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CLASSIFICATION OF SOLUTIONS FOR SECOND ORDER QUASI-LINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

S. Lourdu Marian[1 §], A. George Maria Selvam[2]

[1]Department of Master of Computer Applications
Saveetha Engineering College
Thandalam, Chennai, 602 105, INDIA
e-mail: lm_jerrome@rediffmail.com

[2]Department of Mathematics
Sacred Heart College
Tirupattur, Vellore Dist., Tamil Nadu, 635 601, INDIA
e-mail: agm_shc@rediffmail.com

Abstract: In this paper, we investigate the oscillatory and non-oscillatory behavior of solutions of second order quasi-linear neutral delay difference equation

$$\Delta\left[a(n)\left|\Delta\left(x(n)+p(n)x(n-\tau)\right)\right|^{\alpha-1}\Delta\left(x(n)+p(n)x(n-\tau)\right)\right] + q(n+1)f(x(n+1-\sigma))h(\Delta x(n+1))=0,$$

where $n\geq n_0, \alpha>0, \tau\geq 0$ and $\sigma\geq 0$ are constants, $a(n), q(n), p(n)$ are positive real sequences and $f,h\in C(\mathbb{R}:\mathbb{R})$.

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1. Introduction

In this paper, we consider the second order quasi-linear neutral delay difference equations of the form

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§Correspondence author

$$\Delta\left[a(n)\left|\Delta\left(x(n)+p(n)x(n-\tau)\right)\right|^{\alpha-1}\Delta\left(x(n)+p(n)x(n-\tau)\right)\right]$$
$$+\,q(n+1)f(x(n+1-\sigma))h(\Delta x(n+1))=0,\quad n\geq n_0,\quad (1)$$

where $\alpha>0,\tau\geq 0$ and $\sigma\geq 0$ are constants, $a(n),q(n),p(n)$ are positive real sequences and $f,h\in C(\mathbb{R}:\mathbb{R})$. Throughout this paper, we assume that:

(H_1) $h(u)\geq c>0$, for $u\neq 0, a(n)>0$ for all $n\in\mathbb{N}(n_0)$;

(H_2) $f:\mathbb{R}\to\mathbb{R}$ is continuous and $uf(u)>0$, for $u\neq 0$;

(H_3) $f(xy)\leq f(x)f(y), x,y\in\mathbb{R}$;

(H_4) $f(u)-f(v)=g(u,v)(u-v)$ for all $u,v\neq 0$ where g is a non negative function.

We shall use the following notations: $\mathbb{N}=\{0,1,2,\ldots\}$ the set of natural numbers including zero and $\mathbb{N}(k)=\{k,k+1,k+2,\ldots\}$ where $k\in\mathbb{N}$.

In recent years, the study of the oscillation and non oscillation of solutions of difference equations has drawn extensive attention, see, for example, the monographs [1] and [2] and the references cited therein. The papers [4]-[7] deal with the oscillation and non oscillation of quasilinear difference equations. In [3] and [8], the oscillation of discrete analogues of delay differential equations have been studied. In [4] the author studied the second order quasilinear neutral difference equation

$$\Delta\left[a_n\left|\Delta(x_n+p_nx_{g(n)})\right|^{\alpha-1}\Delta(x_n+p_nx_{g(n)})\right]+f(n,x_{\sigma(n)})=0$$

and established sufficient conditions for the oscillation of the above equation. In [5], the author investigated the oscillation of solution of the second order quasilinear neutral difference equation of the form

$$\Delta\left[a_{n-1}\left|\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right|^{\alpha-1}\Delta\left(x_{n-1}+p_{n-1}x_{n-1-\sigma}\right)\right]+q_nf(x_{n-\tau})=0.$$

By a solution of (1), we mean a sequence $\{x(n)\}$ which is defined for $n\geq n_0-m$ where $m=\max\{\tau,\sigma\}$ and satisfies (1) for $n\geq n_0$. A solution $\{x(n)\}$ of equation (1) is said to be *oscillatory* if the terms $x(n)$ of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *non-oscillatory*. A non-oscillatory solution $x(n)$ of (1) is said to be *weakly oscillatory* if $\Delta x(n)$ changes sign for arbitrarily large values of n.

