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# SOME THREE-STEP ITERATIVE METHODS FREE FROM SECOND ORDER DERIVATIVE FOR FINDING SOLUTIONS OF SYSTEMS OF NONLINEAR EQUATIONS

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**Abstract:** We present some three-step iterative methods to find roots of several variables functions. These methods do not need the evaluation of the second order derivative of the given $n$-valued nonlinear function. The new methods are extensions of iterative methods to find roots of nonlinear equation $f(x) = 0$. Convergency of the methods is proved. We provide numerical results to show the efficiency of the new methods for systems of nonlinear equations. It is observed that new methods take less number of iterations than the Newton's method. We compare the run time of the Newton's method and new three-step algorithms. Residual falls of logarithm of residuals show a high-order convergence of the new methods. Also we apply the new algorithms to solve the Chandrasekhar integral equation in radiative transfer.

**AMS Subject Classification:** 65H10, 65B99
**Key Words:** three step method, root finding, iteration number, nonlinear system, CPU-time, convergency

## 1. Introduction

One of the basic methods to solve nonlinear equation

$$f(x) = 0 \tag{1}$$

is the Newton's method which under certain condition converges quadratically.

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The Newton's method to solve equation (1) is:

For initial guess $x_0$ until convergence compute

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \dots. \tag{2}$$

In recent years, much attention has been given to develop several iterative methods to solve equation (1). Many researchers have expanded the Newton's method or have presented multi-stage Newton-like methods to solve nonlinear equation (1) (see, e.g., [26, 14, 4, 17, 19, 20, 6, 22, 21, 27, 24]).

Now consider the problem of finding a real zero of a nonlinear system

$$F(X) = 0, \tag{3}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is an $n$-variable $n$-valued function. In general, the zeros of several variables functions cannot be expressed in closed form. Therefore, we have to use iterative methods, where by starting from an initial guess $X^{(0)}$, we compute the next approximation $X^{(k)}$, $k = 1, 2, \dots$, recurrently:

$$X^{(k+1)} = G(X^{(k)}), \quad k = 0, 1, \dots.$$

There are many approaches to solve equation (3). The Newton's method to solve (3) is an important and basic method which converges quadratically. The Newton's method to solve (3) is an extension of the iterative method (2), as follows:

**Algorithm 0.** For initial guess $X^{(0)}$ until convergence do

$$X^{(k+1)} = X^{(k)} - J_F(X^{(k)})^{-1} F(X^{(k)}), \quad k = 0, 1, \dots, \tag{4}$$

where $J_F(X^{(k)})$ is the Jacobian of $F$ evaluated at $X^{(k)}$.

In recent years, some third and fourth order iterative methods have been proposed and analyzed for solving systems of nonlinear equations that improve some classical methods such as the Newton's method and Chebyshev-Halley methods. It has been demonstrated that the methods are efficient, and can compete with the Newton's method. For more details, see [1, 5, 10, 11, 13, 15, 16, 25]. Darvishi and Barati [7, 8, 9] presented some high order iterative methods free from second order derivative of function $F$, to solve equation (3). Frontini and Sormani [12] obtained a third-order method based on a quadrature formula to solve systems of nonlinear equations. Babajee et al [2] proposed a fourth order iterative method to solve equation (3).

The aim of this paper is to construct some new efficient iterative methods to solve systems of nonlinear equations. The new methods are based on the proposals of Noor and Noor [22], Noor et al [21] and Yun [27] to solve nonlinear equation (1). In this paper, we extend their methods to obtain some three-step high-order Newton-like methods to find a zero of systems of nonlinear equations.

The paper is organized as follows. In the next section we present the three-step high-order Newton-like methods for several variables functions. In Section 3 we investigate the convergency of our new algorithms. In Section 4 we solve some examples by the Newton's method and three-step methods, also we show the efficiency of new methods to solve the Chandrasekhar integral equation in radiative transfer. Finally we present some conclusions in Section 5.

