 with $\alpha = b$, $\beta = c$ and $\gamma = a + 2\pi$.

(ii) In case $e < f < a < b < c < d$, we obtain scaling sets of the kind in S_3^2 with $\alpha = b$, $\beta = c$ and $\gamma = a + 2\pi$.

(iii) In case $e < f < c < d < a < b$, we obtain scaling sets of the kind in S_3^3 with $\alpha = b$, $\beta = d + 2\pi$ and $\gamma = a + 2\pi$.

Remark 2.2. S_3^1, S_3^2 and S_3^3 partition S_3 and $S_3^1 = -S_3^3$.

3. Wavelet Sets Determined by Three-Interval Scaling Sets

In this section, we determine those subsets of $\mathcal{W}_n^s$ for $n = 3, 4, 5$ or 6 which arise from three-interval scaling sets. Recall S_3^1, S_3^2 and S_3^3.

In order that the wavelet set $W = 2S \backslash S$ determined by a three-interval scaling set S consists of exactly three-intervals, we should have either of the following conditions:

(i) $2\alpha = \gamma$, $\alpha + 2\pi = 2\beta$, and $2\gamma = \beta + 2\pi$ in S_3^1 which provide $\alpha = \frac{6\pi}{7}$, $\beta = \frac{10\pi}{7}$, and $\gamma = \frac{12\pi}{7}$,

(ii) $2\gamma = \alpha + 2\pi, \gamma = 2\beta$, and $2\alpha = \beta$ in S_3^3 which provide $\alpha = \frac{2\pi}{7}$, $\beta = \frac{4\pi}{7}$, and $\gamma = \frac{8\pi}{7}$.

From (i) we obtain the three-interval wavelet set as follows:

$$W_1 = [-\frac{4\pi}{7}, -\frac{2\pi}{7}) \cup [\frac{6\pi}{7}, \frac{10\pi}{7}) \cup [\frac{40\pi}{7}, \frac{48\pi}{7})$$

(which is same for $j = 2, p = 2$ in (A)) while from (ii),

$$W_2 = [-\frac{48\pi}{7}, -\frac{40\pi}{7}) \cup [-\frac{10\pi}{7}, -\frac{6\pi}{7}) \cup [\frac{2\pi}{7}, \frac{4\pi}{7}) \equiv -W_1.$$

Example 3.1. Since the scaling set S for a wavelet set W is unique and $S = \bigcup_{j \in \mathbb{N}} 2^{-j} W$, the scaling set of $W(3, 6) \equiv [-\frac{4\pi}{15}, -\frac{2\pi}{15}] \cup [\frac{14\pi}{15}, \frac{26\pi}{15}] \cup [\frac{208\pi}{15}, \frac{224\pi}{15}]$ comes out to be

$$[-\frac{2\pi}{15}, \frac{14\pi}{15}) \cup [\frac{26\pi}{15}, \frac{28\pi}{15}) \cup [\frac{52\pi}{15}, \frac{56\pi}{15}) \cup [\frac{104\pi}{15}, \frac{112\pi}{15})$$

which consists of four-intervals. Further, since S is a 2π-translation congruent to $[-\frac{2\pi}{15}, \frac{28\pi}{15})$, it follows that $W(3,6)$ is an MRA wavelet set which does not arise through a three-interval scaling set.

Example 3.2. Since a non-MRA wavelet set does not possess a scaling set, $W(j,p)$, when p is odd does not arise from a three-interval scaling set.

