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AVERAGING OF DIFFERENTIAL EQUATIONS WITH HUKUHARA DERIVATIVE WITH MAXIMA

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Abstract: Justification of the averaging method for differential equations with Hukuhara derivative with maxima is presented. Two theorems on substantiates for differential equations with Hukuhara derivative with maxima are established.

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1. Introduction

Investigation of the cluster of trajectories and constructing the set of attainability for the controlled systems is a very important part in researching of optimal control problems.

A.A. Tolstonogov [17] has established relation between the solution of integral funnel equation of the controlled system and the solution of corresponding equation with Hukuhara derivative.

Differential equations with Hukuhara derivative were presented in the works of F.S. De Blasi and F. Iervolino [5], [6]. In these works were also investigated main properties of the solution of the equation with Hukuhara derivative.

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In the works [12], [13] the justifcation of the averaging method for differential equation with Hukuhara derivative with delay are presented.

Many systems with automatic regulations are describe as differential equations systems with maxima. In the papers [3], [8], [9], [15] the differential equations with maximums for the mathematical simulation of some systems with automatic regulation are presented. Application of the averaging method for differential equations with maximums has been studied extensively by many researchers (see [1], [2], [4], [10], [11] and the references therein).

In the present paper two theorems on the justification of the averaging method for differential equations with Hukuhara derivative and with maxima are established.

Let us give some needed further information from the theory of differential equation with Hukuhara derivative.

Definition 1. (see [7]) Let us assume that sets $A, B \in \text{conv}\ (\mathrm{R}^n)$. Hukuhara difference of sets A and B is the set $C \in \text{conv}\ (\mathrm{R}^n)$, where $A = B + C$. Denoted $A \overset{H}{-} B$.

Hukuhara difference, if it exists, can be defined uniquely. The difference operation is continuous regarding the Hausdorf metrics.

Definition 2. (see [7]) Multivalued mapping $X : \mathrm{R}^1 \to \text{conv}\ (\mathrm{R}^n)$ can be differentiated with respect to Hukuhara at $t_0 \in \mathrm{R}^1$, if there exist the differences $X(t_0 + \Delta t) \overset{H}{-} X(t_0)$ and $X(t_0) \overset{H}{-} X(t_0 - \Delta t)$ for all suficiently small $\Delta t > 0$ and exists an element $D_H X(t_0) \in \text{conv}\ (\mathrm{R}^n)$ that

$$\lim_{\Delta t \to 0_+} h\left(\frac{X(t_0 + \Delta t) \overset{H}{-} X(t_0)}{\Delta t}, D_H X(t_0)\right) = \lim_{\Delta t \to 0_+} h\left(\frac{X(t_0) \overset{H}{-} X(t_0 - \Delta t)}{\Delta t}, D_H X(t_0)\right),$$

where $h(A, B)$ – is Hausdorf distance between the A and B sets, whereas $h(A, B) = \min\{d > 0 | A \subset S_d(B),\ B \subset S_d(A)\}, A, B \in comp(\mathrm{R}^n), S_d(A)$ – closed d -neighborhood of set$A$.

In the work [7] the integral for continuous mapping $X : [a, b] \to \text{conv}\ (\mathrm{R}^n)$ is defined and

$$D_H \int_a^t X(s)\, ds = X(t).$$

2. Main Results

The standart form differential equation with Hukuhara derivative with maxima

$$D_H X(t) = \varepsilon F\left(t, X(t), \max_{\tau\in[g(t),\gamma(t)]} |X(\tau)|\right), \quad X(0) = X_0, \tag{1}$$

is considered. Here ε is a small parameter, $X : \mathrm{R}^1 \to \mathrm{conv}\,(\mathrm{R}^n)$, $D_H X(t)$ is a Hukuhara derivative, $F : \mathrm{R}^1 \times \mathrm{conv}\,(\mathrm{R}^n) \times \mathrm{R}^1 \to \mathrm{conv}\,(\mathrm{R}^n)$, $t \in \left[0, L\varepsilon^{-1}\right]$, $|A| = h(\{0\}, A)$, $g(t)$ and $\gamma(t)$ are known functions, $0 \le g(t) \le \gamma(t) \le t$.