## 2. The Three-Step Iterative Methods

Let $F:\mathbb{R}^n \to \mathbb{R}^n$, where $F=(f_1,f_2,\ldots,f_n)^t$ and $f_i:\mathbb{R}^n \to \mathbb{R}$. In this section to solve system (3) we introduce four three-step high-order algorithms. All algorithms are based on extension of similar ones to solve nonlinear equation $f(x)=0$. To find a simple root $\alpha$, that is, $f(\alpha)=0$ and $f'(\alpha)\neq 0$ of

$$f(x)=0, \tag{5}$$

where $f$ is a real valued function on $\mathbb{R}$, Noor and Noor [22] introduced two three-step methods. Their first method is as follows:

**Method 1.** (see [22, Algorithm 2.3]) For a given $x_0$, calculate the approximate solution $x_{k+1}$, by the following iterative scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= -\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f(y_k)}{f'(x_k)} - \frac{f(y_k+z_k)}{f'(x_k)}, \qquad k=0,1,\ldots. \end{aligned} \tag{6}$$

Their second method is:

**Method 2.** (see [22, Algorithm 2.4]) For a given $x_0$, compute the approximate solution $x_{k+1}$, by the following iterative scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= y_k - \frac{f(y_k)}{f'(y_k)}, \\ x_{k+1} &= z_k - \frac{f(z_k)}{f'(z_k)}, \qquad k=0,1,\ldots. \end{aligned} \tag{7}$$

Noor et al [21] introduced the following three-step method for solving equation (5).

**Method 3.** (see [21, Algorithm 2.6]) For a given $x_0$, calculate the approximate solution $x_{k+1}$, by the following iterative scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= -\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k + z_k)}{f'(x_k)}, \qquad k = 0, 1, \ldots . \end{aligned} \tag{8}$$

Finally, the last three-step method is due to Yun [27], who suggested the following scheme:

**Method 4.** (see [27, Equations 17]) For initial guess $x_0$ calculate the approximate solution $x_{k+1}$, by the following iterative steps:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k &= -\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} &= y_k + z_k - \frac{f(y_k + z_k)}{f'(x_k)}, \quad k = 0, 1, \ldots . \end{aligned} \tag{9}$$

We note that Methods 3 and 4 are remarkably different from each other, even though they are very similar.

Now, we extend the above methods to solve systems of nonlinear equations (3). From Method 1 we present the following algorithm to solve systems of nonlinear equations.

**Algorithm 1.** For a given initial guess $X^{(0)}$ until convergence do:

$$\begin{aligned} Y^{(k)} &= X^{(k)} - J_F(X^{(k)})^{-1} F(X^{(k)}), \\ Z^{(k)} &= -J_F(X^{(k)})^{-1} F(Y^{(k)}), \\ X^{(k+1)} &= X^{(k)} - J_F(X^{(k)})^{-1} \left( F(X^{(k)}) + F(Y^{(k)}) + F(Y^{(k)} + Z^{(k)}) \right), \\ &\qquad \text{for } k = 0, 1, \ldots . \end{aligned} \tag{10}$$

The following algorithm is an extension of Method 2.

**Algorithm 2.** For a given initial guess $X^{(0)}$ until convergence do:

$$\begin{aligned} Y^{(k)} &= X^{(k)} - J_F(X^{(k)})^{-1} F(X^{(k)}), \\ Z^{(k)} &= Y^{(k)} - J_F(Y^{(k)})^{-1} F(Y^{(k)}), \\ X^{(k+1)} &= Z^{(k)} - J_F(Z^{(k)})^{-1} F(Z^{(k)}), \qquad k = 0, 1, \ldots . \end{aligned} \tag{11}$$

Similarly, by extension of Method 3 we can obtain our third algorithm as follows:

**Algorithm 3.** For a given initial guess $X^{(0)}$ until convergence do:

$$\begin{aligned} &Y^{(k)} = X^{(k)} - J_F(X^{(k)})^{-1}F(X^{(k)}),\\ &Z^{(k)} = -J_F(X^{(k)})^{-1}F(Y^{(k)}),\\ &X^{(k+1)} = Y^{(k)} - J_F(X^{(k)})^{-1}F(Y^{(k)} + Z^{(k)}), \qquad k = 0, 1, \ldots . \end{aligned} \tag{12}$$

Finally, if we extend Method 4 we can obtain the following algorithm to solve systems of nonlinear equations:

**Algorithm 4.** For a given initial guess $X^{(0)}$ until convergence do:

$$\begin{aligned} &Y^{(k)} = X^{(k)} - J_F(X^{(k)})^{-1}F(X^{(k)}),\\ &Z^{(k)} = -J_F(X^{(k)})^{-1}F(Y^{(k)}),\\ &X^{(k+1)} = Y^{(k)} + Z^{(k)} - J_F(X^{(k)})^{-1}F(Y^{(k)} + Z^{(k)}), \quad k = 0, 1, \ldots . \end{aligned} \tag{13}$$

Note that we can show easily that Method 1 and Method 4 are similar methods, hence their extensions, that is, Algorithm 1 and Algorithm 4 are same, too. Therefore in fact we have only three new algorithms. These new algorithms do not need second order derivative of function $F$. For simplicity we show our algorithms by **Alg1-Alg3**, respectively. Also we call the Newton's method as **Alg0**. In the next section we consider the convergency for our new algorithms. Also in Section 4 we apply these algorithms to solve some systems of nonlinear equations.

## 3. Convergency Analysis

Our aim in this section is to prove convergency of our algorithms. The convergency analysis of the new algorithms are similar. Hence, we only show the convergency of Algorithms 1 and 3. Convergency analysis of Algorithm 2 is similar and straightforward. We need the following preliminaries.

### 3.1. Preliminaries

**Definition 1.** (see [23]) Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$. Then $\overline{X}$ is a point of attraction of the iteration

$$X^{(k+1)} = G(X^{(k)}), \quad k = 0, 1, \ldots . \tag{14}$$

If there is an open neighborhood $S$ of $\overline{X}$ such that $S \subset D$ and for any $X^{(0)} \in S$, the iterates $X^{(k)}$ defined by equation (14) lie all in $D$ and converge to $\overline{X}$.

**Lemma 1.** (see [23, Banach Perturbation Lemma]) *Let $A \in L(\mathbb{R}^n)$ be* *nonsingular. If $E \in L(\mathbb{R}^n)$ and $\|A^{-1}\|.\|E\| < 1$, then $A+E$ is nonsingular and*

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|.\|E\|}.$$

**Theorem 1.** (see [23, Ostrowski Theorem]) *Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function in $\overline{X}$, that is a solution of the system $X = G(X)$. Let $X^{(k)}$ be the sequence of iterations by means of fixed point iteration $X^{(k+1)} = G(X^{(k)}),\ k = 0, 1, 2, ...$ if the spectral radius of $J_G(\overline{X})$ is lower that 1, then $X^{(k)}$ converges to $\overline{X}$.*

### 3.2. Convergency of Algorithm 1

**Theorem 2.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function in $\overline{X}$, where $\overline{X}$ is a solution of the system of nonlinear equations $F(X) = 0$. Let us suppose that $J_F(X)$ is continuous and $J_F(\overline{X})$ is nonsingular. Then function*

$$G(X) = X - J_F(X)^{-1}C(X),$$

*where:*

$$C(X) = F(X) + F(Y) + F(Y+Z),$$

$$Z = -J_F(X)^{-1}F(Y),$$

$$Y = X - J_F(X)^{-1}F(X),$$

*is well-defined in a neighborhood of $\overline{X}$, is differentiable and*

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

*Proof.* Firstly, let us prove that $J_F(X)$ is nonsingular for any $X$ in a neighborhood of $\overline{X}$. Let $\beta = \|J_F(\overline{X})^{-1}\|$ and $\epsilon$ be such that $0 < \epsilon < (2\beta)^{-1}$ is satisfied. By continuity of $J_F(X)$ in $\overline{X}$ there exists a $\delta > 0$ such that $\|J_F(X) - J_F(\overline{X})\| \leq \epsilon$ if $\|X - \overline{X}\| \leq \delta$. Therefore, we have:

$$\|J_F(\overline{X})^{-1}\|\|J_F(X) - J_F(\overline{X})\| \leq \beta\epsilon < \frac{1}{2}.$$

Hence, by Lemma 1 $J_F(X)^{-1}$ exists and we have

$$\begin{aligned}\|J_F(X)^{-1}\| &= \|(J_F(\overline{X}) + (J_F(X) - J_F(\overline{X})))^{-1}\| \\ &\leq \frac{\|J_F(\overline{X})^{-1}\|}{1 - \|J_F(\overline{X})^{-1}\|\|J_F(X) - J_F(\overline{X})\|} \leq \frac{\beta}{1-\beta\epsilon} < 2\beta, \quad (15)\end{aligned}$$

for $X \in S = \{X : \|X - \overline{X}\| \leq \delta\}$. Thus $G(X)$ is well-defined.