Proposition 3.3. *There are seven kinds of four-interval wavelet sets in* $\mathcal{W}_4^s$ *determined by three-interval scaling sets as listed below:*

(i) $\mathcal{W}_{4,1}^s \equiv \Big\{[2(\alpha-2\pi),\ \alpha-2\pi)\cup[\frac{\alpha}{2}, \frac{\alpha}{4}+\pi)\cup[\frac{\alpha}{4}+3\pi, 2\alpha)\cup[2(\frac{\alpha}{2}+2\pi), 2(\frac{\alpha}{4}+3\pi)) : \alpha\in(\frac{12\pi}{7}, 2\pi)\Big\}$,

(ii) $\mathcal{W}_{4,2}^s \equiv \Big\{[2(2\alpha-2\pi),\ 2\alpha-2\pi)\cup[\alpha, 4\alpha-2\pi)\cup[8\alpha-4\pi, \alpha+2\pi)\cup[2(\alpha+2\pi), 8\alpha) : \alpha\in(\frac{2\pi}{3}, \frac{6\pi}{7})\Big\}$,

(iii) $\mathcal{W}_{4,3}^s \equiv \Big\{[2(\frac{\alpha}{4}-\frac{\pi}{2}),\ \frac{\alpha}{4}-\frac{\pi}{2})\cup[\alpha, \frac{\alpha}{2}+\pi)\cup[\frac{\alpha}{4}+\frac{3\pi}{2}, 2\alpha)\cup[2(\alpha+2\pi), 2(\frac{\alpha}{2}+3\pi)) : \alpha\in(\frac{6\pi}{7}, 2\pi)\Big\}$,

(iv) $\mathcal{W}_{4,4}^s \equiv \Big\{[\frac{-8\pi}{3},\ 2(\alpha-2\pi))\cup[\alpha-2\pi, -\frac{2\pi}{3})\cup[\frac{2\pi}{3}, \alpha)\cup[2\alpha, \frac{8\pi}{3}) : \alpha\in(\frac{2\pi}{3}, \frac{4\pi}{3})\Big\}$,

(v) $\mathcal{W}_{4,5}^s = -\mathcal{W}_{4,1}^s$,

(vi) $\mathcal{W}_{4,6}^s = -\mathcal{W}_{4,2}^s$,

(vii) $\mathcal{W}_{4,7}^s = -\mathcal{W}_{4,3}^s$.

Proof. First we consider S_3^1. An element $S\in S_3^1$ provides a wavelet set W of the form

$$W \equiv 2S\backslash S = \big[2(\gamma-2\pi),\ \gamma-2\pi\big)\cup\big[\alpha,\ \beta\big)\cup\big[\gamma,\ 2\alpha\big)\cup\big[2\beta,\ \alpha+2\pi\big)\cup\big[\beta+2\pi,\ 2\gamma\big) \cup\big[2(\alpha+2\pi),\ 2(\beta+2\pi)\big). \quad (3.1)$$

In case $2\alpha>\gamma$, $2\beta<\alpha+2\pi$, and $2\gamma>\beta+2\pi$, W has six-intervals. Therefore, in order that W be a wavelet set having just four-intervals, we should have either of the following conditions:

(i) $2\alpha=\gamma,\ 2\beta=\alpha+2\pi$, and $2\gamma>\beta+2\pi$,

(ii) $2\alpha=\gamma,\ 2\beta<\alpha+2\pi$, and $2\gamma=\beta+2\pi$,

(iii) $2\alpha>\gamma,\ 2\beta=\alpha+2\pi$, and $2\gamma=\beta+2\pi$.

From condition (i) we obtain

$$W = \big[2(\gamma-2\pi),\ \gamma-2\pi\big)\cup\big[\frac{\gamma}{2},\ \frac{\gamma}{4}+\pi\big)\cup\big[\frac{\gamma}{4}+3\pi,\ 2\gamma\big)$$

$$\cup \big[2(\frac{\gamma}{2}+2\pi),\, 2(\frac{\gamma}{4}+3\pi)\big),$$

where $\gamma \in (\frac{12\pi}{7}, 2\pi)$, which is a wavelet set of the kind as described in $\mathcal{W}^s_{4,1}$. From condition (ii) we get the wavelet sets of the kind as described in $\mathcal{W}^s_{4,2}$, while from condition (iii) as described in $\mathcal{W}^s_{4,3}$.

