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A TWO-STEP HIGH ORDER NEWTON-LIKE METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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Abstract: In this work in order to solve systems of nonlinear equations, a two-step high order Newton-like method free from second derivative is presented. We prove that the method is convergent. The computational aspect of the method is studied using some numerical experiments including an application to solve a boundary value problem and to the Chandrasekhar integral equation in radiative transfer. Residual falls of logarithm of errors show cubic convergence of the method.

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1. Introduction

Consider the problem of finding a real zero of a nonlinear system

$$F(\mathbf{x}) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, that is, a system with n equations and n unknowns. Finding roots of systems of nonlinear equations efficiently is of major importance and has widespread applications in numerical and applied mathematics. In fact,

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there are cases where thousands of nonlinear equations depending on some independent variables, must be solved effectively. There are many approaches to solve equation (1). One of the best root-finding methods for systems of nonlinear equations is Newton's method, as it is well known. The second order Newton method is one of the most common iterative methods adopted for finding approximate solutions of systems $F(\mathbf{x}) = 0$, [10]. The Newton's method to solve (1) is an important and basic method which converges quadratically. Third order iterative methods like the Halley and Chebyshev methods [1, 11], despite their cubic convergence, are considered less practical from a computational point of view because of the costly second derivatives. In fact, for a nonlinear system of n equations and n unknowns, the first Fréchet derivative is a matrix with n^2 values while the second Fréchet derivative has n^3 values. This implies a huge amount of operations in order to evaluate every iteration [1].

There are many high-order Newton-like and Chebyshev-like methods free from second derivatives to solve system (1). Babajee and Dauhoo [2], [3] presented some variants of Newton's method with third order convergence. Cordero and Torregrosa [5] in order to solve systems of nonlinear equations, developed some variants of Newton's method based on trapezoidal and midpoint rules of quadrature. Darvishi and Barati [6] presented a third order iterative method based on Adomian decomposition method to solve systems of nonlinear equations. Darvishi and Barati [7] developed a fourth order iterative method free from second order derivative to solve equation (1). They also [8] presented two super cubic iterative methods based on the Adomian decomposition method and based on a quadrature formulae to obtain the inverse of Jacobian matrix. Frontini and Sormani [9] obtained a third-order method based on a quadrature formulae to solve systems of nonlinear equations. Babajee et al [4] proposed a fourth order iterative method to solve equation (1).

In this paper, we present a two-step high order Newton-like method to find a zero of systems of nonlinear equations.

The rest of this paper is organized as follows. In the next section we present the two-step Newton-like method and we prove that the method is convergent. In Section 3 we solve some examples by Newton's method and two-step method. Section 4 shows the power of new method to solve large systems of nonlinear equations, which we solve a boundary value problem and Chandrasekhar integral equation. Finally we present a brief conclusion in Section 5.

2. Two-Step High Order Newton-Like Method

To find a simple root α, that is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ of

$$f(x) = 0, \tag{2}$$

where f is a real valued continuously function on $\mathbb{R}$, the following two-step third order method to solve (2) presented by Weerakoon and Fernando [15]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{3}$$

They proved that, starting with an initial approximation x_0 in the neighborhood of the root α, this method converges cubically to α. This method does not need second order derivative of function f.

In this paper we extend this method to obtain a two-step high order iterative method to solve systems of nonlinear equations.

Let us consider the problem of finding a real root of a function $F : \mathbb{R}^n \to \mathbb{R}^n$, that is a real solution $\overline{\mathbf{x}}$ of the nonlinear equation system $F(\mathbf{x}) = 0$ of n equations with n variables. This solution can be obtained as a fixed point of some function $G : \mathbb{R}^n \to \mathbb{R}^n$ by means of the fixed point iteration method

$$\mathbf{x}^{(k+1)} = G(\mathbf{x}^{(k)}), \quad k = 0, 1, \cdots,$$

where $\mathbf{x}^{(0)}$ is the initial estimation. The best known fixed point method is the classical Newton's method, given by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J_F(\mathbf{x}^{(k)})^{-1} F(\mathbf{x}^{(k)}),$$

where $J_F(\mathbf{x}^{(k)})$ is the Jacobian matrix of the function F evaluated in $\mathbf{x}^{(k)}$. To present the new high order method to solve system (1) we introduce the following algorithm:

Two-Step Newton-Like Algorithm

Start with an approximation $\mathbf{x}^{(0)}$ and for $n = 0, 1, 2, \ldots$, until convergence, do

$$\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - J_F(\mathbf{x}^{(k)})^{-1} F(\mathbf{x}^{(k)}), \tag{4}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - 2\Big(J_F(\mathbf{x}^{(k)}) + J_F(\mathbf{y}^{(k)})\Big)^{-1} F(\mathbf{x}^{(k)}). \tag{5}$$

2.1. Convergency of Two-Step Method

Our aim in this part is to prove that, under certain conditions, two-step method is a convergent method. To do this, some previous results are needed and will be introduced in the following. We state the following important definition and lemma which are proved in [14, 13].

Definition. (see [14, pp. 299]) Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$. Then $\overline{\mathbf{x}}$ is a point of attraction of the iteration

$$\mathbf{x}^{(k+1)} = G(\mathbf{x}^{(k)}), \quad k = 0, 1, \dots . \tag{6}$$

If there is an open neighborhood S of $\overline{\mathbf{x}}$ such that $S \subset D$ and for any $\mathbf{x}^{(0)} \in S$, the iterates $\{\mathbf{x}^{(k)}\}$ defined by equation (6) lie all in D and converge to $\overline{\mathbf{x}}$.

Lemma. (Banach Perturbation Lemma) *Let $A \in L(\mathbb{R}^n)$ be nonsingular. If $E \in L(\mathbb{R}^n)$ and $\|A^{-1}\|\,\|E\| < 1$, then $A + E$ is nonsingular and*

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\,\|E\|}.$$

Now, by this lemma we state and prove the following theorem.

Theorem 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function at $\overline{\mathbf{x}}$, where $\overline{\mathbf{x}}$ is a solution of the system of nonlinear equations $F(\mathbf{x}) = 0$. Let us suppose that J_F is continuous and $J_F(\overline{\mathbf{x}})$ is nonsingular. Then the function*

$$G(\mathbf{x}) = \mathbf{x} - 2C(\mathbf{x})^{-1}F(\mathbf{x}), \text{ where } C(\mathbf{x}) = J_F(\mathbf{x}) + J_F(\mathbf{x} - J_F(\mathbf{x})^{-1}F(\mathbf{x}))$$

is well-defined in a neighborhood of $\overline{\mathbf{x}}$, is differentiable and

$$J_G(\overline{\mathbf{x}}) = I - J_F(\overline{\mathbf{x}})^{-1}J_F(\overline{\mathbf{x}}) = 0.$$

Proof. Firstly, let us prove that $C(\mathbf{x})$ is nonsingular for any $\mathbf{x}$ in a neighborhood of $\overline{\mathbf{x}}$. Let $\|J_F^{-1}(\overline{\mathbf{x}})\| = \beta$ and ε be such that $0 < \varepsilon < (2\beta)^{-1}$. By continuity of $J_F(\mathbf{x})$ in $\overline{\mathbf{x}}$ there exists a $\delta > 0$ such that $\|J_F(\mathbf{x}) - J_F(\overline{\mathbf{x}})\| \leq \varepsilon$ if $\|\mathbf{x} - \overline{\mathbf{x}}\| \leq \delta$.

Let us consider $\mathbf{z} = \mathbf{x} - J_F(\mathbf{x})^{-1}F(\mathbf{x})$. By the convergency of classical Newton's method, it can be assured that $\|\mathbf{z} - \overline{\mathbf{x}}\| \leq \delta$, and $\|J_F(\mathbf{z}) - J_F(\overline{\mathbf{x}})\| \leq \varepsilon$.

Then by using Banach's Lemma it can be stated that $C(\mathbf{x})$ is nonsingular and

$$\begin{aligned}\|C(\mathbf{x})^{-1}\| &= \|J_F(\mathbf{x}+\mathbf{z})^{-1}\| = \|(J_F(\overline{\mathbf{x}}) + (J_F(\mathbf{x}+\mathbf{z}) - J_F(\overline{\mathbf{x}})))^{-1}\| \\ &\leq \frac{\|J_F(\overline{\mathbf{x}})^{-1}\|}{1 - \|J_F(\overline{\mathbf{x}})^{-1}\| \cdot \|J_F(\mathbf{x}+\mathbf{z}) - J_F(\overline{\mathbf{x}})\|} \leq \frac{\beta}{1 - \beta\varepsilon} < 2\beta,\end{aligned}$$

for $\|\mathbf{x}-\overline{\mathbf{x}}\| \leq \delta$. So, the function $G(\mathbf{x})$ is well-defined in the neighborhood of $\overline{\mathbf{x}}$, $S=\{\mathbf{x} : \|\mathbf{x}-\overline{\mathbf{x}}\| \leq \delta\}$.