Also $J_F(\overline{X}) = J_C(\overline{X})$, because

$$J_C(\overline{X}) = J_F(\overline{X}) + J_Y(\overline{X})J_F(\overline{X}) + J_{Y+Z}(\overline{X})J_F(\overline{X}),$$

where

$$J_Y(\overline{X}) = J_Z(\overline{X}) = 0.$$

Now, we prove that $C(X)$ is differentiable in a neighborhood of $\overline{X}$. We have

$$C(X) - C(\overline{X}) - J_F(\overline{X})(X - \overline{X}) = F(Y + Z) - F(\overline{X}) - J_F(\overline{X})(Y + Z - \overline{X})$$
$$- (J_F(X) - J_F(\overline{X}))(Y + Z - \overline{X}) + (J_F(X) - J_F(\overline{X}))(X - \overline{X}),$$

therefore

$$\|C(X)-C(\overline{X})-J_F(\overline{X})(X-\overline{X})\| \leq \|F(Y+Z)-F(\overline{X})-J_F(\overline{X})(Y+Z-\overline{X})\|$$
$$+ \|J_F(X) - J_F(\overline{X})\|\|Y + Z - \overline{X}\| + \|J_F(X) - J_F(\overline{X})\|\|X - \overline{X}\|.$$

By continuity of $J_F$ for any $X \in S$, differentiability of $F$ at $\overline{X}$ and convergency of the second Chebyshev-like method (CL2)[3], $C(X)$ is differentiable in a neighborhood of $\overline{X}$ and

$$\|C(X) - C(\overline{X}) - J_C(\overline{X})(X - \overline{X})\| \leq \epsilon\|X - \overline{X}\|. \tag{16}$$

Finally, we prove that

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

From (15), (16) and continuity of $J_F$, for any $X \in S$ we have

$$\begin{aligned}
&\|G(X) - G(\overline{X}) - (I - J_F(\overline{X})^{-1}J_C(\overline{X}))(X - \overline{X})\| \\
&\ = \|J_F(\overline{X})^{-1}J_F(\overline{X})(X - \overline{X}) - J_F(X)^{-1}C(X)\| \\
&\leq \|J_F(X)^{-1}(C(X) - C(\overline{X}) - J_F(\overline{X})(X - \overline{X}))\| \\
&\ +\|J_F(X)^{-1}(J_F(X) - J_F(\overline{X})J_F^{-1}(\overline{X})J_F(\overline{X})(X - \overline{X})\| \\
&\leq \|J_F(X)^{-1}\|\|C(X) - C(\overline{X}) - J_C(\overline{X})(X - \overline{X})\| \\
&\ +\|J_F(X)^{-1}\|\|(J_F(X) - J_F(\overline{X}))(X - \overline{X})\| \\
&\leq 2\beta\epsilon\|X - \overline{X}\| + 2\beta\epsilon\|X - \overline{X}\| = 4\beta\epsilon\|X - \overline{X}\|.
\end{aligned}$$

As $\epsilon$ is arbitrary and $\beta$ is constant, it can be concluded from the above inequality that G is differentiable in $X$ and also

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

This completes the proof. □

**Theorem 3.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at each point of an open neighborhood $S$ of $\overline{X}$, that is a solution of the system $F(X) = 0$. Let $J_F(X)$ is Lipschitz continuous in $S$ and nonsingular in $X$. Then the sequence $X^{(k+1)} = G(X^{(k)}),\ k = 0, 1, ...$ obtained by the iterative scheme (10) converges* *to $\overline{X}$ and*

$$\lim_{k\to\infty} \frac{\|X^{(k+1)} - \overline{X}\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

*Proof.* Theorem 2 ensures that $G(X)$ is well-defined in a neighborhood of $\overline{X}$, differentiable in $\overline{X}$ and spectral radius of $J_G(X)$ is less than 1.

