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ADMISSIBILITY OF DEGENERATE DIFFERENCE EQUATIONS

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Abstract: We give characterization of the admissibility of degenerate difference equations in terms of solvability of the operator equations $AX - BXS_{\mathcal{M}} = -\delta_0^{\mathcal{M}}$ and $AX - BXS_{\mathcal{M}} = C$.

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1. Introduction

In this paper, we study the admissibility of the nonhomogeneous equation $Bx_{n+1} = Ax_n + y_n$ where A and B are closed, densely defined, linear, generally unbounded operators.

The question of regular admissibility of a subspace $\mathcal{M}$ in $BUC(\mathbb{R}, E)$ (the space of all bounded and uniformly continuous functions on $\mathbb{R}$ with values in the Banach space E) plays an important role in the study of asymptotic behavior of solutions of differential equations. A classical approach to the question of regular admissibility of a subspace $\mathcal{M}$ is to use the so called Green's function. An alternative way to study the admissibility is to use a method introduced by Vu et al [8], which connects the regular admissibility of the space $\mathcal{M}$ with the solvability of the operator equation of the form

$$AX - X\mathcal{D}_{\mathcal{M}} = C, \tag{1}$$

where $\mathcal{D}$ is the differentiation operator, $\mathcal{D}_{\mathcal{M}}$ is its restriction to the space $\mathcal{M}$,

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C is a bounded operator from $\mathcal{M}$ to E and $X : \mathcal{M} \to E$ is the unknown bounded operator. We generalize their result to the nonhomogeneous equation $Bx_{n+1} = Ax_n + y_n$. A sequence $(x_n)_{n\in\mathbb{Z}} \subset E$ is called a solution, if the equation is satisfied for all $n, n \in \mathbb{Z}$.

2. The Generalized Spectrum and Resolvent

Assume that $D := D(A) \cap D(B)$ is dense in E. Let λ be a complex number, such that $(\lambda B - A)$ is one-to-one on D. Define C_λ by

$$D(C_\lambda) = \{x \in D : \text{ there exists a unique } y \in D$$
$$\text{such that } Bx = \lambda By - Ay\}$$
$$C_\lambda x = y.$$

Define the resolvent set of (A, B) by

$$\rho(A, B) := \{\lambda \in \mathbb{C} : C_\lambda \text{ is densely defined and bounded}\},$$

and the spectrum by $\sigma(A, B) := \mathbb{C} \setminus \rho(A, B)$. For $\lambda \in \rho(A, B)$, define the generalized resolvent by $R_\lambda :=$ closure of the operator C_λ. Thus, R_λ is a bounded operator on X.

Moreover, $\rho(A, B)$ is an open set, R_λ satisfies the resolvent identity and is an analytic function in $\rho(A, B)$.

We denote by $Sp(\mathbf{x})$ the spectrum of $\mathbf{x}$. That is $Sp(\mathbf{x})$ consists of λ ($|\lambda| = 1$) such that for every neighborhood $\mathcal{U}$ of λ there exists a numerical sequence $\varphi = (\varphi_n) \in l^1$ with $supp(\hat{\varphi}) \subset \mathcal{U}$ and $\varphi * \mathbf{x}$ is not identically zero, where

$$(\varphi * \mathbf{x})_n \equiv \sum_{i=-\infty}^{\infty} \varphi_{n-i} x_i, n \in \mathbb{Z}.$$

We recall the following theorem from Alsulami et al [1]:

Theorem 1. *Assume that* $\mathbf{x} = (x_n)_{n=-\infty}^{\infty}$ *is a bounded solution of* $Bx_{n+1} = Ax_n + y_n$. *Then,* $Sp(\mathbf{x}) \subset \sigma(A, B) \cup Sp(\mathbf{y})$.

3. Operator Equation $AX - BXD = C$

Let A and B be closed operators on a Banach space E with $D(A) \cap D(B)$ dense in E, D be a closed operator on Banach space F and C be a bounded operator from F to E.

Definition 2. A bounded operator $X : F \to E$ is called a *bounded solution* of the operator equation

$$AX - BXD = C \tag{2}$$

if $Range(X) \subseteq D(B)$ and for each $f \in D(D)$, $Xf \in D(A)$, and $AXf - BXDf = Cf$.

Note that the operator equation $AX - XD = C$ is a special case of equation (2), where B is the identity operator. If B is bounded and invertible, then we can convert equation (2) into equation of the form

$$AX - XB = C \tag{3}$$

by multiplying both sides of (2) by B^{-1}.