Considering S_3^2 and proceeding likewise we get four-interval wavelet sets of the kind as described in $\mathcal{W}^s_{4,4}$. Members of S_3^3 provide wavelet sets as described in $\mathcal{W}^s_{4,5}$, $\mathcal{W}^s_{4,6}$ and $\mathcal{W}^s_{4,7}$. □

Example 3.4. The four-interval wavelet set given by

$$[-2\pi, -\frac{5\pi}{4}) \cup [-\frac{5\pi}{8}, -\frac{\pi}{2}) \cup [\frac{3\pi}{4}, \frac{11\pi}{8}) \cup [\frac{11\pi}{2}, 6\pi)$$

which arises from the four-interval scaling set

$$[-\pi, -\frac{5\pi}{8}) \cup [-\frac{\pi}{2}, \frac{3\pi}{4}) \cup [\frac{11\pi}{8}, \frac{3\pi}{2}) \cup [\frac{11\pi}{4}, 3\pi)$$

is an MRA wavelet set while Journé wavelet set given by

$$[-\frac{32}{7}\pi, -4\pi) \cup [-\pi, -\frac{4}{7}\pi) \cup [\frac{4}{7}\pi, \pi) \cup [4\pi, \frac{32}{7}\pi)$$

is a non-MRA wavelet set both of which do not arise from three-interval scaling sets.

Proposition 3.5. *There are eight kinds of five-interval wavelet sets in* $\mathcal{W}^s_5$ *determined by three-interval scaling sets as listed below*:

$$\text{(i) } \mathcal{W}^s_{5,1} \equiv \Big\{ \big[2(\gamma-2\pi),\, \gamma-2\pi\big) \cup \big[\frac{\gamma}{2},\, \beta\big) \cup \big[2\beta,\, \frac{\gamma}{2}+2\pi\big) \cup \big[\beta+2\pi,\, 2\gamma\big) \cup \big[2(\frac{\gamma}{2}+2\pi),\, 2(\beta+2\pi)\big) : 2\beta < \frac{\gamma}{2}+2\pi, 2\gamma > \beta+2\pi, \gamma < 2\beta \Big\},$$

$$\text{(ii) } \mathcal{W}^s_{5,2} \equiv \Big\{ \big[2(\gamma-2\pi),\, \gamma-2\pi\big) \cup \big[\alpha,\, 2\gamma-2\pi\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[4\gamma-4\pi,\, \alpha+2\pi\big) \cup \big[2(\alpha+2\pi),\, 4\gamma\big) : 2\alpha > \gamma, 4\gamma < \alpha+6\pi, \alpha+2\pi < 2\gamma \Big\},$$

$$\text{(iii) } \mathcal{W}^s_{5,3} \equiv \Big\{ \big[2(\gamma-2\pi),\, \gamma-2\pi\big) \cup \big[\alpha,\, \frac{\alpha}{2}+\pi\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[\frac{\alpha}{2}+3\pi,\, 2\gamma\big) \cup \big[2(\alpha+2\pi),\, 2(\frac{\alpha}{2}+3\pi)\big) : 2\alpha > \gamma, 2\gamma > \frac{\alpha}{2}+3\pi \Big\},$$

$$\text{(iv) } \mathcal{W}^s_{5,4} \equiv \Big\{ \big[2(\alpha-2\pi),\, 2(\beta-2\pi)\big) \cup \big[2(2\alpha-2\pi),\, \alpha-2\pi\big) \cup \big[\beta-2\pi,\, 2\alpha-2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[2\beta,\, 4\alpha\big) : \alpha < \frac{2\pi}{3}, \beta < 2\alpha \Big\},$$

$$\text{(v) } \mathcal{W}^s_{5,5} \equiv \Big\{ \big[2(\alpha-2\pi),\, 2(\beta-2\pi)\big) \cup \big[\beta-2\pi,\, \frac{\alpha}{2}-\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\frac{\alpha}{2}+\pi,\, 2\alpha\big)$$

$$\cup \big[2\beta,\, \alpha+2\pi\big) : \alpha > \frac{2\pi}{3}, 2\beta < \alpha+2\pi\Big\},$$