In this paper, we discuss the oscillatory and non-oscillatory behavior of solutions of (1) in two cases: $q\geq 0$ and q changing sign for all large n. We obtain some sufficient conditions in order that every solution of equation (1) is oscillatory and investigate the asymptotic nature of non-oscillatory solutions of (1).

Let S denote the set of all non-trivial solutions of (1). In view of their asymptotic behavior, all solutions of (1) may be a priori divided in to the following classes:

— $M^+ = \{x(n) \in S :$ there exists an integer $N \in \mathbb{N}$ such that $x(n)\Delta x(n) \geq 0$ for all $n \in \mathbb{N}(N)\}$;

— $M^- = \{x(n) \in S : x(n)$ is non-oscillatory and there exists an integer $N \in \mathbb{N}$ such that $x(n)\Delta x(n) \leq 0$ for all $n \in \mathbb{N}(N)\}$;

— $\mathrm{OS} = \{x(n) \in S :$ for every integer $N \in \mathbb{N}$, there exists $n \geq N$ such that $x(n)x(n+1) \leq 0\}$;

— $\mathrm{WOS} = \{x(n) \in S : x(n)$ is non-oscillatory and for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ there exists $n \geq N$ such that $\Delta x(n)\Delta x(n+1) \leq 0\}$.

In the following results we shall provide sufficient conditions which ensure the existence and non-existence of the solutions of (1) in the above four classes when $q(n)$ is non-negative, and changes sign for large $n \in \mathbb{N}$. For the sake of convenience, the sequence $z(n)$ is defined by

$$z(n) = x(n) + p(n)x(n-\tau).$$

2. Main Results

Theorem 1. *Assume that conditions:*

(C_1) $-1 < -r \leq p(n)$;

(C_2) $q(n) \geq 0$, $\limsup\limits_{n\to\infty} \sum\limits_{i=n_0}^{n} q(i+1) = \infty$;

(C_3) $\sum\limits_{i=n_0}^{\infty} \frac{1}{(a(i))^{\frac{1}{\alpha}}} = \infty$

hold. Then for equation (1), we have $M^+ = \emptyset$.

Proof. Suppose that (1) has a solution $x(n) \in M^+$. There is no loss of generality in assuming that there exists $n \in \mathbb{N}(n_0)$ such that $x(n) > 0, \Delta x(n) \geq 0, x(n-m) > 0, \Delta x(n-m) \geq 0$ for all $n \in \mathbb{N}(n_0)$ (the proof is similar if $x(n) < 0, \Delta x(n) \leq 0$ for large n).

If $p(n) \geq 0$, we have $z(n) \geq x(n) > 0$.

If $-1 < -r \leq p(n)$, we claim that $z(n) > 0$ for all $n \in \mathbb{N}(n_1)$.

Otherwise there is a $n_2 \in \mathbb{N}(n_1)$ such that $z(n_2) \leq 0$.

Then $x(n_2) + p(n_2)x(n_2-\tau) \leq 0$, $x(n_2) - rx(n_2-\tau) \leq 0$,

$$x(n_2) \le rx(n_2-\tau),$$
$$x(n_2+\tau) \le rx(n_2) \text{ replacing } n_2 \text{ by } n_2+\tau.$$

By induction,

$$x(n_2+k\tau) \le 0 \text{ for large } k.$$

This contradicts the fact that $x(n) > 0, \Delta x(n) \ge 0$, for $n \in \mathbb{N}(n_1)$. Hence $z(n) > 0$ for all $n \in \mathbb{N}(n_1)$.

From the equation (1) it follows that

$$\Delta[a(n)|\Delta z(n)|^{\alpha-1}\Delta z(n)] = -q(n+1)f(x(n+1-\sigma))h(\Delta x(n+1)) \le 0, \quad n \in \mathbb{N}(n_1). \tag{2}$$

We claim that $\Delta z(n) \ge 0$ for $n \in \mathbb{N}(n_1)$. Otherwise there exists a $n_3 \in \mathbb{N}(n_1)$ such that $\Delta z(n_3) < 0$. From (2), it follows that $\Delta[a(n)|\Delta z(n)|^{\alpha-1}\Delta z(n)] \le 0$.