Using Ostrowski's Theorem it can be concluded that $\{X^{(k)}\}_{k\geq 1}$ converges to $\overline{X}$, moreover, as $G$ is differentiable in $\overline{X}$ and

$$\lim_{k\to\infty} \frac{\|G(X^{(k)}) - G(\overline{X}) - J_G(\overline{X})(X^{(k)} - \overline{X})\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

From $J_G(\overline{X}) = 0$ we have

$$\lim_{k\to\infty} \frac{\|G(X^{(k)}) - G(\overline{X})\|}{\|X^{(k)} - \overline{X}\|} = \lim_{k\to\infty} \frac{\|X^{(k+1)} - \overline{X}\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

This completes the proof. □

### 3.3. Convergency of Algorithm 3

**Theorem 4.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function in $\overline{X}$, where $\overline{X}$ is a solution of the system of nonlinear equations $F(X) = 0$. Let us suppose that $J_F(X)$ is continuous and $J_F(\overline{X})$ is nonsingular. Then function*

$$G(X) = X - J_F(X)^{-1}C(X),$$

*where*

$$\begin{aligned} C(X) &= F(X) + F(Y + Z), \\ Z &= -J_F(X)^{-1}F(Y), \\ Y &= X - J_F(X)^{-1}F(X), \end{aligned}$$

*is well-defined in a neighborhood of $\overline{X}$, is differentiable and*

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

*Proof.* Firstly, we prove that $J_F(X)$ is nonsingular for any $X$ in a neighborhood of $\overline{X}$, let $\beta = \|J_F(\overline{X})^{-1}\|$ and $\epsilon$ be such that $0 < \epsilon < (2\beta)^{-1}$ is satisfied. By continuity of $J_F(X)$ in $\overline{X}$ there exists a $\delta > 0$ such that $\|J_F(X) - J_F(\overline{X})\| \leq \epsilon$ if $\|X - \overline{X}\| \leq \delta$. Therefore, we have

$$\|J_F(\overline{X})^{-1}\|\|J_F(X) - J_F(\overline{X})\| \leq \beta\epsilon < \frac{1}{2}.$$

By Lemma 1 $J_F(X)^{-1}$ exists and satisfies the followings

$$\|J_F(X)^{-1}\| = \|(J_F(\overline{X}) + (J_F(X) - J_F(\overline{X})))^{-1}\|$$

$$\leq \frac{\|J_F(\overline{X})^{-1}\|}{1-\|J_F(\overline{X})^{-1}\|\|J_F(X)-J_F(\overline{X})\|} \leq \frac{\beta}{1-\beta\epsilon} < 2\beta, \quad (17)$$

for $X \in S = \{X : \|X-\overline{X}\| \leq \delta\}$. Thus $G(X)$ is well-defined. Also $J_F(\overline{X}) = J_C(\overline{X})$, because

$$J_C(\overline{X}) = J_F(\overline{X}) + J_{Y+Z}(\overline{X})J_F(\overline{X}),$$

where $J_Y(\overline{X}) = J_Z(\overline{X}) = 0$. On the other hand, by differentiability of $F$ in $\overline{X}$ for $X \in S$ we have

$$\|F(X) - F(\overline{X}) - J_F(\overline{X})(X-\overline{X})\| \leq \epsilon\|X-\overline{X}\|.$$

Also, by the convergency of the second Chebyshev-like method (CL2) in [3], it can be assured that $\|Y+Z-\overline{X}\| \leq \delta$, therefore $C(X)$ is differentiable in a neighborhood of $\overline{X}$ and

$$\|C(X) - C(\overline{X}) - J_C(\overline{X})(X-\overline{X})\| \leq \epsilon\|X-\overline{X}\|. \quad (18)$$