(vi) $\mathcal{W}^s_{5,6} = -\mathcal{W}^s_{5,1}$,

(vii) $\mathcal{W}^s_{5,7} = -\mathcal{W}^s_{5,2}$,

(viii) $\mathcal{W}^s_{5,8} = -\mathcal{W}^s_{5,3}$,

where $0 < \alpha < \beta < \gamma < 2\pi$.

Proof. Recall equation (3.1). In order that W be a wavelet set having just five-intervals, we should have either of the following conditions:

(i) $2\alpha = \gamma,\ 2\beta < \alpha + 2\pi$, and $2\gamma > \beta + 2\pi$,

(ii) $2\alpha > \gamma,\ 2\beta < \alpha + 2\pi$, and $2\gamma = \beta + 2\pi$,

(iii) $2\alpha > \gamma,\ 2\beta = \alpha + 2\pi$, and $2\gamma > \beta + 2\pi$.

From condition (i) we obtain $W \equiv \big[2(\gamma-2\pi),\, \gamma-2\pi\big) \cup \big[\frac{\gamma}{2},\, \beta\big) \cup \big[2\beta,\, \frac{\gamma}{2} + 2\pi\big) \cup \big[\beta+2\pi,\, 2\gamma\big) \cup \big[2(\frac{\gamma}{2}+2\pi),\, 2(\beta+2\pi)\big)$, where $2\beta < \frac{\gamma}{2} + 2\pi$, $\gamma < 2\beta$, and $2\gamma > \beta + 2\pi$, which is a wavelet set of the kind as described in $\mathcal{W}^s_{5,1}$. From condition (ii) we get the wavelet sets of the kind as described in $\mathcal{W}^s_{5,2}$, while from condition (iii) as described in $\mathcal{W}^s_{5,3}$.

Considering S^2_3 and proceeding likewise, we get five-interval wavelet sets of the kind as described in $\mathcal{W}^s_{5,4}$ and $\mathcal{W}^s_{5,5}$. Members of S^3_3 provide wavelet sets as described in $\mathcal{W}^s_{5,6}$, $\mathcal{W}^s_{5,7}$ and $\mathcal{W}^s_{5,8}$. $\square$

Example 3.6. The five-interval wavelet set given by

$$[-\frac{9\pi}{4}, -\frac{5\pi}{4}) \cup [-\frac{5\pi}{8}, -\frac{9\pi}{16}) \cup [\frac{3\pi}{4}, \frac{11\pi}{8}) \cup [\frac{23\pi}{16}, \frac{3\pi}{2}) \cup [\frac{11\pi}{2}, \frac{23\pi}{4})$$

which arises from the four-interval scaling set

$$[-\frac{9\pi}{8}, -\frac{5\pi}{8}) \cup [-\frac{9\pi}{16}, \frac{3\pi}{4}) \cup [\frac{11\pi}{8}, \frac{23\pi}{16}) \cup [\frac{11\pi}{4}, \frac{23\pi}{8})$$

is an MRA wavelet set while the wavelet set given by

$$[-\frac{8\pi}{3}, -2\pi) \cup [-\pi, -\frac{2\pi}{3}) \cup [\frac{2\pi}{5}, \frac{2\pi}{3}) \cup [\frac{8\pi}{3}, 3\pi) \cup [6\pi, \frac{32\pi}{5})$$

is a non-MRA wavelet set [1] both of which do not arise from three-interval scaling sets.