On the other hand, by differentiability of F in $\overline{\mathbf{x}}$, it can be assumed that δ is small enough to

$$\|F(\mathbf{x})-F(\overline{\mathbf{x}})-J_F(\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})\| \leq \varepsilon\|\mathbf{x}-\overline{\mathbf{x}}\|, \quad \forall \mathbf{x} \in S.$$

Then

$$\begin{aligned}
&\|G(\mathbf{x})-G(\overline{\mathbf{x}})-\Big(I-2C(\overline{\mathbf{x}})^{-1}J_F(\overline{\mathbf{x}})\Big)(\mathbf{x}-\overline{\mathbf{x}})\| \\
&= 2\|C(\overline{\mathbf{x}})^{-1}J_F(\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})-C(\mathbf{x})^{-1}F(\mathbf{x})\| \\
&\leq 2\{\|C(\mathbf{x})^{-1}(F(\mathbf{x})-F(\overline{\mathbf{x}})-J_F(\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}}))\| \\
&\quad +\|C(\mathbf{x})^{-1}(C(\mathbf{x})-C(\overline{\mathbf{x}}))C(\overline{\mathbf{x}})^{-1}J_F(\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})\|\} \\
&\leq 2\{\|C(\mathbf{x})^{-1}\| \cdot \|F(\mathbf{x})-F(\overline{\mathbf{x}})-J_F(\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})\| \\
&\quad +\|C(\mathbf{x})^{-1}\| \cdot \|C(\mathbf{x})-C(\overline{\mathbf{x}})\| \cdot \|\mathbf{x}-\overline{\mathbf{x}}\|\} \\
&\leq 8\beta\varepsilon\|\mathbf{x}-\overline{\mathbf{x}}\|
\end{aligned}$$

for any $\mathbf{x} \in S$. As ε is arbitrary and β is constant, it can be concluded from the previous inequality that G is differentiable in $\overline{\mathbf{x}}$ and also

$$J_G(\overline{\mathbf{x}}) = I-2C(\overline{\mathbf{x}})^{-1}J_F(\overline{\mathbf{x}}) = 0,$$

because, $C(\overline{\mathbf{x}})^{-1} = \frac{1}{2}J_F(\overline{\mathbf{x}})^{-1}$. □

The following result is the Ostrowski's Theorem on the convergence of the fixed point method.

Theorem 2. (see [13, Ostrowski Theorem]) *Suppose that $G : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differentiable function in $\overline{\mathbf{x}}$, that is a solution of the system $\mathbf{x} = G(\mathbf{x})$. Let $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ be the sequence of iterates obtained by means of fixed point iteration, $\mathbf{x}^{(k+1)} = G(\mathbf{x}^{(k)})$, $k = 0, 1, \cdots$. If the spectral radius of $J_G(\overline{\mathbf{x}})$ is lower than 1, then $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ converges to $\overline{\mathbf{x}}$.*

Finally, the main result of this section states the following:

Theorem 3. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be differentiable at each point of an open neighborhood D of $\overline{\mathbf{x}} \in \mathbb{R}^n$, that is a solution of the system $F(\mathbf{x}) = 0$. Let us suppose that $J_F(\mathbf{x})$ is continuous and nonsingular in $\overline{\mathbf{x}}$. Then the sequence $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ obtained using the two-step iterative method converges to $\overline{\mathbf{x}}$ and*

$$\lim_{k\rightarrow\infty} \frac{\|\mathbf{x}^{(k+1)}-\overline{\mathbf{x}}\|}{\|\mathbf{x}^{(k)}-\overline{\mathbf{x}}\|} = 0.$$

Proof. Firstly, Theorem 1 allows us to assure that

$$G(\mathbf{x}) = \mathbf{x}-2J_F(\mathbf{x}+\mathbf{z})^{-1}F(\mathbf{x}),$$

where $\mathbf{z} = \mathbf{x}-J_F(\mathbf{x})^{-1}F(\mathbf{x})$ is well-defined in a neighborhood of $\overline{\mathbf{x}}$, is differentiable in $\overline{\mathbf{x}}$ and $J_G(\overline{\mathbf{x}}) = 0$, and also that $\|J_F(\mathbf{x}+\mathbf{z})^{-1}\| < 2\beta$, where

$\beta = \|J_F(\overline{\mathbf{x}})^{-1}\|$.