Let us consider the following averaged equation

$$D_H Y(t) = \varepsilon \overline{F}\left(Y(t), \max_{\tau\in[g(t),\gamma(t)]} |Y(\tau)|\right), \quad Y(0) = X_0, \tag{2}$$

for the equation (1). Here

$$\overline{F}(X, z) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} F(s, X, z)ds, \quad t \text{ is some constant.} \tag{3}$$

In (3) integral of multivalued mapping is considered in respect of Riemann-Hukuhara, convergence in is considered in respect of Hausdorf metrics, i.e.

$$\lim_{T\to\infty} h\left(\overline{F}(X, z), \frac{1}{T} \int_t^{t+T} F(s, X, z)ds\right) = 0. \tag{4}$$

Theorem 1. In $Q = \{t \ge 0;\ X, Y \in D \subset \mathrm{conv}\ (\mathrm{R}^n)\}$ the following conditions hold:

1) $F(t, X, z)$ is a continuous mapping on t and

$$|F(t, X, z)| \le M \qquad (|A| = h(\{0\}, A)),$$

$$h\left(F\left(t, X', z'\right), F\left(t, X'', z''\right)\right) \le \lambda\left[h\left(X', X''\right) + |z' - z''|\right];$$

2) limit (3) exists evenly with respect to $X \in D$ and z if $t \ge 0$;

3) $g(t)$ and $\gamma(t)$ are evenly continuous functions and $t \ge 0$ and $0 \le g(t) \le \gamma(t) \le t$, $t \ge 0$;

4) the solution $Y(t)$ of the equation (2) at $\varepsilon \in (0, \sigma], t \ge 0, Y(0) = X_0 \subset D' \subset D$ together with its ρ-neighbourhood belong to D.

Then for any $\eta > 0$, and any $L > 0$ exists $\varepsilon(\eta, L) \in (0, \sigma]$ for which while $\varepsilon \in (0, \varepsilon_0]$ and $t \in \left[0, L\varepsilon^{-1}\right]$ the following estimate is correct:

$$h(X(t), Y(t)) \le \eta, \tag{5}$$

where $X(t), Y(t)-$ are solutions of equations (1) and (2) accordingly such, that

$X(0) = Y(0) = X_0 \in D'$.

Proof. First, we will get that multivalued mapping $\overline{F}(Y, Z)$ satisfies inequalities:

$$\left|\overline{F}(X, z)\right| \le M$$

and

$$h\left(\overline{F}(X', z'), \overline{F}(X'', z'')\right) \le \lambda\left[h\left(X', X''\right) + |z' - z''|\right].$$

From assumption 2) of Theorem 1, for any $\delta > 0$ there exists such $T(\delta) > 0$ that for all $T \ge T(\delta)$ the following estimates hold:

$$h\left(\overline{F}(X, z), \{0\}\right) \le h\left(\overline{F}(X, z), \frac{1}{T}\int_0^T F(s, X, z)ds\right) + h\left(\frac{1}{T}\int_0^T F(s, X, z)ds, \{0\}\right) < \delta + \frac{1}{T}\int_0^T h\left(F(s, X, z), \{0\}\right)ds \le \delta + M,$$

$$h\left(\overline{F}(X', z'), \overline{F}(X'', z'')\right) \le h\left(\overline{F}(X', z'), \frac{1}{T}\int_0^T F(s, X', z')ds\right) + h\left(\frac{1}{T}\int_0^T F(s, X', z')ds, \frac{1}{T}\int_0^T F(s, X'', z'')ds\right) + h\left(\overline{F}(X'', z''), \frac{1}{T}\int_0^T F(s, X'', z'')ds\right)$$

$$< 2\delta + \frac{1}{T}\int_0^T h\left(F(s, X', z'), F(s, X'', z'')\right)ds \le 2\delta + \lambda\left[h\left(X', X''\right) + |z' - z''|\right].$$