Equation (3) has been considered by many authors. It was first studied intensively for bounded operators by Daleckii et al [3] and Rosenblum [5]. For unbounded operators, the case when A and B are generators of C_0-semigroups was considered in Arendt et al [2] and Vu [7] and the general case was considered in Ruess et al [6] and Vu et al [9]. The following Theorem is a brief summary of known results about the unique solvability of equation (3)

Theorem 3. *(1) If A and D are bounded operators, the equation (3) has a unique solution for every bounded C if and only if $\sigma(A) \cap \sigma(D) = \emptyset$. In this case, the solution is given by*

$$X = -\frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} C (\lambda - D)^{-1} d\lambda, \tag{4}$$

where Γ is a Cauchy contour which separates $\sigma(A)$ and $\sigma(D)$ such that $\sigma(D)$ is inside of Γ.

Note. *If we take Γ as a contour around $\sigma(A)$, then the solution is the same integral (now over new Γ) but with positive sign, i.e.*

$$X = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} C (\lambda - D)^{-1} d\lambda. \tag{5}$$

(2) If A and $-D$ are generators of C_0-semigroup $T(t)$ and $S(t)$ with growth bound $\omega(A)$ and $\omega(-D)$ respectively such that $\omega(A) + \omega(-D) < 0$, then for every bounded C, equation (3) has a unique solution, which is given by

$$X = -\int_0^\infty T(t) C S(t) dt. \tag{6}$$

(3) If A and $-D$ are generators of C_0-semigroups with $\sigma(A) \cap \sigma(D) = \emptyset$ and if one of them is the generator of an analytic semigroup, then equation (3)

has a unique solution.

(4) If A is the generator of an exponentially dichotomic C_0-semigroup $T(t)$ and $-D$ is the generator of an isometric C_0-group $S(t)$, then for every C, equation (3) *has a unique solution given by*

$$X = -\int_{-\infty}^{\infty} G_A(t)CS(t)dt, \tag{7}$$

where

$$G_A(t) = \begin{cases} T(t)P, & t \geq 0, \\ -T(t)(I-P), & t < 0, \end{cases}$$

is the Green function. Here, P denotes the dichotomic projection.

(5) If A and D are closed operators with disjoint spectra and if one of them is bounded, say D, then for every C, equation (3) *has a unique solution given by*

$$X = -\frac{1}{2\pi i}\int_{\Gamma}(\lambda - A)^{-1}C(\lambda - D)^{-1}d\lambda,$$

where Γ is a Cauchy contour around $\sigma(D)$ and disjoint from $\sigma(A)$.

(6) If for every bounded operator C, equation (3) *has a unique solution, then $\sigma(A) \cap \sigma(D) = \emptyset$.*

(7) If A and D are closed unbounded operators, then the condition $\sigma(A) \cap \sigma(D) = \emptyset$ is, in general, not sufficient for the solvability of (3)*.*

If B of equation (2) is not invertible, then the situation is quite different, even if B is bounded. We recall the following theorem from Lan [4, Corollary 3.5]:

Theorem 4. *If for every bounded operator $C : F \mapsto E$, the equation*

$$AX - BXD = C$$

has a unique bounded solution, then $\sigma(A, B) \cap \sigma(D) = \emptyset$.

The converse of the above theorem is generally false, even for the case when $B = I$ (see Vu [7, Example 9]). However, it holds in some particular cases. For example, when $B = I$, and A and D are generators of C_0 semigroups, one of which is analytic (see Vu [7]), or both are eventually norm continuous (see Arendt et al [2]).

4. Admissibility of Degenerate Difference Equations

Consider the following equation:

$$Bx_{n+1} = Ax_n + y_n \; \forall n \in \mathbb{Z}. \tag{8}$$

Let $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$, $\mathbf{y} := (y_n)_{n\in\mathbb{Z}}$ and $\mathbf{z} := (z_n)_{n\in\mathbb{Z}}$. We also use the notation $(\mathbf{x})_n \equiv x_n$.

Let $l^\infty(\mathbb{Z}, E)$ be the Banach space of all bounded sequences $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$ with the sup-norm. Consider the shift operator $S : l^\infty(\mathbb{Z}, E) \mapsto l^\infty(\mathbb{Z}, E)$ via $S : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ (thus, $(S\mathbf{x})_n = x_{n+1}$). Below, for convenience of notation, we also use the notation $\varphi_m = S^m$, i.e. $(\varphi_m \mathbf{x})_n = (\mathbf{x})_{n+m}$. Let $\mathcal{M}$ be a subspace of $l^\infty(\mathbb{Z}, E)$ which is translation invariant, i.e. invariant with respect to the shift operator S. We denote the restriction of S on $\mathcal{M}$ by $S_\mathcal{M}$ and define the Dirac operator $\delta_0 : l^\infty(\mathbb{Z}, E) \to E$ by $\delta_0 \mathbf{x} := x_0$.