If the sequence $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ is obtained by means of fixed point iteration on G, using Ostrowski's Theorem it can be concluded that $\{\mathbf{x}^{(k)}\}_{k\geq 0}$ converges to $\overline{\mathbf{x}}$. Moreover, as G is differentiable in $\overline{\mathbf{x}}$,

$$\lim_{k\to\infty} \frac{\|G(\mathbf{x}^{(k)}) - G(\overline{\mathbf{x}}) - J_G(\overline{\mathbf{x}})(\mathbf{x}^{(k)} - \overline{\mathbf{x}})\|}{\|\mathbf{x}^{(k)} - \overline{\mathbf{x}}\|} = 0,$$

but $J_G(\overline{\mathbf{x}}) = 0$, so this limit is equivalent to:

$$\lim_{k\to\infty} \frac{\|G(\mathbf{x}^{(k)}) - G(\overline{\mathbf{x}})\|}{\|\mathbf{x}^{(k)} - \overline{\mathbf{x}}\|} = \lim_{k\to\infty} \frac{\|\mathbf{x}^{(k+1)} - \overline{\mathbf{x}}\|}{\|\mathbf{x}^{(k)} - \overline{\mathbf{x}}\|} = 0. \qquad \square$$

3. Numerical Examples

In this section we will check the effectiveness of the two-step method, in order to find the zeros of several nonlinear functions. Hence, we solve some systems of nonlinear equations by the Newton's method (NM) and by our two-step method (TSM). As the results show, the new method works as well. Figures of residuals fall show a high-order convergence for all test problems, (see Figures 1-4). In all test problems the vector-valued function F is as $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x}))$. Numerical computations have been carried out in *MATLAB*, all with a common outline: every iterate is obtained from the previous one by means of an iterative expression:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - A^{-1}\mathbf{b},$$

where $\mathbf{x}^{(k)} \in \mathbb{R}^n$, A is a real $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. The matrix A and the vector $\mathbf{b}$ are different according to the method used, but in any case the inverse calculation $-A^{-1}\mathbf{b}$ is carried out solving the linear system $A\mathbf{y} = -\mathbf{b}$, using Gaussian elimination. So, the new estimation is easily obtained by the addition of the solution of the linear system and the previous iterate: $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{y}$. The stopping criteria for all examples is $\|F(\mathbf{x}^{(k)})\|_2 < 10^{-13}$, where $\mathbf{x}^{(k)}$ is the k-th approximation of the solution and $\|\cdot\|_2$ shows the Euclidean norm. In any test problem an exact solution or an approximate solution of the system is presented. Table 1 shows the numerical results.

Example 1. Consider the system $F(\mathbf{x}) = 0$ as:

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 - 9,\ x_1x_2x_3 - 1,\ x_1 + x_2 - x_3^2).$$

This system has a zero close to

$$\alpha = (-2.0902946,\ 2.1402581,\ -0.2235251)^t.$$

Example 2. Consider the system:

$$F(x_1, x_2) = (-x_1 + x_2^2 - 0.3,\ x_1^2 - x_2 - 0.2) = 0.$$

This has a zero close to

$$\alpha = (-0.28603216362886041412 8, -0.11818560136979282260 1)^t.$$

Example 3. The next example has the following form:

$$F(x_1, x_2, x_3) = (x_1^4 + x_2^4 + x_3^4 - 3,\ x_1^3 - x_2^3 + x_3^3 - 1,\ x_1^2 + x_2^2 - x_3^2 - 1) = 0,$$

we would like to obtain an approximation of exact root $\alpha = (1., 1., 1.)^t$.