Proposition 3.7. *There are three kinds of six-interval wavelet sets in* $\mathcal{W}^s_6$ *determined by three-interval scaling sets as listed below*:

(i) $\mathcal{W}^s_{6,1} \equiv \Big\{ \big[2(\gamma-2\pi),\, \gamma-2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, \alpha+2\pi\big)$
$$\cup \big[\beta+2\pi,\, 2\gamma\big) \cup \big[2(\alpha+2\pi),\, 2(\beta+2\pi)\big) : 2\alpha > \gamma,\ 2\beta < \alpha+2\pi,$$
$$2\gamma > \beta + 2\pi\Big\},$$

(ii) $\mathcal{W}^s_{6,2} \equiv \Big\{ \big[2(\alpha-2\pi),\, 2(\beta-2\pi)\big) \cup \big[2(\gamma-2\pi),\, \alpha-2\pi\big) \cup \big[\beta-2\pi,\, \gamma-2\pi\big) \cup$

$\big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, 2\gamma\big) : 2\alpha > \gamma, 2\gamma < \alpha + 2\pi \Big\},$

(iii) $\mathcal{W}^s_{6,3} = -\mathcal{W}^s_{6,1}$,

where $0 < \alpha < \beta < \gamma < 2\pi$.

Proof. Recall equation (3.1). In case $2\alpha > \gamma,\ 2\beta < \alpha+2\pi$ and $2\gamma > \beta+2\pi$, W has six-intervals as given in $\mathcal{W}^s_{6,1}$. Considering S_3^2 and proceeding likewise, we get six-interval wavelet sets of the kind as described in $\mathcal{W}^s_{6,2}$. Members of S_3^3 provide wavelet sets as described in $\mathcal{W}^s_{6,3}$. ☐

Example 3.8. The six-interval wavelet set given by

$$[-\frac{9\pi}{4}, -\frac{3\pi}{2}) \cup [-\frac{3\pi}{4}, -\frac{9\pi}{16}) \cup [\frac{3\pi}{4}, \frac{5\pi}{4}) \cup [\frac{23\pi}{16}, \frac{3\pi}{2}) \cup [\frac{5\pi}{2}, \frac{11\pi}{4}) \cup [\frac{11\pi}{2}, \frac{23\pi}{4})$$

which arises from the four-interval scaling set

$$[-\frac{9\pi}{8}, -\frac{3\pi}{4}) \cup [-\frac{9\pi}{16}, \frac{3\pi}{4}) \cup [\frac{5\pi}{4}, \frac{23\pi}{16}) \cup [\frac{11\pi}{4}, \frac{23\pi}{8})$$

is an MRA wavelet set while the wavelet set given by

$$[-4\pi, -\frac{24\pi}{7}) \cup [-\frac{8\pi}{7}, -\pi) \cup [-\frac{6\pi}{7}, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\pi, \frac{8\pi}{7}) \cup [\frac{24\pi}{7}, 4\pi)$$

is a non-MRA wavelet set [1] both of which do not arise from three-interval scaling sets.

Remark 3.9. Dai and Larson [2, Chapter 4] obtained the following wavelet sets.

$$\text{(i) } E_{ab} = [-2\pi, -2\pi+2a) \cup [-2\pi+2b, -\pi) \cup [-\pi+a, -\pi+b) \cup [\pi, \pi+a) \cup [\pi+b, 2\pi) \cup [2\pi+2a, 2\pi+2b),\ \text{where } 0 < a < b < \frac{\pi}{2},$$

$$\text{(ii) } E_{abc} = [-2\pi+2a, -2\pi+2b) \cup [-2\pi+2c, -\pi+a) \cup [-\pi+b, -\pi+c) \cup [\pi+a, \pi+b) \cup [\pi+c, 2\pi+2a) \cup [2\pi+2b, 2\pi+2c),\ \text{where } 0 < a < b < c < \frac{\pi}{2},$$

$$\text{(iii) } F_a = [-\frac{8\pi}{3}+2a, -2\pi) \cup [-\frac{4\pi}{3}-2a, -\frac{4\pi}{3}+a) \cup [-\pi, -\frac{2\pi}{3}-a) \cup [\frac{2\pi}{3}+a, \pi) \cup [\frac{4\pi}{3}-a, \frac{4\pi}{3}+2a) \cup [2\pi, \frac{8\pi}{3}-2a),\ \text{where } 0 \le a \le \frac{\pi}{3},$$