Definition 5. We call $\mathcal{M} \subset l^\infty(\mathbb{Z}, E)$ a regularly admissible subspace with respect to (8) if for every $\mathbf{y} = (y_n)_{n=-\infty}^{\infty} \in \mathcal{M}$, there exists a unique solution $\mathbf{x} = (x_n)_{n=-\infty}^{\infty} \in \mathcal{M}$ of equation (8).

We make the following assumption: $\mathcal{M}$ is invariant under any bounded linear operator commuting with S (and hence with all φ_n, $n \in \mathbb{Z}$).

Lemma 6. *Let $\mathcal{M}$ be admissible with respect to (8) and assume that the above assumption holds. Let X be the bounded solution operator defined as $X\mathbf{y} = x_0$ where $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$ is the solution of (8) for given $\mathbf{y} := (y_n)_{n\in\mathbb{Z}}$ and S be the shift operator.*

Then, X and S are commute in the sense that for $\mathbf{y} \in \mathcal{M}$, $XS\mathbf{y} = SX\mathbf{y}$.

Proof. Given $\mathbf{y} \in \mathcal{M}$, there exist a unique $\mathbf{x}$ satisfies (8). Also, it is not difficult to see that $(S\mathbf{x})$ is the unique solution of the same equation with $\mathbf{y}$ replaced by $(S\mathbf{y}) := (y_{n+1})_{n\in\mathbb{Z}}$.

Let $\mathbf{y} := (y_n)_{n\in\mathbb{Z}} \in \mathcal{M}$. Then

$$SX\mathbf{y} = (Sx_0) = x_1$$

and

$$XS\mathbf{y} := X(S\mathbf{y})_n = X(\mathbf{y})_{n+1} = (\mathbf{x})_1 = x_1,$$

which implies $XS\mathbf{y} = SX\mathbf{y}$. □

Theorem 7. *Let $\mathcal{M}$ be as above. Then, the following are equivalent:*

(i) $\mathcal{M}$ is regularly admissible.

(ii) The operator equation

$$AX - BXS_{\mathcal{M}} = -\delta_0^{\mathcal{M}} \tag{E1}$$

has a unique bounded solution.

(iii) For every bounded linear operator $C : \mathcal{M} \longrightarrow E$

$$AX - BXS_{\mathcal{M}} = C \tag{E2}$$

has a unique bounded solution.

Proof. (i)$\Longrightarrow$(ii). Let $\mathcal{M}$ be admissible and $G : \mathcal{M} \longrightarrow \mathcal{M}$ be the bounded operator defined by $G\mathbf{y} = \mathbf{x}$ where $\mathbf{x}$ is the unique solution in $\mathcal{M}$ of the equation (8) with given $\mathbf{y} \in \mathcal{M}$ and $(G\mathbf{y})_n = (\mathbf{x})_n$. It is not difficult to see that G is linear and closed. Thus, by the Closed Graph Theorem, G is a bounded linear operator on $\mathcal{M}$. It is also easy to see that G commutes with $S_{\mathcal{M}}$. Define $X\mathbf{y} = (G\mathbf{y})_0 := x_0$. It can been seen that G commutes with φ_n; in particular, $(G\mathbf{y})_n = X\varphi_n\mathbf{y}$. Therefore, from $\mathbf{y} \in D(S_{\mathcal{M}})$ it follows that $(G\mathbf{y})_n \in D(S_{\mathcal{M}})$ and

$$BS_{\mathcal{M}}(G\mathbf{y})_n = (G\mathbf{y})_{n+1} = x_{n+1} = Ax_n + y_n = A(G\mathbf{y})_n + y_n.$$

In particular, by putting $n = 0$ we obtain

$$BS_{\mathcal{M}}(G\mathbf{y})_0 = A(G\mathbf{y})_0 + y_0,$$

i.e.

$$BS_{\mathcal{M}}X\mathbf{y} = AX\mathbf{y} + \delta_0^{\mathcal{M}}\mathbf{y} \ \ \forall \mathbf{y} \in D(S_{\mathcal{M}}).$$

By Lemma 6 we have

$$BXS_{\mathcal{M}}\mathbf{y} = AX\mathbf{y} + \delta_0^{\mathcal{M}}\mathbf{y} \ \ \forall \mathbf{y} \in D(S_{\mathcal{M}}).$$

Hence

$$AX\mathbf{y} - BXS_{\mathcal{M}}\mathbf{y} = -\delta_0^{\mathcal{M}}\mathbf{y} \ \ \forall \mathbf{y} \in D(S_{\mathcal{M}}).$$

Thus, X is a bounded solution of the operator equation (E1).

