International Journal of Pure and Applied Mathematics
Volume 57 No. 4 2009, 459-474

PARAMETER-UNIFORM RITZ-GALERKIN FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH DELAY IN CONVECTION TERM

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Abstract: In this paper, we have presented a Ritz-Galerkin finite element method for solving a class of singularly perturbed delay differential equation with small delay in convection term. We have taken a Shishkin mesh to resolve the boundary layer and used hat functions as a basis function for the given method which leads to a tri-digonal linear system. We prove that the method is almost second order convergent independently of the perturbation parameters. Numerical experiments support these theoretical results.

AMS Subject Classification: 34K26, 76N20, 65L10, 65M60
Key Words: singularly perturbed, boundary layer, second order ODEs, Shishkin mesh, delay differential equation, two point boundary value problems, Ritz-Galerkin finite element method

Received: October 11, 2009

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1. Introduction

We consider the boundary-value problems for a class of singularly perturbed differential equations of the convection-diffusion type with small delay

$$\epsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \tag{1}$$

$\forall x \in \Omega = (0,1)$ and subject to the interval conditions

$$y(x) = \psi(x) \quad \text{on} -\delta \leq x \leq 0, \quad y(1) = \gamma, \tag{2}$$

where $a(x)$, $b(x) \leq -\theta < 0$, $f(x)$ and $\psi(x)$ are sufficiently smooth functions. The model problem (1) and (2) exhibits layer behavior on the left side of the interval $[0,1]$, if $a(x) \geq \alpha > 0$ throughout the interval $[0,1]$, where α is a constant. The singular perturbation parameter ϵ is small ($0 < \epsilon \ll 1$) and δ is also a small shifting parameter, $0 < \delta = o(\epsilon)$ such that $\epsilon - \delta a(x) > 0$ for all $x \in [0,1]$ and γ is a constant, let $M_l = \epsilon - \delta\alpha > 0$. Delay differential equations (DDEs) have been used for many years in control theory and only recently have been applied to biological models.

There are applications of delay in active vibration and noise control [10]. Some other applications of delay differential systems in modeling and analysis are reported in [5] for conveyor belts, metal rolling system, population models, economic systems, remote control, urban traffic, electric transmission line, heat exchangers, control systems for nuclear reactors with time delay [1], artificial nueral networks, manufacturing systems [3], and capacity management [2]. Singularly perturbed delay differential equations arises in various practical problems in bio-informatics and physics. In this paper, we present the numerical study of boundary value problems for singularly perturbed delay differential equations of the convection-diffusion type with delay in the convection term which was initiated in [6]. In 1965, Stein [11] gave a realistic model for the stochastic activity of neurons.

Lange and Miura gave an asymptotic approach to solve boundary value problems for the second order singularly perturbed differential difference equation with small shifts [7], [8]. It is well known that classical methods, fail to provide reliable numerical results for such problems (in the sense that the parameter ϵ and the mesh size h cannot vary independently). There are essentially two strategies to design schemes which have small truncation errors inside the layer region(s). The first approach which forms the class of fitted mesh methods consists in choosing a fine mesh in the layer region(s). The second approach is in the context of the fitted operator methods in which the mesh remains uniform and the difference schemes reflect the qualitative behavior of the solution(s) inside the layer region(s). A nice discussion using one or both of the above strategies can be found in Miller et al [9]. The work contained in this paper falls under the first category. We use Ritz-Galerkin finite element method on a fitted mesh to resolve the boundary layer. We first present some analytical results for the case when there is one boundary layer at the left end of the interval. The case when the layer occurs at right end can be analyzed similarly. However, we have presented some numerical results for the later case also.

The rest of the paper is organized as follows. In Section 2, we discuss the continuous problem and show the boundedness of the exact solution of the continuous problem and prove the minimum principle. In Section 3, we introduce a mesh selection strategy. In Section 4, the Ritz-Galerkin finite element method has been discussed and we have shown that it has almost second order ϵ-uniform convergence. Several numerical examples are given in Section 5. Finally, Section 6 contains conclusion.

Throughout the paper, we use C as a generic positive constant independent of ϵ, which may take different values in different equations.