Finally, we prove that

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

From (17), (18) and continuity of $J_F$ for any $X \in S$ we have

$$\begin{aligned}
&\|G(X) - G(\overline{X}) - (I - J_F(\overline{X})^{-1}J_C(\overline{X}))(X-\overline{X})\| \\
&= \|J_F(\overline{X})^{-1}J_F(\overline{X})(X-\overline{X}) - J_F(X)^{-1}C(X)\| \\
&\leq \|J_F(X)^{-1}(C(X) - C(\overline{X}) - J_F(\overline{X})(X-\overline{X}))\| \\
&\quad + \|J_F(X)^{-1}(J_F(X) - J_F(\overline{X})J_F(\overline{X})^{-1}J_F(\overline{X})(X-\overline{X})\| \\
&\leq \|J_F(X)^{-1}\|\|C(X) - C(\overline{X}) - J_C(\overline{X})(X-\overline{X})\| \\
&\quad + \|J_F(X)^{-1}\|\|(J_F(X) - J_F(\overline{X}))(X-\overline{X})\| \\
&\leq 2\beta\epsilon\|X-\overline{X}\| + 2\beta\epsilon\|X-\overline{X}\| = 4\beta\epsilon\|X-\overline{X}\|.
\end{aligned}$$

As $\epsilon$ is arbitrary and $\beta$ is constant, it can be concluded from the above inequality that G is differentiable in $X$ and also

$$J_G(\overline{X}) = I - J_F(\overline{X})^{-1}J_C(\overline{X}) = 0.$$

This completes the proof. □

**Theorem 5.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at each point of an open neighborhood $S$ of $\overline{X}$, that is a solution of the system $F(X) = 0$. Let $J_F(X)$ is Lipschitz continuous in $S$ and nonsingular in $X$. Then the sequence $X^{(k+1)} = G(X^{(k)}), k = 0, 1, ...$ obtained by the iterative scheme (12) converges to $\overline{X}$ and*

$$\lim_{k\to\infty} \frac{\|X^{(k+1)} - \overline{X}\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

*Proof.* Theorem 4 ensures that $G(X)$ is well-defined in a neighborhood of $\overline{X}$, differentiable in $\overline{X}$ and spectral radius of $J_G(X)$ is less than 1. Using Ostrowski's Theorem it can be concluded that $\{X^{(k)}\}_{k\geq 1}$ converges to $\overline{X}$, in addition, as $G$ is differentiable in $\overline{X}$

$$\lim_{k\to\infty} \frac{\|G(X^{(k)}) - G(\overline{X}) - J_G(\overline{X})(X^{(k)} - \overline{X})\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

As $J_G(\overline{X}) = 0$ we have

$$\lim_{k\to\infty} \frac{\|G(X^{(k)}) - G(\overline{X})\|}{\|X^{(k)} - \overline{X}\|} = \lim_{k\to\infty} \frac{\|X^{(k+1)} - \overline{X}\|}{\|X^{(k)} - \overline{X}\|} = 0.$$

Hence the iterative scheme (12) is convergent. □

## 4. Numerical Results

In this section, we solve some systems of nonlinear equations by the Newton's method (**Alg0**) and by Algorithm 1-Algorithm 3 (**Alg1-Alg3**). As the results show, the new methods work very well. In all test problems we have a vector-valued function such as $F(X) = (f_1(X), f_2(X), \cdots, f_n(X))$ for $X \in \mathbb{R}^n$.

Numerical computations have been carried out by *MATLAB*, all with a common outline: every iterate is obtained from the previous one by means of an iterative expression: $X^{(k+1)} = X^{(k)} - A^{-1}\mathbf{b}$, where $X^{(k)} \in \mathbb{R}^n$, $A$ is a real $n\times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. The matrix $A$ and the vector $\mathbf{b}$ are different according to the method used, but in any case the inverse calculation $-A^{-1}\mathbf{b}$ is carried out by solving the linear system $A\mathbf{u} = -\mathbf{b}$, using the Gaussian elimination without pivotting. So, the new estimation is easily obtained by the addition of the solution of the linear system and the previous iterate: $X^{(k+1)} = X^{(k)} + \mathbf{u}$. The stopping criteria is $\|F(X^{(k)})\|_2 < 10^{-13}$, where $\|\cdot\|_2$ shows the Euclidean norm. In each test problem an exact solution or an approximate solution of the system is presented by $X^*$. Table 1 shows the numerical results, in which for any test problem the number of iterations and CPU time are reported.

**Example 1.** (see [12, Example 1])

$$F(x_1, x_2) = (-x_1 + x_2^2 - 0.3,\ x_1^2 - x_2 - 0.2),$$

$X^* = (-0.28603216362886041412 8, -0.11818560136979282260 1)^t$.