$$\text{(iv) } G_a = [-\frac{8\pi}{3}, -\frac{8\pi}{3}+2a) \cup [-\frac{4\pi}{3}+a, -\frac{2\pi}{3}) \cup [\frac{2\pi}{3}, \frac{2\pi}{3}+a) \cup [\frac{4\pi}{3}+2a, \frac{8\pi}{3}),\ \text{where } 0 \le a \le \frac{\pi}{3}.$$

It can be noted that each E_{ab} lies in $\mathcal{W}^s_{6,2}$, where $\alpha = \pi, \beta = \pi+a$, and $\gamma = \pi+b$, each E_{abc} lies in $\mathcal{W}^s_{6,2}$, where $\alpha = \pi + a, \beta = \pi + b$, and $\gamma = \pi + c$, each F_a lies in $\mathcal{W}^s_{6,2}$, where $\alpha = \frac{2\pi}{3} + a, \beta = \pi$, and $\gamma = \frac{4\pi}{3} - a$, and each G_a lies in $\mathcal{W}^s_{4,4}$, where $\alpha = \frac{2\pi}{3} + a$.

4. Contractibility of Families of Wavelet Sets

It is known that the metric of $L^2(\mathbb{R})$ arose through its usual norm when restricted to its subspace of characteristic functions $\mathcal{C}$ becomes equivalent to the metric d given by $d(f, g) = |supp\, f \Delta supp\, g|$, where Δ denotes the symmetric difference [7]. We denote the collection of all measurable sets of $\mathbb{R}$ having finite measure equipped with the symmetric difference metric by $\mathcal{M}$. It is straightforward to see that the convergence of functions in $\mathcal{C}$ amounts to that of their supports in $\mathcal{M}$. Also, we can identify $\mathcal{M}$ with $\mathcal{C}$ through characteristic functions. In this section, with the notation described in the Section 3, we show that $\mathcal{W}^s_{4,\,i}$, $i = 1, 2, \cdots, 7$; $\mathcal{W}^s_{5,\,j}$, $j = 1, 2, \cdots, 8$ and $\mathcal{W}^s_{6,\,k}$, $k = 1, 2, 3$ are contractible subspaces of $(\mathcal{M}, \Delta)$. First, we establish a lemma.

Lemma. *(i) Let a, b, a_n, $b_n \in \mathbb{R}$ with $(b - a) > 0$ and $(b_n - a_n) > 0$, where $n \in \mathbb{N}$. Then $\lim_{n\to\infty} [a_n, b_n) = [a, b)$ in $\mathcal{M}$ iff $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.*

(ii) Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{W}^s_{6,1}$, where

$$\begin{aligned} w_n =& \big[2(\gamma_n - 2\pi),\, \gamma_n - 2\pi\big) \cup \big[\alpha_n,\, \beta_n\big) \cup \big[\gamma_n,\, 2\alpha_n\big) \cup \big[2\beta_n,\, \alpha_n + 2\pi\big) \cup \\ & \big[\beta_n + 2\pi,\, 2\gamma_n\big) \cup \big[2(\alpha_n + 2\pi),\, 2(\beta_n + 2\pi)\big) \\ \equiv& \cup_{i=1}^{6} (w_n)_i \qquad \textit{(say)}, \end{aligned}$$

and let

$$\begin{aligned} w =& \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, \alpha + 2\pi\big) \cup \big[\beta + 2\pi,\, 2\gamma\big) \cup \\ & \big[2(\alpha + 2\pi),\, 2(\beta + 2\pi)\big) \\ \equiv& \cup_{i=1}^{6} (w)_i \qquad \textit{(say)}, \end{aligned}$$

be in $\mathcal{W}^s_{6,1}$. Then $\lim_{n\to\infty} w_n = w$ iff $\lim_{n\to\infty} (w_n)_i = (w)_i$, for each $i = 1, 2, ..., 6$ in $\mathcal{M}$.