2. Continuous Problem

The term containing delay argument is approximated by Taylor's series. Since the solution of the boundary value problem (1) and (2) is sufficiently smooth, so by Taylor's series

$$y'(x-\delta) \approx y'(x) - \delta y''(x). \tag{3}$$

Using equation (3) in (1), we get

$$(\epsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{4}$$

$$y(0) = \psi(0) = \psi_0, \qquad y(1) = \gamma. \tag{5}$$

The differential operator corresponding to the boundary value problem (1) and (2) is defined by

$$\mathcal{L}_l\pi(x) \equiv (\epsilon - \delta a(x))\pi''(x) + a(x)\pi'(x) + b(x)\pi(x)$$

A Priori Estimates when the Layer is on the Left Side. We first establish a priori bounds for the solution and its derivatives. The differential operator $\mathcal{L}_l$ satisfies the following minimum principle:

Lemma 2.1. (Continuous Minimum Principle) *Let π be any sufficiently smooth function satisfying $\pi(0) \geq 0$, and $\pi(1) \geq 0$. Then $\mathcal{L}_l\pi(x) \leq 0$, $\forall\, x \in \Omega$*

implies that $\pi(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.

Proof. The proof is by contradiction. If possible suppose that there is a point $x^* \in [0,1]$ such that $\pi(x^*) < 0$ and $\pi(x^*) = \min_{x\in\bar{\Omega}} \pi(x)$. It is clear from the given conditions $x^* \notin \{0,1\}$, therefore $\pi'(x^*) = 0$ and $\pi''(x^*) \geq 0$. Thus we have

$$\begin{aligned}
\mathcal{L}_l\pi(x)\mid_{x=x^*} &= (\epsilon - \delta a(x))\pi''(x) + a(x)\pi'(x) + b(x)\pi(x)\mid_{x=x^*} \\
&= (\epsilon - \delta a(x^*))\pi''(x^*) + a(x^*)\pi'(x^*) + b(x^*)\pi(x^*) > 0,
\end{aligned}$$

a contradiction. It follows that $\pi(x^*) \geq 0$ and so $\pi(x) \geq 0\ \forall\, x \in \bar{\Omega}$. □

Lemma 2.2. (Stability) *Let $y(x)$ be the solution of the problem (4) and (5), then we have*

$$\|y\| \leq \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|).$$

Proof. We construct two barrier function $\pi^{\pm}$ defined by

$$\pi^{\pm}(x) = \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm y(x).$$

Then we have

$$\begin{aligned}
\pi^{\pm}(0) &= \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm y(0) \\
&= \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm \psi_0 \geq 0,
\end{aligned}$$

$$\begin{aligned}
\pi^{\pm}(1) &= \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm y(1) \\
&= \theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm \gamma \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_l\pi^{\pm}(x) &= (\epsilon - \delta a(x))(\pi^{\pm}(x))'' + a(x)(\pi^{\pm}(x))' + b(x)(\pi^{\pm}(x)) \\
&= b(x)(\theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|) \pm \mathcal{L}_l y(x) \\
&= b(x)[\theta^{-1}\|f\| + \max(|\psi_0|, |\gamma|] \pm f(x) \\
&\leq (-\|f\| \pm f(x)) + b(x)(\max(|\psi_0|, |\gamma|) \quad \text{as} \quad b(x) \leq -\theta < 0 \\
&\leq 0.
\end{aligned}$$

Therefore, from Lemma 2.1, we obtain $\pi^{\pm}(x) \geq 0$ for all $x \in [0,1]$ which gives the required estimate. □

Lemma 2.1 implies that the solution is unique and since the problem under consideration is linear, the existence of the solution is implied by its uniqueness. Further, the boundedness of the solution is implied by (2.2). It is easy to prove:

Lemma 2.3. *Let $y(x)$ be the solution of the problem (4) and (5). Then*

$y(x)$ satisfies

$$|y(x)| \leq C\left[1+\exp\left(-\frac{\alpha x}{M_l}\right)\right]$$

and

$$|y^k(x)| \leq CM_l^{-k}\exp\left(-\frac{\alpha x}{M_l}\right), \quad \forall\, k \geq 1,$$

where C is a constant independent of M_l.

This result is useful in proving the following lemma.