Summing the above inequality from n_3 to $n-1$, we obtain

$$a(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n) - a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3) \le 0,$$
$$a(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n) \le a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3),$$
$$a(n)\,[\Delta z(n)]^{\alpha} \le a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3),$$
$$[\Delta z(n)]^{\alpha} \le a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3)\frac{1}{a(n)},$$
$$\Delta z(n) \le \left[a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3)\right]^{\frac{1}{\alpha}} \frac{1}{(a(n))^{\frac{1}{\alpha}}}.$$

Summing the above inequality from n_3 to $n-1$, we obtain

$$z(n) - z(n_3) \le \left[a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3)\right]^{\frac{1}{\alpha}} \sum_{i=n_3}^{n-1} \frac{1}{(a(i))^{\frac{1}{\alpha}}},$$
$$z(n) \le z(n_3) - \left[-a(n_3)\,|\Delta z(n_3)|^{\alpha-1}\,\Delta z(n_3)\right]^{\frac{1}{\alpha}} \sum_{i=n_3}^{n-1} \frac{1}{(a(i))^{\frac{1}{\alpha}}}, \quad n \in \mathbb{N}(n_1).$$

By using (C_3), we have $\lim_{n\to\infty} z(n) = -\infty$, which contradicts the fact that $z(n) > 0$ for $n \in \mathbb{N}(n_1)$. So

$$\Delta z(n) \ge 0, \quad n \in \mathbb{N}(n_1). \tag{3}$$

From equation (1) together with (H_1) and (H_2), we obtain

$$\sum_{s=n_1}^{n-1} \frac{\Delta\left[a(s)\,(\Delta z(s))^{\alpha}\right]}{f(x(s-\sigma))} \le -c \sum_{s=n_1}^{n-1} q(s+1),$$

$$\frac{a(n)\,[\Delta z(n)]^{\alpha}}{f(x(n-\sigma))} - \frac{a(n_1)\,[\Delta z(n_1)]^{\alpha}}{f(x(n_1-\sigma))}$$
$$+\sum_{s=n_1}^{n-1} a(s)\,[\Delta z(s)]^{\alpha}\,\frac{g(x(s+1-\sigma),x(s-\sigma))\Delta x(s-\sigma)}{f(x(s+1-\sigma))f(x(s-\sigma))} \le -c\sum_{s=n_1}^{n-1} q(s+1),$$

$$\frac{a(n)\,[\Delta z(n)]^{\alpha}}{f(x(n-\sigma))} \le \frac{a(n_1)\,[\Delta z(n_1)]^{\alpha}}{f(x(n_1-\sigma))} - c\sum_{s=n_1}^{n-1} q(s+1), n\in\mathbb{N}(n_1).$$

From (C_2) we obtain,

$$\lim_{n\to\infty}\inf\frac{a(n)\,[\Delta z(n]^{\alpha}}{f(x(n-\sigma))} = -\infty$$

which contradicts $\Delta z(n)\ge 0$ for $n\in\mathbb{N}(n_1)$.

The proof is complete. □

Theorem 2. *Assume that conditions:*

(C_4) $p(n)$ *is a non-negative and non-decreasing sequence for all* $n\in\mathbb{N}(n_0)$*;*

(C_5) $\lim\limits_{n\to\infty}\sup\sum\limits_{s=n_0}^{n} q(s) = \infty$

hold. Then for equation (1) we have $M^+ = \emptyset$.

Proof. Suppose that (1) has a solution $x(n)\in M^+$.

There is no loss of generality in assuming that there exists $n_1\in\mathbb{N}(n_0)$ such that $x(n)>0, \Delta x(n)\ge 0, x(n-m)>0, \Delta x(n-m)\ge 0$ for all $n\in\mathbb{N}(n_0)$ (the proof is similar if $x(n)<0, \Delta x(n)\le 0$ for all large n).