Proof. For (i), suppose $\lim_{n\to\infty} [a_n, b_n) = [a, b)$ in $\mathcal{M}$. Then for $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|[a_n, b_n)\Delta[a, b)| < \epsilon$, whenever $n > N$, or equivalently, on account of the fact that $[a_n, b_n) \cap [a, b) \neq \phi$, $|a_n - a| + |b_n - b| < \epsilon$, whenever $n > N$. From this we infer that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. For the converse, suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then for $\epsilon > 0$, choose an $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ and $|b_n - b| < \epsilon/2$ whenever $n > N$. This provides

$[a_n,\, b_n) \cap [a,b) \neq \phi$, and hence $|[a_n,\, b_n)\Delta[a,b)| < \epsilon$, whenever $n > N$. Thus $\lim\limits_{n\to\infty}[a_n,\, b_n) = [a,b)$.

For (ii), suppose $\lim\limits_{n\to\infty}(w_n)_i = (w)_i$, for each $i = 1,2,...,6$ in $\mathcal{M}$. Passing to the characteristic functions, we have $\lim\limits_{n\to\infty}\chi_{(w_n)_i} = \chi_{(w)_i}$, for each $i = 1,2,...,6$ in $\mathcal{C}$. Therefore, $\lim\limits_{n\to\infty}\sum_{i=1}^{6}\chi_{(w_n)_i} = \sum_{i=1}^{6}\chi_{(w)_i}$. Thus, $\lim\limits_{n\to\infty}\chi_{\cup_{i=1}^{6}(w_n)_i} = \chi_{\cup_{i=1}^{6}(w)_i}$. Hence $\lim\limits_{n\to\infty} w_n = w$. For the converse, suppose $\lim\limits_{n\to\infty} w_n = w$. Then for $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|(\cup_{i=1}^{6}(w_n)_i)\Delta(\cup_{i=1}^{6}(w)_i)| < \epsilon$, or $|\cup_{i=1}^{6}((w_n)_i\Delta(w)_i)| < \epsilon$, whenever $n > N$, or equivalently, on account of the fact that $(w_n)_i$ intersects only $(w)_i$, for each $i = 1,2,...,6$, $\sum_{i=1}^{6}|(w_n)_i\Delta(w)_i| < \epsilon$, whenever $n > N$. Hence, $\lim\limits_{n\to\infty}(w_n)_i = (w)_i$, for each $i = 1,2,...,6$ in $\mathcal{M}$. □

Proposition 4.1. *$\mathcal{W}^s_{4,\,i}, i = 1,2,\cdots,7$; $\mathcal{W}^s_{5,\,j}, j = 1,2,\cdots,8$ and $\mathcal{W}^s_{6,\,k}, k = 1,2,3$ are contractible subspaces of $(\mathcal{M},\Delta)$.*

Proof. We prove only that $\mathcal{W}^s_{6,1}$ is contractible. Recall $\mathcal{W}^s_{6,1}$. Choose an element w_0 in $\mathcal{W}^s_{6,1}$ given by

$$\begin{aligned} w_0 =& [-\frac{4\pi}{7},\, -\frac{2\pi}{7}) \cup [\frac{13\pi}{14},\, \frac{9\pi}{7}) \cup [\frac{12\pi}{7},\, \frac{13\pi}{7}) \cup [\frac{18\pi}{7},\, \frac{41\pi}{14}) \cup [\frac{23\pi}{7},\, \frac{24\pi}{7}) \cup \\ & [\frac{41\pi}{7},\, \frac{46\pi}{7}) \\ \equiv & \cup_{i=1}^{6}(w_0)_i \qquad \text{(say)} \end{aligned}$$