Lemma 2.4. *The solution $y(x)$ of (4) and (5) admits the decomposition*

$$y(x) = v_{l,\epsilon}(x) + w_{l,\epsilon}(x),$$

where the regular (smooth) component $v_{l,\epsilon}(x)$ satisfies

$$|v_{l,\epsilon}(x)| \leq C\left[1+\exp\left(-\frac{\alpha x}{M_l}\right)\right],$$

$$|v_{l,\epsilon}^k(x)| \leq C\left[1+M_l^{(2-k)}\exp\left(-\frac{\alpha x}{M_l}\right)\right], \quad \forall\, k \geq 1,$$

and the singular component $w_{l,\epsilon}(x)$ satisfies

$$|w_{l,\epsilon}^k(x)| \leq CM_l^{-k}\exp\left(-\frac{\alpha x}{M_l}\right), \quad \forall\, k \geq 0.$$

Proof. Plugging the three term asymptotic expansion for $v_{l,\epsilon}(x)$, viz.

$$v_{l,\epsilon}(x) = v_0(x) + M_l v_1(x) + M_l^2 v_2(x) \tag{6}$$

into equation (1) and (2), we obtain the relations

$$a(x)v_0'(x) + b(x)v_0(x) = f(x), \quad v_0(1) = y(1),$$

$$a(x)v_1'(x) + b(x)v_1(x) = -v_0''(x), \quad v_1(1) = 0,$$

$$v_2''(x) = 0, \quad v_2(0) = 0, \quad v_2(1) = 0,$$

$$\mathcal{L}_l v_{l,\epsilon}(x) = f, \quad v_{l,\epsilon}(0) = v_0(0) + M_l v_1(0), \quad v_{l,\epsilon}(1) = y(1), \tag{7}$$

and

$$\mathcal{L}_l w_{l,\epsilon}(x) = 0, \quad w_{l,\epsilon}(0) = y(0) - v_{l,\epsilon}(0), \quad w_{l,\epsilon}(1) = 0. \tag{8}$$

Since v_0 and v_0 are independent of M_l, we obtain

$$|v_i^k(x)| \leq C, \quad \forall k \geq 0 \quad \text{and} \quad i = 0, 1. \tag{9}$$

Lemma 2.3 then implies

$$|v_2(x)| \leq C\left[1+\exp\left(-\frac{\alpha x}{M_l}\right)\right] \tag{10}$$

and

$$|v_2^k(x)| \leq CM^{-k}\exp\left(-\frac{\alpha x}{M_l}\right), \quad \forall\, k \geq 1. \tag{11}$$

Using (9), (10) and (11) into (6) and the equations which are obtained via the successive differentiation of (6), we will get the desired bounds on the smooth component. To obtain the bounds on the singular components w_ϵ, consider two barrier functions $\eta^\pm$ defined by

$$\eta^\pm(x) = (|y(0)| + |v_{l,\epsilon}(0)|)\exp\left(-\frac{\alpha x}{M_l}\right) \pm w_{l,\epsilon}(x),$$

clearly, $\eta^\pm(0)$ and $\eta^\pm(1)$ are non-negative. Further $L_\epsilon w_{l,\epsilon} = 0$ implies that $L_\epsilon \eta^\pm \leq 0$. Therefore , Lemma 2.1 implies that $\eta^\pm \geq 0$ which gives

$$|w_{l,\epsilon}(x)| \leq C\exp\left(-\frac{\alpha x}{M_l}\right), \quad \text{where} \quad C = (|y(0)| + |v_{l,\epsilon}(0)|). \tag{12}$$

Now to find the bound on the derivatives of $w_\epsilon(x)$, we introduce

$$\chi(x) = \frac{\int_0^x \exp\left(-\frac{A(t)}{M_l}\right)dt}{\int_0^1 \exp\left(-\frac{A(t)}{M_l}\right)dt}, \quad \text{where} \quad A(x) = \int_0^x a(s)ds. \tag{13}$$

Clearly, $\chi(0) = 0$, $\chi(1)$ and $x \in [0,1] \Rightarrow \chi(x) \leq 1$. Further, $\mathcal{L}_l\chi(x) \leq 0$. Therefore, from Lemma 2.1, we have $\chi(x) \geq 0$. Thus $0 \leq \chi(x) \leq 1$. We write $w_{l,\epsilon}(x)$ as

$$w_{l,\epsilon}(x) = C_1\chi(x) + C_2(1-\chi(x)).$$

Using (8) and the above values of $\chi(x)$ at 0 and 1, we get $C_1 = 0$ and $C_2 = y(0) - v_{l,\epsilon}(0)$. This implies that

$$w_{l,\epsilon}(x) = (y(0) - v_{l,\epsilon}(0))(1-\chi(x)) \tag{14}$$

and therefore,

$$|w'_{l,\epsilon}(x)| \leq C|\chi'(x)| \quad \text{where} \quad C = y(0) - v_{l,\epsilon}(0). \tag{15}$$