By condition (C_4), we see that

$$z(n)>0, \Delta z(n)\ge 0, n\in\mathbb{N}(n_1). \tag{4}$$

Similar to the proof of Theorem 1, we obtain

$$\lim_{n\to\infty}\inf\frac{a(n)\,[\Delta z(n]^{\alpha}}{f(x(n-\sigma))} = -\infty$$

which is a contradiction to (4). The proof is complete. □

Theorem 3. *Assume that* $\tau\le\sigma$*. If the function* $\frac{1}{(f(u))^{\frac{1}{\alpha}}}$ *is locally integrable on* $(0,\epsilon)$ *and* $(-\epsilon,0)$ *for all* $\epsilon>0$ *and*

(C_6) $\int\limits_0^{\epsilon}\frac{1}{(f(u))^{\frac{1}{\alpha}}}du<\infty, \int\limits_{-\epsilon}^{0}\frac{1}{(f(u))^{\frac{1}{\alpha}}}du>-\infty;$

(C_7) $p(n)$ *is a non-negative and non-increasing function for all* $n\in\mathbb{N}(n_0)$*;*

(C_8) $\lim_{n\to\infty}\sup \sum_{s=n_1}^{n} \frac{1}{(a(s)f(1+p(s)))^{\frac{1}{\alpha}}} \left(\sum_{i=n_1}^{s-1} q(i+1)\right)^{\frac{1}{\alpha}} = \infty$, $n_1 \in \mathbb{N}(n_0)$,

hold, then for equation (1) we have $M^- = \emptyset$.

Proof. Suppose that equation (1) has a solution $x(n) \in M^-$.

There is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x(n) > 0, \Delta x(n) \leq 0, x(n-m) > 0, \Delta x(n-m) \leq 0$ for all $n \in \mathbb{N}(n_1)$ (the proof is similar if $x(n) < 0, \Delta x(n) \geq 0$ for all large n).

In view of (C_7), we obtain $z(n) > 0, \Delta z(n) \leq 0, n \in \mathbb{N}(n_1)$.

From equation (1) together with (H_1) and (H_2), we obtain

$$\sum_{i=n_1}^{n-1} \frac{\Delta\left[-a(i)\left(-\Delta z(i)\right)^{\alpha}\right]}{f(x(i-\sigma))} \leq -c\sum_{i=n_1}^{n-1} q(i+1),$$

$$\frac{-a(n)\left[-\Delta z(n)\right]^{\alpha}}{f(x(n-\sigma))} - \frac{-a(n_1)\left[-\Delta z(n_1)\right]^{\alpha}}{f(x(n_1-\sigma))} + \sum_{i=n_1}^{n-1} a(i)\left[-\Delta z(i)\right]^{\alpha} \frac{g(x(i+1-\sigma), x(i-\sigma))\Delta x(i-\sigma)}{f(x(i+1-\sigma))f(x(i-\sigma))} \leq -c\sum_{i=n_1}^{n-1} q(i+1),$$

$$\frac{\left[-a(n)\left(-\Delta z(n)\right)^{\alpha}\right]}{f(x(n-\sigma))} \leq -c\sum_{i=n_1}^{n-1} q(i+1), \quad n \in \mathbb{N}(n_1). \tag{5}$$

By (H_1), we have

$$\frac{\left[\left(-\Delta z(n)\right)^{\alpha}\right]}{f(x(n-\sigma))} \geq c\frac{\sum_{i=n_1}^{n-1} q(i+1)}{a(n)}, \quad n \in \mathbb{N}(n_1). \tag{6}$$

But $x(n)$ is non increasing and $\tau \leq \sigma$. Hence $z(n) \leq x(n-\sigma) + p(n)x(n-\sigma) \leq (1+p(n))x(n-\sigma)$.

By using (H_3), we have

$$\begin{aligned} f(z(n)) &\leq f(1+p(n))f(x(n-\sigma)),\\ \frac{f(z(n))}{f(1+p(n))} &\leq f(x(n-\sigma)),\\ f(x(n-\sigma)) &\geq \frac{f(z(n)}{f(1+p(n))}. \end{aligned}$$

Substituting in (6), we get

$$\frac{\left[-\Delta z(n)\right]^{\alpha}}{f(z(n)} \geq c\frac{1}{a(n)f[1+p(n)]}\sum_{i=n_1}^{n-1} q(i+1)$$

$$\frac{-\Delta z(n)}{(f(z(n)))^{\dfrac{1}{\alpha}}} \geq c^{\frac{1}{\alpha}} \left[\frac{\sum\limits_{i=n_1}^{n-1} q(i+1)}{a(n)f[1+p(n)]} \right]^{\frac{1}{\alpha}}, \quad n \in \mathbb{N}(n_1). \tag{7}$$