On the other hand, if X is a bounded solution of the operator equation (E1), then for every $\mathbf{y} \in D(S_{\mathcal{M}})$, the vector $\mathbf{x} \in \mathcal{M}$ defined by $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$ where $x_n = X\varphi_n\mathbf{y} \ \forall n \in \mathbb{Z}$ is a solution of equation (8). Indeed, we have

$$\begin{aligned} BS_{\mathcal{M}}x_n &= BS_{\mathcal{M}}X\varphi_n\mathbf{y} = BXS_{\mathcal{M}}\varphi_n\mathbf{y} \\ &= (AX + \delta_0^{\mathcal{M}})\varphi_n\mathbf{y} = AX\varphi_n\mathbf{y} + \varphi_n\delta_0^{\mathcal{M}}\mathbf{y} = Ax_n + y_n. \end{aligned}$$

Thus, for every $\mathbf{y} \in \mathcal{M}$ the sequence $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$, where $x_n = X\varphi_n\mathbf{y} \ \forall n \in \mathbb{Z}$, is a solution in $\mathcal{M}$ of (8).

Since the solution in $\mathcal{M}$ is unique for every $\mathbf{y} \in \mathcal{M}$, it follows that the solution X of the operator equation (E1) is unique.

(ii)$\Longrightarrow$(iii). It follows from the uniqueness of the solution of the operator equation (E1) that $X = 0$ is the only solution of the following operator equation:

$$AX - BXS_{\mathcal{M}} = 0.$$

From this it follows that a solution of the operator equation $AZ - BZS_{\mathcal{M}} = C$ is unique, if it exists. Let X be the unique bounded solution of the operator equation (E1) and C be given bounded linear operator. By the assumption above, we can define operator

$$Z : \mathcal{M} \to E \text{ by } Z\mathbf{y} = X\widetilde{\mathbf{y}},$$

where $(\widetilde{\mathbf{y}})_n = -C\varphi_n\mathbf{y}$.

Thus

$$AZ\mathbf{y} - BZS_{\mathcal{M}}\mathbf{y} = -AX(C\varphi_n\mathbf{y}) + BXS_{\mathcal{M}}(C\varphi_n\mathbf{y}) = \delta_0(C\varphi_n\mathbf{y}) = C\mathbf{y}, \quad \forall \mathbf{y} \in D(Z),$$

i.e., $AZ - BZS_{\mathcal{M}} = C$.

(iii)$\Longrightarrow$(i). Since for every C there exists a unique solution X of the operator equation (E2), then it follows that $\sigma(A) \cap \sigma(S_{\mathcal{M}}) = \emptyset$ see [2].

But, from part ((i)$\Longrightarrow$(ii)), the sequence $\mathbf{x} := (x_n)_{n\in\mathbb{Z}}$, where $(\mathbf{x})_n = X\varphi_n\mathbf{y}$, is a solution in $\mathcal{M}$ of (8). We need to show that the solution is unique. Suppose there exist two solutions $\mathbf{v} := (v_n)_{n\in\mathbb{Z}}$ and $\mathbf{w} := (w_n)_{n\in\mathbb{Z}}$ in $\mathcal{M}$ of (8). Consider $\mathbf{z} := (z_n)_{n\in\mathbb{Z}}$ where $z_n = v_n - w_n$. Then, $\mathbf{z}$ is a solution of equation $Bx_{n+1} = Ax_n$ in $\mathcal{M}$. Thus, by Theorem 1, $Sp(\mathbf{z}) \subset \sigma(A, B)$. Since $\mathbf{z} \in \mathcal{M}$, we have $Sp(\mathbf{z}) \subset \sigma(S_{\mathcal{M}})$. By Corollary 4, it follows from (iii) that $\sigma(A, B) \cap \sigma(S_{\mathcal{M}}) = \emptyset$. Hence, we have $Sp(\mathbf{z}) = \emptyset$, which implies that $\mathbf{z} = 0$.

Therefore, the solution is unique in $\mathcal{M}$ and $\mathcal{M}$ is regularly admissible. □

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