(by taking $\alpha = \frac{13\pi}{14}$, $\beta = \frac{9\pi}{7}$ and $\gamma = \frac{12\pi}{7}$ in $\mathcal{W}^s_{6,1}$) and a general element w in $\mathcal{W}^s_{6,1}$ given by

$$\begin{aligned} w =& \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, \alpha + 2\pi\big) \cup \big[\beta + 2\pi,\, 2\gamma\big) \cup \\ & \big[2(\alpha + 2\pi),\, 2(\beta + 2\pi)\big) \\ \equiv & \cup_{i=1}^{6}(w)_i \qquad \text{(say)}, \end{aligned}$$

where $2\alpha > \gamma,\ 2\beta < \alpha + 2\pi$ and $2\gamma > \beta + 2\pi$.

Define $F : \mathcal{W}^s_{6,1} \times [0,1] \longrightarrow \mathcal{W}^s_{6,1}$ by

$$F(w,t) = \cup_{i=1}^{6}((1-t)(w_0)_i + t(w)_i),$$

where $t \in [0,1]$. Note that $F(w,0) = w_0$ and $F(w,1) = w$. Since for $a = \frac{13\pi}{14}(1-t) + \alpha t$, $b = \frac{9\pi}{7}(1-t) + \beta t$ and $c = \frac{12\pi}{7}(1-t) + \gamma t$, we have $0 < a < b < c < 2\pi$, $c < 2a$, $2b < a + 2\pi$, and $b + 2\pi < 2c$, it is seen that $F(w,t) \in \mathcal{W}^s_{6,1}$.

For the continuity of F, we choose a sequence $\big((w_n, t_n)\big)_{n\in\mathbb{N}}$ in $\mathcal{W}^s_{6,1} \times [0,1]$ converging to (w,t) in it. Then by repeated use of the lemma, we find that $\lim\limits_{n\to\infty} F(w_n, t_n) = F(w,t)$. Thus F describes the homotopy between the identity map and the constant map on $\mathcal{W}^s_{6,1}$ collapsing $\mathcal{W}^s_{6,1}$ to w_0. □

Corollary 4.2. *The families of wavelets* $\mathcal{W}_{4,i}, i = 1, 2, \cdots, 7$; $\mathcal{W}_{5,j}, j = 1, 2, \cdots, 8$ *and* $\mathcal{W}_{6,k}, k = 1, 2, 3$ *are contractible.*

Acknowledgments

The authors thank their supervisor Professor K.K. Azad for his help and encouragement.

The first author is supported by CSIR, New Delhi.

The second author is supported by UGC, New Delhi.

References

[1] M. Bownik, Z. Rzeszotnik, D. Speegle, A characterization of dimension functions of wavelets, *Appl. Comput. Harmon. Anal.*, **10** (2001), 71-92.

[2] X. Dai, D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Amer. Math. Soc.*, **134**, No. 640 (1998), MR 98m: 47067.

[3] Qing Gu, Deguang Han, On multiresolution analysis (MRA) wavelets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.*, **6** (2000), 437-447.

[4] Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee, Jin Keun Seo, Unimodular wavelets for L^2 and the Hardy space H^2, *Michigan Math. J.*, **41** (1994), 345-361.

[5] E. Hernández, G.L. Weiss, *A First Course on Wavelets*, CRC Press (1996).

[6] E. Ionascu, D. Larson, C. Pearcy, On wavelet sets, *J. Fourier Anal. Appl.*, **4** (1998), 711-721.

[7] D. Speegle, The s-elementary wavelets are path-connected, *Proc. Amer. Math. Soc.*, **127** (1999), 223-233.

[8] N.K. Shukla, G.C.S. Yadav, A characterization of three-interval scaling sets, *Real Anal. Exchange*, **35**, No. 1 (2009).

[9] The Wutam Consortium, Basic properties of wavelets, *J. Fourier Anal. Appl.*, **4** (1998), 575-594.