Now for $z(n+1) \leq u \leq z(n)$, we have $\frac{1}{f(u)} \geq \frac{1}{f(z(n))}$. Hence it follows that

$$\int\limits_{z(n+1)}^{z(n)} \frac{du}{(f(u))^{\frac{1}{\alpha}}} \geq -\frac{\Delta z(n)}{(f(z(n))^{\frac{1}{\alpha}}}. \tag{8}$$

Summing the inequality (7) from n_1 to n and using (8), we obtain

$$\int\limits_{z(n+1)}^{z(n_1)} \frac{du}{(f(u))^{\frac{1}{\alpha}}} \geq c^{\frac{1}{\alpha}} \sum_{s=n_1}^{n} \left[\frac{1}{(a(s)f(1+p(s)))^{\frac{1}{\alpha}}} \left(\sum_{i=n_1}^{s-1} q(i+1) \right)^{\dfrac{1}{\alpha}} \right],$$
$$n \in \mathbb{N}(n_1).$$

By using (C_8), we have

$$\lim_{n\to\infty} \sup \int\limits_{z(n+1)}^{z(n_1)} \frac{du}{(f(u))^{\frac{1}{\alpha}}} = \infty$$

which contradicts (C_6).

The proof is complete. ☐

Theorem 4. *Let $q(n) \geq 0$ for all $n \in \mathbb{N}(n_0)$. If:*

(C_9) $p(n) \equiv p > 0$ *for all* $n \geq n_0$,

then for equation (1) we have $WOS = \emptyset$.

Proof. Let $x(n)$ be a weakly oscillatory solution of (1).

There is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x(n) > 0, x(n-m) > 0$ for all $n \geq n_1$ (the proof is similar if $x(n) < 0$ for all large n).

Then $z(n) > 0$ for all $n \in \mathbb{N}(n_1)$ and weakly oscillatory.

Equation (1) takes the form

$$\Delta \left[a(n) \left| z(n) \right|^{\alpha-1} \Delta z(n) \right] + q(n+1)f(x(n+1-\sigma))h(\Delta x(n+1)) = 0$$

Put $F(n) = a(n)|\Delta z(n)|^{\alpha-1}\Delta z(n)$. Hence $F(n)$ is an oscillatory sequence.

For $n \in \mathbb{N}(n_0)$, $\Delta F(n) = -q(n+1)f(x(n+1-\sigma))h(\Delta x(n+1)) \leq 0$. Hence

F is non-increasing. This is a contradiction to the fact that F is oscillatory.

The proof is complete. □

Theorem 5. *Assume the conditions* (C_3), (C_5) *and* (C_9) *hold. Then every solution of equation (1) is either oscillatory or weakly oscillatory.*

Proof. From Theorem 2, it follows that for equation (1), $M^+ = \emptyset$.

In order to complete the proof it suffices to show that for (1), $M^- = \emptyset$.

Suppose that (1) has a solution $x(n) \in M^-$.

There is no loss of generality in assuming that there exists $n_1 \geq n_0$ such that $x(n) > 0$, $\Delta x(n) \leq 0, x(n-m) > 0, \Delta x(n-m) \leq 0$ for all $n \in \mathbb{N}(n_1)$ (the proof is similar if $x(n) < 0, \Delta x(n) \geq 0$ for all large n). By using (C_9), we see that $z(n) > 0, \Delta z(n) \leq 0, n \in \mathbb{N}(n_1)$. From (2), it follows that

$$\Delta[a(n)|\Delta z(n)|^{\alpha-1}\Delta z(n)] \leq 0 \quad \text{for} \quad n \in \mathbb{N}(n_0).$$

Hence $a(n)|\Delta z(n)|^{\alpha-1}\Delta z(n)$ is a decreasing sequence.

Therefore there exists $n_1 \in \mathbb{N}(n_0)$ such that $\Delta z(n_1) < 0$. Thus:

$$\begin{aligned}
a(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n) \leq& a(n_1)\,|\Delta z(n_1)|^{\alpha-1}\,\Delta z(n_1) \equiv k, \quad n \in \mathbb{N}(n_1),\\
a(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n) \leq& k,\\
-a(n)\,[-\Delta z(n)]^{\alpha} \leq& k,\\
[-\Delta z(n)]^{\alpha} \geq& -\frac{k}{a(n)},\\
[-\Delta z(n)] \geq& \left(\frac{-k}{a(n)}\right)^{\frac{1}{\alpha}},\\
\Delta z(n) \leq& -\left(\frac{-k}{a(n)}\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Summing the above inequality from n_1 to $n-1$, we obtain

$$z(n) - z(n_1) \leq -\sum_{j=n_1}^{n-1} \frac{(-k)^{\frac{1}{\alpha}}}{(a(j))^{\frac{1}{\alpha}}}, \quad z(n) \leq z(n_1) - (-k)^{\frac{1}{\alpha}} \sum_{j=n_1}^{n-1} \frac{1}{(a(j))^{\frac{1}{\alpha}}}.$$