Equation (13) implies that

$$|w'_{l,\epsilon}(x)| \leq CM_l^{-1}\exp\left(-\frac{\alpha x}{M_l}\right). \tag{16}$$

Now, using the relation

$$w''_{l,\epsilon}(x) = -a(x)w'_{l,\epsilon}(x) + b(x)w_\mu(x),$$

the bounds on $w''_{l,\epsilon}(x)$ can be obtained via (12) and (16). Finally the successive differentiation in (8) and the use of the bounds on the derivatives obtained at each earlier steps gives the desired bounds on $w^k_{l,\epsilon}(x)$. This completes the proof. □

A Priori Estimates when the Layer is on the Right Side. The model problem (1) and (2) exhibits layer behavior on the right side of the interval $[0, 1]$, if $a(x)$ is replaced by $-a(x)$ which leads to

$$\begin{aligned} \mathcal{L}_r y(x) &\equiv -(\epsilon + \delta a(x))y''(x) + a(x)y'(x) + b^*(x)y(x) = f^*(x), \quad (17)\\ y(0) &= \psi_0, \quad y(1) = \gamma, \quad (18) \end{aligned}$$

where $b^*(x) = -b(x)$, $f^*(x) = -f(x)$ and $M_r = (\epsilon + \delta\alpha) > 0$. Here the differential operator $\mathcal{L}_r$, satisfies the continuous maximum principle which is stated in the following lemma.

Lemma 2.5. (Continuous Maximum Principle) *Let π be any sufficiently smooth function satisfying $\pi(0) \geq 0$, and $\pi(1) \geq 0$. Then $\mathcal{L}_r\pi(x) \geq 0$, $\forall\, x \in \Omega$ implies that $\pi(x) \geq 0$, $\forall\, x \in \bar{\Omega}$.*

Proof. The proof is by contradiction. If possible suppose that there is a point $x^* \in [0, 1]$ such that $\pi(x^*) < 0$ and $\pi(x^*) = \min_{x\in\bar{\Omega}} \pi(x)$. It is clear from the given conditions $x^* \notin \{0, 1\}$, therefore $\pi'(x^*) = 0$ and $\pi''(x^*) \geq 0$. Thus we have

$$\begin{aligned} \mathcal{L}_r\pi(x) \mid_{x=x^*} &= -(\epsilon + \delta a(x))\pi''(x) + a(x)\pi'(x) + b^*(x)\pi(x) \mid_{x=x^*}\\ &= -(\epsilon + \delta a(x))\pi''(x^*) + a(x^*)\pi'(x^*) + b^*(x^*)\pi(x^*) < 0, \end{aligned}$$

a contradiction. It follows that $\pi(x^*) \geq 0$ and so $\pi(x) \geq 0\ \forall\, x \in \bar{\Omega}$. □

We state the following results without proofs (because in this case the boundary layer will be at the right end of the interval, therefore, by transforming x to $1-x$ and proceeding in the similar manner one can easily get the proofs).

Lemma 2.6. *Let $y(x)$ be the solution of the problem (17) and (18). Then $y(x)$ satisfies*

$$|y(x)| \leq C\left[1 + \exp\left(-\frac{\alpha(1-x)}{M_r}\right)\right]$$

and

$$|y^k(x)| \leq CM_r^{-k}\exp\left(-\frac{\alpha(1-x)}{M_r}\right), \quad \forall\, k \geq 1.$$

Here C is a constant independent of M_r.

This result is useful in proving the following lemma.

Lemma 2.7. *The solution $y(x)$ of (17) and (18) admits the decomposition*

$$y(x) = v_{r,\epsilon}(x) + w_{r,\epsilon}(x),$$

where the regular (smooth) component $v_{r,\epsilon}(x)$ satisfies

$$|v_{r,\epsilon}(x)| \leq C\left[1 + \exp\left(-\frac{\alpha(1-x)}{M_r}\right)\right],$$

$$|v_{r,\epsilon}^{k}(x)| \leq C\left[1 + M_r^{(2-k)}\exp\left(-\frac{\alpha(1-x)}{M_r}\right)\right], \quad \forall\, k \geq 1,$$

and the singular component $w_{r,\epsilon}(x)$ satisfies

$$|w_{r,\epsilon}^{k}(x)| \leq CM_r^{-k}\exp\left(-\frac{\alpha(1-x)}{M_r}\right), \quad \forall\, k \geq 0.$$