The value of δ is arbitrary. Than we get:

$$h\left(\overline{F}(X, z), \{0\}\right) \le M,$$

$$h\left(\overline{F}(X', z'), \overline{F}(X'', z'')\right) \le \lambda\left[h\left(X', X''\right) + |z' - z''|\right].$$

The solutions of equations (1), (2) exist (see [5], [6]) and are equivalent to integral equations:

$$X(t) = X_0 + \varepsilon \int\limits_0^t F\left(s, X(s), \max_{\tau\in[g(s),\gamma(s)]} |X(\tau)|\right) ds, \qquad (6)$$

$$Y(t) = X_0 + \varepsilon \int\limits_0^t \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds. \qquad (7)$$

than we can write

$$\begin{aligned}
&h(X(t), Y(t)) \\
=&\varepsilon h\left(\int\limits_0^t F\left(s, X(s), \max_{\tau\in[g(s),\gamma(s)]} |X(\tau)|\right) ds, \int\limits_0^t \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds\right) \\
=&\varepsilon h\left(\int\limits_0^t F\left(s, X(s), \max_{\tau\in[g(s),\gamma(s)]} |X(\tau)|\right) ds, \int\limits_0^t F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds\right) \\
+&\varepsilon h\left(\int\limits_0^t F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds, \int\limits_0^t \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds\right) \\
&\le \varepsilon\lambda \int\limits_0^t h(X(s), Y(s))\, ds + \varepsilon\lambda \int\limits_0^t \max_{\tau\in[g(s),\gamma(s)]} \big| |X(\tau)| + |Y(\tau)| \big| ds + \mu(\varepsilon) \\
&= \varepsilon\lambda \int\limits_0^t h(X(s), Y(s))\, ds + \varepsilon\lambda \int\limits_0^t \max_{\tau\in[g(s),\gamma(s)]} \big| h(X(\tau), \{0\}) + h(Y(\tau), \{0\}) \big| ds + \mu(\varepsilon) \\
&= \varepsilon\lambda \int\limits_0^t h(X(s), Y(s))\, ds + \varepsilon\lambda \int\limits_0^t \max_{\tau\in[g(s),\gamma(s)]} \big| h(X(\tau), Y(\tau)) \big| ds + \mu(\varepsilon), \qquad (8)
\end{aligned}$$

where $\mu(\varepsilon) = \max_t \beta(t, \varepsilon)$,

$$\beta(t, \varepsilon) = \varepsilon h\left(\int\limits_0^t F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds, \int\limits_0^t \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]} |Y(\tau)|\right) ds\right).$$

Note that

$$\delta\left(t\right)=\max_{0\leq s\leq t}h\left(X(s),Y(s)\right). \tag{9}$$

is the uniform metric.

Then, using this notation and (8) we get:

$$\delta\left(t\right)\leq 2\lambda\varepsilon\int\limits_{0}^{t}\delta(s)ds+\mu\left(\varepsilon\right). \tag{10}$$

We consider $t_i=\frac{iL}{m\varepsilon}$, $i=0,1,...,m$. Let $t\in[t_k,t_{k+1}]$. Then, using properties of the Hausdorf metric, we get:

$$\begin{aligned}
\beta\left(t,\varepsilon\right)=\varepsilon h\Bigg(&\int\limits_{0}^{t}F\left(s,Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds,\\
&\int\limits_{0}^{t}\overline{F}\left(Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds\Bigg)\\
=\varepsilon h\Bigg(&\sum_{i=0}^{k-1}\int\limits_{t_i}^{t_{i+1}}F\left(s,Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds\\
&+\int\limits_{t_k}^{t}F\left(s,Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds,\\
&\sum_{i=0}^{k-1}\int\limits_{t_i}^{t_{i+1}}\overline{F}\left(Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds\\
&+\int\limits_{t_k}^{t}\overline{F}\left(Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds\Bigg)\\
\leq\sum_{i=0}^{k-1}\varepsilon h\Bigg(&\int\limits_{t_i}^{t_{i+1}}F\left(s,Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds,\\
&\int\limits_{t_i}^{t_{i+1}}\overline{F}\left(Y\left(s\right),\max_{\tau\in[g(s),\gamma(s)]}\left|Y(\tau)\right|\right)ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon h\left(\int\limits_{t_k}^{t} F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds,\right.\\
&\qquad\left.\int\limits_{t_k}^{t} \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds\right)\\
&\leq \varepsilon\sum_{i=0}^{k-1}\left\{h\left(\int\limits_{t_i}^{t_{i+1}} F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds,\right.\right.\\
&\qquad\left.\int\limits_{t_i}^{t_{i+1}} F\left(s, Y(t_i), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds\right)\\
&+h\left(\int\limits_{t_i}^{t_{i+1}} \overline{F}_0\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds,\right.\\
&\qquad\left.\int\limits_{t_i}^{t_{i+1}} \overline{F}\left(Y(t_i), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds\right)\\
&+h\left(\int\limits_{t_i}^{t_{i+1}} F\left(s, Y(t_i), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds,\right.\\
&\qquad\left.\left.\int\limits_{t_i}^{t_{i+1}} \overline{F}\left(Y(t_i), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds\right)\right\}\\
&+\varepsilon h\left(\int\limits_{t_k}^{t} F\left(s, Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds,\right.\\
&\qquad\left.\int\limits_{t_k}^{t} \overline{F}\left(Y(s), \max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\right)ds\right)\\
&\leq 2\lambda\sum_{i=0}^{k-1}\varepsilon\left\{\int\limits_{t_i}^{t_{i+1}}\left[h\left(Y(s), Y(t_i)\right)\right.\right. \qquad (11)\\
&\qquad\left.+\left|\max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|-\max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right|\right]ds
\end{aligned}
$$

$$+h\left(\int\limits_{t_i}^{t_{i+1}} F\left(s, Y\left(t_i\right), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds,\right.$$

$$\left.\left.\int\limits_{t_i}^{t_{i+1}} \overline{F}\left(Y\left(t_i\right), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds\right)\right\} + 2M\frac{L}{m}. \qquad (12)$$

Let us estimate every component in (12).

$$\varepsilon\int\limits_{t_i}^{t_{i+1}} h\left(Y\left(s\right), Y\left(t_i\right)\right)ds$$

$$= \varepsilon\int\limits_{t_i}^{t_{i+1}} h\left(Y\left(t_i\right) + \varepsilon\int\limits_{t_i}^{s}\overline{F}\left(Y\left(\tau\right), \max_{\varsigma\in[g(\tau),\gamma(\tau)]}|Y(\varsigma)|\right)d\tau,\ Y\left(t_i\right)\right)ds$$

$$= \varepsilon^2\int\limits_{t_i}^{t_{i+1}}\left|\int\limits_{t_i}^{s}\overline{F}\left(Y\left(\tau\right), \max_{\varsigma\in[g(\tau),\gamma(\tau)]}|Y(\varsigma)|\right)d\tau\right|ds$$

$$\leq \varepsilon^2\int\limits_{t_i}^{t_{i+1}}\int\limits_{t_i}^{s}\left|\overline{F}\left(Y\left(\tau\right), \max_{\varsigma\in[g(\tau),\gamma(\tau)]}|Y(\varsigma)|\right)\right|d\tau ds \leq$$

$$\leq \frac{\varepsilon^2 M}{2}\left(\frac{L}{\varepsilon m}\right)^2 = \frac{ML^2}{2m^2}. \qquad (13)$$

$$\varepsilon\int\limits_{t_i}^{t_{i+1}}\left|\max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)| - \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right|ds$$

$$\leq 2\varepsilon\int\limits_{t_i}^{t_{i+1}}\max_{\tau\in[g(s),\gamma(s)]}|Y(\tau)|\,ds \leq \frac{2ML}{m}. \qquad (14)$$