By using (C_3), we have

$$\lim_{n\to\infty} z(n) = -\infty.$$

This is a contradiction and the proof is complete. □

Theorem 6. *Let* $q(n) \geq 0$ *for all* $n \in \mathbb{N}(n_0)$ *and conditions* (C_3), (C_5) *and* (C_9) *hold. Then every solution of equation (1) is oscillatory.*

Proof. Proof follows from Theorems 4 and 5. □

Theorem 7. *Assume conditions* (C_7) *and* (C_8) *are satisfied. Then for every solution* $x(n) \in M^-$ *we have* $\lim_{n\to\infty} x(n) = 0$.

Proof. The assertion follows from the same argument as given in the proof of Theorem 3.

Equation (6) implies $\lim_{n\to\infty} z(n) = 0$. Also from the definition of $z(n)$, we find $z(n) \geq x(n)$ for all $n \in \mathbb{N}(n_1)$. Hence $\lim_{n\to\infty} x(n) = 0$.

The proof is complete. □

3. Examples

In this section we provide examples to illustrate the results obtained in the above section.

Example 1. In Theorem 1, some of the assumptions cannot be dropped. We consider the difference equation

$$\Delta\left[n^2 \left|\Delta\left(x(n) - 2x(n-1)\right)\right|^{\alpha-1} \Delta\left(x(n) - 2x(n-1)\right)\right] + \frac{2n+1}{n-2} x(n-2)\left[\Delta x(n+1)\right]^2 = 0, \quad n \in \mathbb{N}(3), \quad (9)$$

for which (C_2) is satisfied whereas (C_1) and (C_3) are violated. The equation (9) has a solution $x(n) = n \in M^+$.

Example 2. We consider the following difference equation

$$\Delta\left[2^{n(\alpha+1)} \left|\Delta\left(x(n) + \frac{1}{2}x(n-1)\right)\right|^{\alpha-1} \Delta\left(x(n) + \frac{1}{2}x(n-1)\right)\right] + 2^{n(3+\alpha)} 2^{4-2\alpha} x^{\alpha}(n-2)\left[\Delta x(n+1)\right]^2 = 0, \quad n \in \mathbb{N}(0), \quad (10)$$

for which (C_7) and (C_8) are satisfied. Thus by Theorem 7, for every solution $x(n) \in M^-$ we have $\lim_{n\to\infty} x(n) = 0$. A solution of (10) is $x(n) = \frac{1}{2^n} \in M^-$ such that $\lim_{n\to\infty} x(n) = 0$.

Example 3. Consider the quasi-linear neutral delay difference equation

$$\Delta\left[\frac{1}{n}\left|\Delta\left(x(n) + 2x(n-1)\right)\right|^{\alpha-1} \Delta\left(x(n) + 2x(n-1)\right)\right] + 2^{\alpha-2}\frac{2n+1}{n(n+1)} x(n-2)\left[\left(\Delta x(n+1)\right)^2\right] = 0, \quad n \in \mathbb{N}(1). \quad (11)$$

Here $n_0 = 3$, $\tau = 1$, $\sigma = 3$, $f(u) = u$, $h(u) = u^2$, $p(n) \equiv 2 > 0$, $a(n) = \frac{1}{n} > 0$

and satisfies (C_3), $q(n) = 2^{\alpha-2}\frac{2n+1}{n(n+1)} > 0$ for $n \in \mathbb{N}(1)$,

$$\lim_{n\to\infty}\sum_{s=3}^{n-1} q(s) = \infty$$

satisfies (C_5). All conditions of Theorem 6 are satisfied. Hence equation (11) is oscillatory. In fact $x(n) = (-1)^n$ is an oscillatory solution of equation (10).

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