3. Mesh Selection Strategy

To obtain the discrete counterpart of the singularly perturbed problem (1) and (2), first we discretize the domain $\bar{\Omega} = [0,1]$ as $\bar{\Omega}^N = \{0 = x_0 < x_1 < .... < x_N = 1\}$ and the piecewise uniform meshes are defined as follows. Let N be a positive integer and multiple of 2; to define the mesh points, we divide the interval $[0,1]$ into two subintervals $[0,\tau]$ and $[\tau,1]$, where the transition parameter is given by

$$\tau = \min\Big(\frac{1}{2}, \frac{2M_l}{\alpha}\ln N\Big).$$

Then, we place $N/2$ mesh points each in $[0,\tau]$ and $[\tau,1]$ respectively. We will denote the step size by

$$h_i = \begin{cases} H_1 = 2\tau/N, & \text{if} \quad i = 1,2,...,N/2, \\ H_2 = 2(1-\tau)/N, & \text{if} \quad i = N/2+1,\cdots N. \end{cases}$$

The mesh points are given by

$$x_i = \begin{cases} iH_1, & \text{if} \quad i = 0,1,2,\cdots N/2, \\ \tau + (i-N/2)H_2, & \text{if} \quad i = N/2+1,\cdots N. \end{cases}$$

If the boundary layer in right hand side. We divide the interval $[0,1]$ into two subintervals $[0,1-\tau]$ and $[1-\tau,1]$, where the transition parameter is given by

$$\tau = \min\Big(\frac{1}{2}, \frac{2M_r}{\alpha}\ln N\Big).$$

Then, we place $N/2$ mesh points each in $[0, 1-\tau]$ and $[1-\tau, 1]$ respectively. We will denote the step size by

$$h_i = \begin{cases} H_1 = 2(1-\tau)/N, & \text{if} \quad i = 1, 2, ..., N/2, \\ H_2 = 2\tau/N, & \text{if} \quad i = N/2+1, \cdots N. \end{cases}$$

The mesh points are given by

$$x_i = \begin{cases} iH_1, & \text{if} \quad i = 0, 1, 2, \cdots N/2, \\ 1-\tau+(i-N/2)H_2, & \text{if} \quad i = N/2+1, \cdots N. \end{cases}$$

where $x_{i+1} = x_i + h_{i+1}, \quad \text{for} \quad i = 0, 1, ..., N-1.$

4. Finite Element Method and Convergence

The theoretical Ritz-Galerkin method is a powerful one. The approximations are very good and the method is easy to implement. To obtain better and better approximations, we choose hat functions as basis function, due to which we get a matrix of great many zeros such as band matrices.

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & \text{for} x \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-x}{h_{i+1}}, & \text{for} x \in [x_i, x_{i+1}], \\ 0, & \text{for} x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{19}$$

The finite element subspace $\bar{V_1}(\bar{\Omega}^N)$ is the space of standard piecewise linear polynomials on a piecewise uniform fitted mesh $\bar{\Omega}^N$. The piecewise uniform fitted mesh $\bar{\Omega}^N = \{x_i\}_0^N$ condensing at the boundary point $x = 0$.

The piecewise linear polynomials are continuous on $\bar{\Omega}$ and are required to vanish at the boundary points $x = 0$ and $x = 1$. It is clear that $\bar{V_1}(\bar{\Omega}^N)$ is a subspace of $H_0^1(\Omega)$. The standard basis for $\bar{V_1}(\bar{\Omega}^N)$ is $\{\phi_i\}_1^{N-1}$, where ϕ_i is the usual hat function for the mesh point x_i.

$$\mathcal{L}_l y(x) \equiv (\epsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1],$$

with

$$y(0) = \psi(0) = \psi_0, \qquad y(1) = \gamma.$$

Let

$$l(x) = \gamma\frac{x-0}{1-0} + \psi_0\frac{1-x}{1-0},$$

where $y(x) = u(x) + l(x)$ and

$$F(x) = f(x) - \left((\epsilon - \delta a(x))\frac{\gamma-\psi_0}{1-0}\right)' - a(x)\left(\frac{\gamma-\psi_0}{1-0}\right)$$

$$-b(x)\left[\gamma\frac{x-0}{1-0}+\psi_0\frac{1-x}{1-0}\right].$$

Then $u(x)$ satisfies the problem.