Example 4. The fourth example is taken as

$$F(x_1, x_2, x_3) = (x_1^3 + x_2^3 + x_3^3 - 3,\ x_1^2 + x_2^2 - x_3^2 - 1,\ x_1^3 - x_2^4 + x_3^4 - 1) = 0,$$

we want to obtain an approximation of exact zero $\alpha = (1., 1., 1.)^t$.

Example 5. Consider $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), f_4(\mathbf{x}))$, where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)^t$ and $f_i : \mathbb{R}^4 \to \mathbb{R},\ i = 1, \cdots, 4$, such that

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= x_1^5 + e^{x_2^3} + x_3 x_4 - x_4^4, \\ f_2(x_1, x_2, x_3, x_4) &= e^{x_1 x_4} + x_2 x_4 + x_3^4 - x_4^2, \\ f_3(x_1, x_2, x_3, x_4) &= x_1 x_2 x_3 x_4 + e^{x_3} + x_4, \\ f_4(x_1, x_2, x_3, x_4) &= x_1^2 + x_2^2 - x_3^2 + x_1 x_2 x_4, \end{aligned}$$

we try to obtain an approximation of the exact solution $\alpha = (0, 0, 0, -1)^t$ for the above system.

Example 6. The last example is taken as

$$F(x_1, x_2, x_3) = (\cos x_2 - \sin x_1,\ x_3^{x_1} - \frac{1}{x_2},\ e^{x_1} - x_3^2) = 0,$$

which has a zero close to

$$\alpha = (0.90956949450, 0.6612268323, 1.5758341440)^t.$$

In Table 1 we report number of iterations for the Newton's method and two-step method. As we can see from Table 1 for all examples the number of iterations for two-step method is less than number of iterations for the Newton's method.

4. Large Systems of Nonlinear Equations

To show power of our method we apply the method on two large systems of nonlinear equations.

The stopping criteria used in these cases is

$$\|\mathbf{y}^{(k+1)} - \mathbf{y}^{(k)}\|_2 + \|F(\mathbf{y}^{(k)})\|_2 < 10^{-11},$$

where $\mathbf{y}^{(k)}$ is the k-th approximation of the solution. We solve both systems by the Newton's method (NM) and our new two-step method (TSM).

Example 7. We consider the following boundary value problem, which is given in [5]:

$$\begin{aligned} &y'' = -(y')^2 - y + \ln x, \quad x \in [1,2],\\ &y(1) = 0, \quad y(2) = \ln 2. \end{aligned}$$

The exact solution of this problem is $y(x) = \ln x$. To discretize the problem we use the second order finite difference method. The zeros of the following nonlinear functions will provide us an estimation of the solution of the problem: $F(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_{n-1}(\mathbf{y}))$, where $\mathbf{y} = (y_1, y_2, \cdots, y_{n-1})^t$ and for $i = 1, 2, \cdots, n-1$ we have $f_i : \mathbb{R}^{n-1} \to \mathbb{R}$, such that

$$\begin{aligned} f_1 &= 4y_2 + y_2^2 + 4y_1(h^2 - 2) - 4h^2 \ln x_2,\\ f_i &= 4(y_{i+1} + y_{i-1}) + (y_{i+1} - y_{i-1})^2 + 4y_i(h^2 - 2) - 4h^2 \ln x_{i+1},\\ &\qquad\qquad \text{for } i = 2, 3, \cdots, n-2,\\ f_{n-1} &= 4(\ln 2 + y_{n-2}) + (\ln 2 - y_{n-2})^2 + 4y_{n-1}(h^2 - 2) - 4h^2 \ln x_n. \end{aligned}$$

In the above system, the second order approximations are used for $y'(x_i)$ and $y''(x_i)$ by step $h = \frac{1}{n}$. By this step, we set the nodes x_i as: $x_i = 1 + ih$, $i = 0, 1, \cdots, n$. Also, y_i, $i = 1, \cdots, n-1$ denotes the unknown $y(x_i)$. In Table 2 some results can be observed: they have obtained by applying the Newton's method (NM) and two-step method (TSM). For every method and initial estimation, we analyze the number of iterations needed to converge to the solution. The initial estimations used are [5]:

$$\begin{aligned} &p = (p_1, \cdots, p_{n-1}), \quad p_i = 0 + i\tfrac{\ln 2}{n}, \quad i = 1, 2, \cdots, n-1\\ &q = (q_1, \cdots, q_{n-1}), \quad q_i = 2, \qquad\quad i = 1, 2, \cdots, n-1. \end{aligned}$$

As we can see from Table 2 for all cases the number of iterations for two-step method is less than number of iterations for the Newton's method.