From assumption 2) of Theorem 1 it follows there exists a decreasing function $\Theta(t) \underset{t\to\infty}{\longrightarrow} 0$ such that

$$\varepsilon h\left(\int\limits_{t_i}^{t_{i+1}} F\left(s, Y\left(t_i\right), \max_{\tau\in[g(t_i),\gamma(t_i)]}|Y(\tau)|\right)ds,\right.$$

$$\int_{t_i}^{t_{i+1}} F_0\left(Y(t_i), \max_{\tau\in[g(t_i),\gamma(t_i)]} |Y(\tau)|\right) ds \Bigg)$$

$$\le \frac{L}{m}\Theta\left(\frac{L}{\varepsilon m}\right). \tag{15}$$

So,

$$\beta(t,\varepsilon) \le \frac{\lambda M L^2}{m} + 4\lambda ML + \frac{2ML}{m} + 2\lambda L\Theta\left(\frac{L}{\varepsilon m}\right). \tag{16}$$

Let choose m_0 such that the following inequality holds

$$\frac{\lambda ML^2}{m_0} + \frac{2ML}{m_0} < \frac{\eta}{2e^{2\lambda L}}. \tag{17}$$

Then we can choose ε_0 such that the following inequality holds

$$4\lambda ML + 2\lambda L\Theta\left(\frac{L}{\varepsilon m_0}\right) < \frac{\eta}{2e^{2\lambda L}}. \tag{18}$$

From (17), (18) and (10) and applying Gronwall-Bellman Lemma, we can get estimate (5).

Theorema 1 is proved. □

Now we consider partially averaged equation

$$D_H Y(t) = \varepsilon G(t, Y(t), Y(\alpha(t))), \tag{19}$$

for the equation (1). Here

$$\lim_{T\to\infty} h\left(\frac{1}{T}\int_t^{t+T} F(s,X,Z)ds, \frac{1}{T}\int_t^{t+T} G(s,X,Z)ds\right) = 0, \tag{20}$$

T is the some constant.

Theorem 2. *Let in $Q = \{t \ge 0;\ X, Y \in D \subset \text{conv}\,(\mathrm{R}^n)\}$ the following conditions hold:*

1) $F(t,X,Z)$,$G(t,X,Z)$ are continuous mappings on t, and

$$h\left(F\left(t,X',Z'\right), F\left(t,X'',Z''\right)\right) \le \lambda\left[h\left(X',X''\right) + h\left(Z',Z''\right)\right],$$

$$h\left(G\left(t,X',Z'\right), G\left(t,X'',Z''\right)\right) \le \lambda\left[h\left(X',X''\right) + h\left(Z',Z''\right)\right],$$

$$|F(t,X,Z)| \le M, \quad |G(t,X,Z)| \le M;$$

2) the limit (20) exists uniformly with respect $X, Z \in D;\ t \ge 0$;

3) $g(t)$ and $\gamma(t)$ are evenly continuous functions and $t \ge 0$ and $0 \le g(t) \le \gamma(t) \le t$, $t \ge 0$;

4) solution $Y(t)$ *of the equation (19) at* $\varepsilon \in (0,\sigma]$, $Y(0) = X_0 \in D' \subset D$; $t \geq 0$ *together with its* ρ *-neighborhood belong to* D.

Then for any $\eta > 0$, *and for any* $L > 0$ *exists such* $\varepsilon(\eta, L) \in (0,\sigma]$ *that for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in \left[0, L\varepsilon^{-1}\right]$ *the following estimate holds:*

$$h\left(X(t), Y(t)\right) \leq \eta,$$

where $X(t), Y(t)$ – *are solutions of equations (1) and (19) accordingly,* $X(0) = Y(0) = X_0 \in D'$.

The proof of Theorem 2 fulfil by analogy with the proof of Theorem 1.

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