$$(\epsilon-\delta a(x))u''(x)+a(x)u'(x)+b(x)u(x)=F(x),\quad x\in[0,1], \tag{20}$$

with

$$u(0)=0,\quad u(1)=0.$$

Divide the interval $[0,1]$ up to N subintervals, so there will be $N-1$ interior grid points given by $x_i=x_{i-1}+h_i;\quad i=1,2,3....,N-1$. Set $x_0=0$ and $x_N=1$. Construct the hat functions ϕ_i, $i=1,2,....,N-1$. Then

$$\bar{u}(x)=\sum_{j=1}^{N-1}c_j\phi_j(x), \tag{21}$$

where $c=(c_1,c_2.....c_{n-1})'$ solves the linear system

$$\sum_{j=1}^{N-1}(\phi_i,\phi_j)_A c_j=(f,\phi_i),\quad 1\le i\le N-1, \tag{22}$$

where

$$(\phi_i,\phi_j)_A=\int_0^1\Big(-(\epsilon-\delta a(x))\phi_i^{'}(x)\phi_j^{'}(x)+a(x)\phi_i(x)\phi_j^{'}(x)+b(x)\phi_i(x)\phi_j(x)\Big)\,dx$$

and

$$(f,\phi_i)=\int_0^1 f(x)\phi_i(x)dx$$

$i=1,2,3.....,N-1$. The above system of equations gives a tridiagonal matrix in $c_j's$. Once the $c_j's$ have been determined from (22), the approximation $y_n(x)$ to the solution, $y(x)$ of (1.1) and (1.2) is given by

$$y_n(x)=\bar{u}(x)+l(x).$$

The proof of parameter-uniform estimate of $\bar{u}(x)-u(x)$ in the maximum norm where $\bar{u}(x)$ is the $\bar{V}_1(\bar{\Omega}^N)$-interpolant of the exact solution $u(x)$ of (20) is given by the following theorem.

Theorem 1. *Let $\bar{u}(x)$ be the $\bar{V}_1(\Omega^N)$ interpolant of the solution $u(x)$ of (20) on the fitted mesh $\bar{\Omega}^N$. Then*

$$\sup_{0<\epsilon\le 1}\|\bar{u}(x)-u(x)\|_{\bar{\Omega}}\le CN^{-2}(\ln N^2).$$

Proof. The estimate is obtained on each subinterval $\Omega_i=(x_{i-1},x_i)$ separately. Let any function g on Ω_i

$$\bar{g}(x)=g_{i-1}(x)\phi_{i-1}(x)+g_i(x)\phi_i(x)$$

	N=100		N=1000	
δ	Results in[6]	Our Results	Results in[6]	Our Results
0.01	1.1824E-02	1.8672e-004	1.2299E-03	1.8650e-006
0.03	1.5156E-02	3.0591e-004	1.5935E-03	3.0564e-006
0.06	2.5848E-02	4.8968e-004	2.8161E-03	9.2395e-006
0.08	8.3131E-02	4.6655e-004	1.1103E-02	1.1120e-005

Table 1: Comparison of maximum absolute errors for Example 1, $\epsilon = 0.1$

	N=100		N=1000	
δ	Results in[6]	Our Results	Results in[6]	Our Results
0.001	9.0928E-02	4.5484e-004	1.2290E-02	1.1019e-005
0.003	9.0928E-02	4.5268e-004	1.5626E-02	1.1001e-005
0.006	1.2845E-01	4.4941e-004	2.6314E-02	1.0973e-005
0.008	1.0150E-01	4.4722e-004	4.8347E-02	1.0954e-005

Table 2: Comparison of maximum absolute errors for Example 1, $\epsilon = 0.01$

	N=100		N=1000	
δ	Results in[6]	Our Results	Results in[6]	Our Results
0.000	1.7855E-01	1.1554E-03	2.3879E-02	2.4786E-05
0.007	1.1763E-01	1.1436E-03	1.3951E-02	2.4836E-05
0.015	8.3515E-02	1.1394E-03	9.4403E-03	2.4930E-05
0.025	6.1475E-02	1.1388E-03	6.7886E-03	2.5091E-05

Table 3: Comparison of maximum absolute errors for Example 2, $\epsilon = 0.01$

	N=100		N=1000	
δ	Results in[6]	Our Results	Results in[6]	Our Results
0.0007	2.1339E-01	1.18325E-03	2.8978E-02	2.4895E-05
0.0015	1.2312E-01	1.17947E-03	1.4627E-02	2.4851E-05
0.0025	8.0964E-02	1.175147E-03	9.1153E-03	2.48195E-05