Remark. As we can see from Figures 1-4 in first iterations of two-step method, convergence is linear. But after some iterations the cubic convergence appears.

Example 8. (Chandrasekhar H-Equation, see [12]) The Chandrasekhar integral equation [12] which arises from radiative transfer theory is a nonlinear integral equation which gives a full nonlinear system of equations if discretized. The Chandrasekhar integral equation is given by

$$F(P, c) = 0, \quad P : [0,1] \to \mathbb{R} \tag{7}$$

Example#	Initial guess	Method	Iterations
1	$(-2.5, 1, 1)^t$	NM	7
		TSM	4
2	$(-1, 1)^t$	NM	7
		TSM	4
3	$(1.25, 1.25, 1.25)^t$	NM	5
		TSM	4
4	$(1.5, 1.5, 1.5)^t$	NM	6
		TSM	4
5	$(0.6, 0.6, 0.6, -0.3)^t$	NM	26
		TSM	15
6	$(1, 0.5, 1)^t$	NM	6
		TSM	4

Table 1: The numerical results for Examples 1-6

Initial estimation	n	Method	Iterations
p	50	NM	3
		TSM	2
p	300	NM	3
		TSM	2
q	10	NM	9
		TSM	6
q	100	NM	11
		TSM	7
q	200	NM	12
		TSM	7
q	300	NM	12
		TSM	8

Table 2: The numerical results for Example 7

with parameter c and the operator F as

$$F(P, c)(y) = P(y) - \left(1 - \frac{c}{2}\int_0^1 \frac{y\ P(v)}{y+v}dv\right)^{-1}. \tag{8}$$

If we discretize the integral equation (8) using the mid-point integration rule with n grid points

$$\int_0^1 f(x)\ dx = \frac{1}{n}\sum_{j=1}^{n} f(x_j), \quad x_j = (j-0.5)h, \quad h = \frac{1}{n}, \quad j = 1, \cdots, n,$$

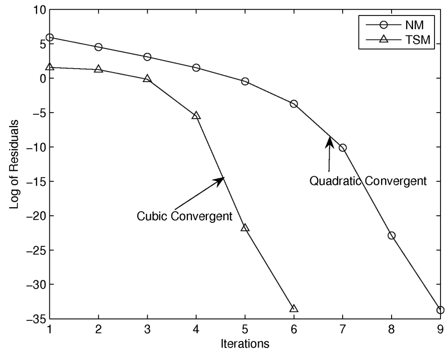

Figure 1: Residual fall for Example 7 with $n = 10$ and starting estimation q

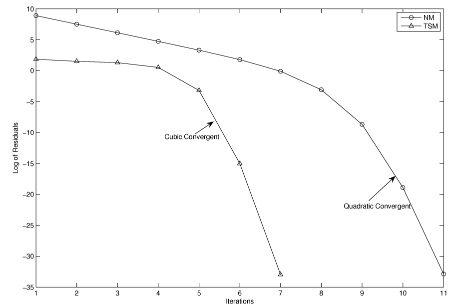

Figure 2: Residual fall for Example 7 with $n = 100$ and starting estimation q

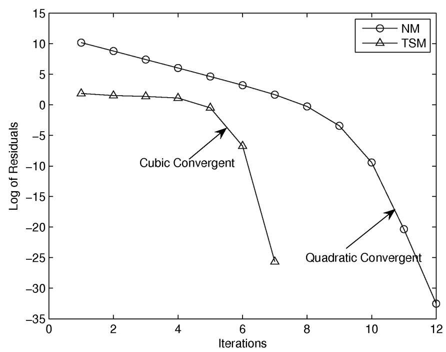

Figure 3: Residual fall for Example 7 with $n = 200$ and starting estimation q

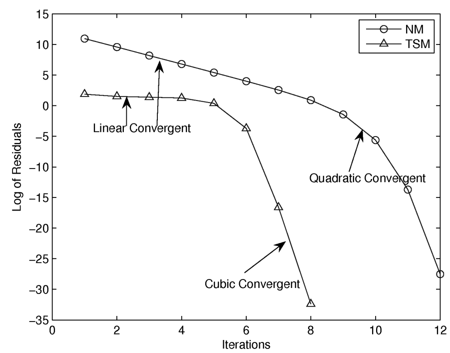

Figure 4: Residual fall for Example 7 with $n = 300$ and starting estimation q

we obtain the following system of nonlinear equations:

$$F_i(\mathbf{y}, c) = y_i - \left(1 - \frac{c}{2n}\sum_{j=1}^{n}\frac{x_i y_i}{x_i + x_j}\right)^{-1}, \quad i = 1, \cdots, n. \tag{9}$$

When starting with $(1, 1, \ldots.., 1)^t$ vector, the system (9) has a solution for all $c \in (0, 1)$. The c were equally spaced with $\Delta c = 0.02$ in the interval $c \in [0, 0.7]$ and we choose $n = 12, 30, 50, 80$. We note that in these cases the Jacobian is a full matrix. In Figures 5-8, we find as c tends to 0.7, the number of iterations to reach convergence increases in both methods. We also observe that the TSM gives better results in terms of number of iterations than second the order

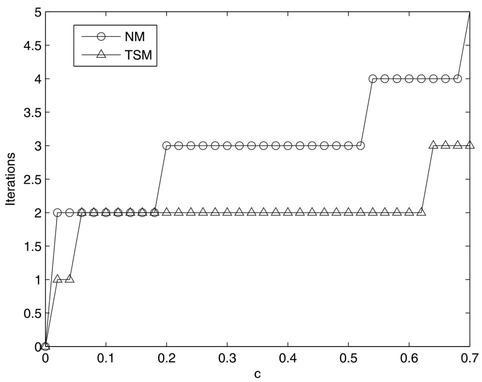

Figure 5: Convergence results for Example 8 with $n = 12$

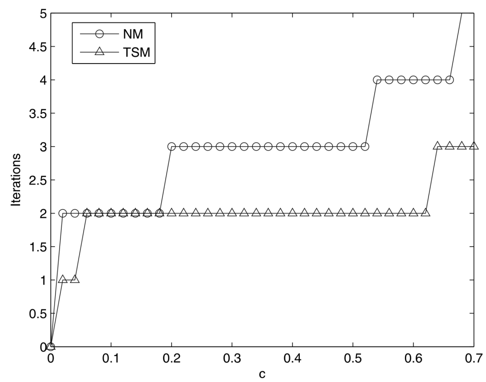

Figure 6: Convergence results for Example 8 with $n = 40$

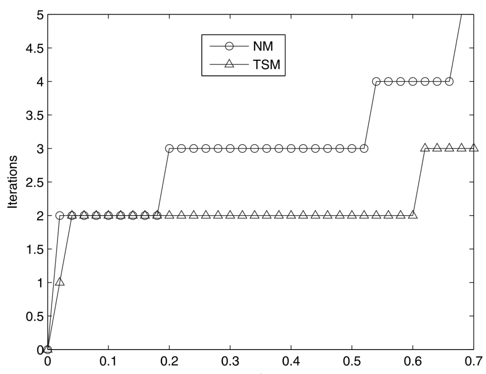

Figure 7: Convergence results for Example 8 with $n = 50$

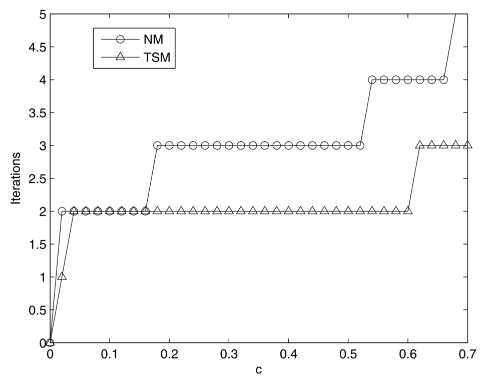

Figure 8: Convergence results for Example 8 with $n = 80$

Newton's method.

5. Conclusions

In this paper, we presented a two-step high order Newton-like method to solve systems of nonlinear equations. The numerical results and figures show that our new method can find zeros of nonlinear systems only by some iterations. The method works very well for large systems of nonlinear equations which arise from a boundary value problem and from the Chandrasekhar integral equation.

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