Table 4: Comparison of maximum absolute errors for Example 2, $\epsilon = 0.001$

and so it is obvious that, on Ω_i

$$\bar{g}(x) \leq \max_{\Omega_i} g(x)[\phi_{i-1}(x) + \phi_i(x)].$$

$\delta\downarrow$	N=32	N=64	N=128	N= 256	N=512
0.01	1.84E-03	4.57E-04	1.14E-04	2.85E-05	7.11E-06
	2.00	2.00	2.00	2.00	
0.02	2.32E-03	5.76E-04	1.43E-04	3.59E-05	8.97E-06
	2.00	2.01	1.99	2.00	
0.04	2.72E-03	1.01E-03	2.53E-04	6.32E-05	1.58E-05
	1.41	2.01	2.00	2.00	

Table 5: Maximum absolute errors and order of convergence for Example 1, $\epsilon = 0.1$

$\delta\downarrow$	N=32	N=64	N=128	N= 256	N=512
0.01	2.92E-03	7.22E-04	1.80E-04	4.51E-05	1.13E-05
	2.01	2.00	2.00	2.00	
0.02	2.45E-03	6.14E-04	1.53E-04	3.83E-05	9.57E-06
	2.00	2.00	2.00	2.00	
0.04	1.85E-03	4.60E-04	1.15E-04	2.87E-05	7.17E-06
	2.00	2.00	2.01	1.99	

Table 6: Maximum absolute errors and order of convergence for Example 2, $\epsilon = 0.1$

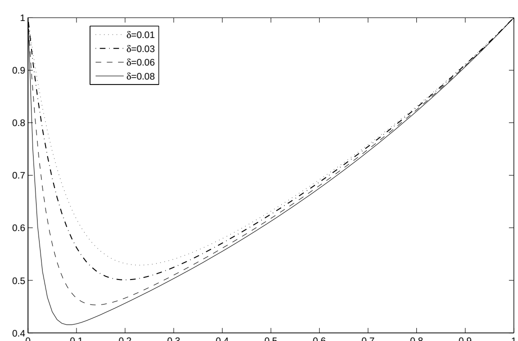

Figure 1: Numerical solutions of Example 1, for $\epsilon = 10^{-1}$ and different values of δ

Taking maximum norm of both sides, we get

$$|\bar{g}(x)| \leq \max_{\Omega_i} |g(x)| \tag{23}$$

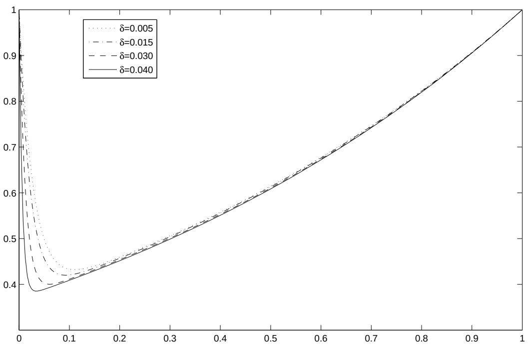

Figure 2: Numerical solutions of Example 1, for $\epsilon = 5 * 10^{-2}$ and different values of δ

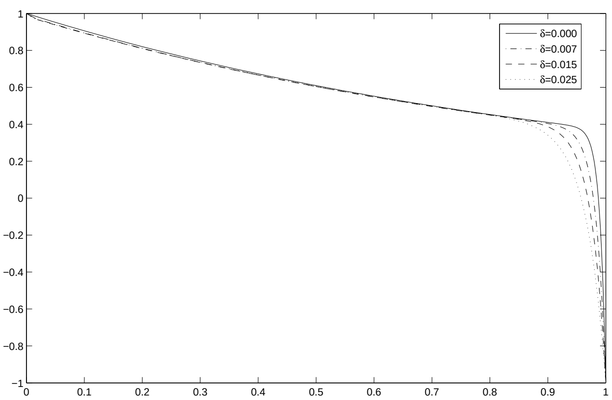

Figure 3: Numerical solutions of Example 2, for $\epsilon = 10^{-2}$ and different values of δ

and by appropriate Taylor expansions it is easy to see that

$$|\bar{g}(x) - g(x)| \leq Ch_i^2 \max_{\Omega_i} |g^{''}(x)|. \tag{24}$$

From (24) and Lemma 2.3, on Ω_i,

$$|\bar{u}(x) - u(x)| \leq Ch_i^2 \max_{\Omega_i} |u^{''}(x)| \leq Ch_i^2/M_l^2. \tag{25}$$

Also, using Lemma 2.4, (24) and (25), on Ω_i,

$$\begin{aligned} |\bar{u}(x) - u(x)| &\leq |\bar{v}_{l,\epsilon}(x) + \bar{w}_{l,\epsilon}(x) - v_{l,\epsilon}(x) - w_{l,\epsilon}(x)| \\ &\leq |\bar{v}_{l,\epsilon}(x) - v_{l,\epsilon}(x)| + |\bar{w}_{l,\epsilon}(x)| + |w_{l,\epsilon}(x)| \end{aligned}$$

$$\leq \quad Ch_i^2 \max_{\Omega_i} |\bar{v}''_{l,\epsilon}(x)| + 2 \max_{\Omega_i} |w_{l,\epsilon}(x)|$$
$$\leq \quad C(h_i^2 + e^{-\alpha x/M_l}). \qquad (26)$$

The argument now depends on whether $\frac{2M_l}{\alpha} \ln N \geq 1/2$ or $\frac{2M_l}{\alpha} \ln N \leq 1/2$. In the first case $1/M_l \leq C \ln N$ and the results follows at once from (25)

$$|\bar{u}(x) - u(x)| \leq CN^{-2}(\ln N)^2.$$

Now in the second case $\tau = \frac{2M_l}{\alpha} \ln N$. Suppose that i satisfies $1 \leq i \leq N/2$. Then $h_i = \frac{4M_l \ln N}{\alpha N}$

$$\frac{h_i}{M_l} = \frac{4}{\alpha} N^{-1} \ln N = CN^{-1} \ln N.$$

Now from (25)

$$|\bar{u}(x) - u(x)| \leq CN^{-2}(\ln N)^2.$$

If i satisfied $N/2 < i \leq N$. Then $\tau \leq x_i \leq 1$

$$e^{-\frac{\alpha x_i}{M_l}} \leq e^{-\frac{\alpha \tau}{M_l}} = e^{-2 \ln N} = N^{-2}.$$

Using this in (26) gives the required result. □

For the right hand boundary layer problem proof of uniform convergence is the same as the proof of the left hand boundary layer problem. The only difference is x is replaced by $1 - x$ in the above proof.

5. Numerical Experiments

To demonstrate the efficiency of the method, we solve two examples having a boundary layer at left and right hand side.

Example 1. Consider the problem ([6], Example 1)

$$\epsilon y''(x) + y'(x - d) - y(x) = 0,$$

with

$$y(0) = 1, \quad \text{if } -\delta \leq x \leq 0, \quad y(1) = 1.$$

Boundary layer exist at left end.

Example 2. Consider the problem ([6], Example 3)

$$\epsilon y''(x) - y'(x - d) - y(x) = 0,$$

with

$$y(0) = 1 \quad \text{if } -\delta \leq x \leq 0, \quad y(1) = -1.$$

Boundary layer exist at right end.

Since the exact solution for considered problems are not available, the maximum absolute errors $E^N_{\epsilon,\delta}$ are evaluated using the double mesh principle [4] for the given method.

$$E^N_{\epsilon,\delta} = \max_{0\leq j\leq N} |v^N_j - v^{2N}_{2j}|,$$

where v^N_j and v^{2N}_{2j} are the computed solutions at x_j and x_{2j} taking N and $2N$ points respectively. The numerical rates of convergence are computed using the formula

$$R_N = \log_2(E^N_{\epsilon,\delta}/E^{2N}_{\epsilon,\delta}).$$

6. Conclusion

A two point boundary value problem for a second-order singularly perturbed delay differential equation with delay in convection is considered. We have discussed Ritz-Galerkin finite element method on piecewise-uniform (Shishkin mesh). Here we have discussed both the cases, when boundary layer is on the left side and when boundary layer is on the right side of the interval. From the graphs, we observed that, when δ increases, the width of the boundary layer decreases in the case of left boundary layer problems and in the case of right boundary layer problems the width of the boundary layer increases when delta increases. This shows clearly the effect of δ on the boundary layers. It has been found that the proposed algorithm gives highly accurate numerical results and higher order of convergence than the scheme given in [6]. Two numerical examples are presented, which demonstrate that the proposed scheme on piecewise-uniform mesh gives more accurate approximate solution.

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