**Example 2.** (see [5, Example (g)]) The second test problem is

$$f_1 = x_1^2 + x_2^2 + x_3^2 - 9,$$
$$f_2 = x_1x_2x_3 - 1,$$
$$f_3 = x_1 + x_2 - x_3^2.$$

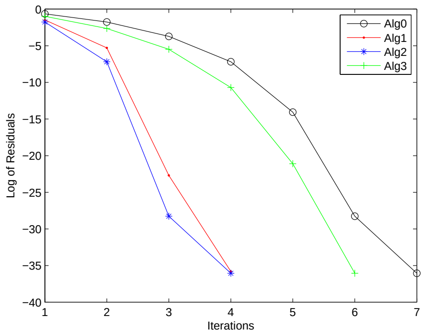


Figure 1: Residual fall for Example 4

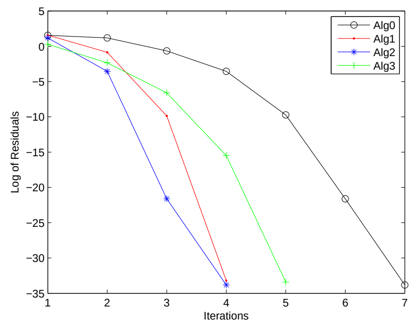


Figure 2: Residual fall for Example 4.

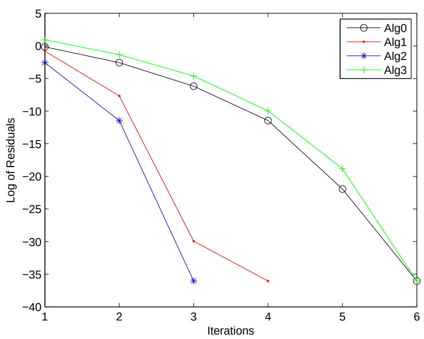


Figure 3: Residual fall for Example 4

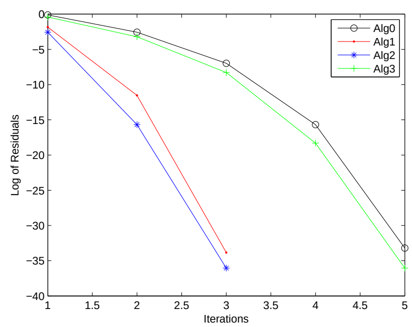


Figure 4: Residual fall for Example 4.

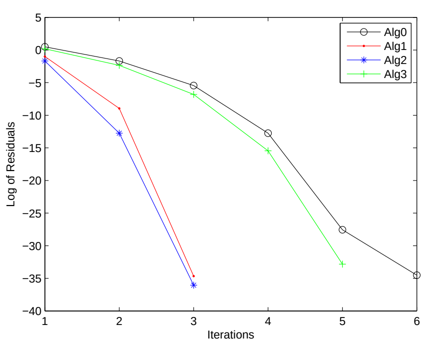


Figure 5: Residual fall for Example 4

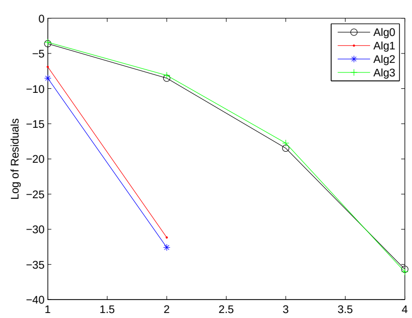


Figure 6: Residual fall for Example 4.

| Example# | $X^{(0)}$ | Algorithm | iter# | CPU-time(S) |
|---|---|---|---|---|
| 1 | $(-1,1)^t$ | **Alg0** | 7 | 0.109375 |
| | | **Alg1** | 4 | 0.078125 |
| | | **Alg2** | 4 | 0.078125 |
| | | **Alg3** | 6 | 0.093750 |
| 2 | $(-2.5,1,1)^t$ | **Alg0** | 7 | 0.125000 |
| | | **Alg1** | 4 | 0.093750 |
| | | **Alg2** | 4 | 0.109375 |
| | | **Alg3** | 5 | 0.109375 |
| 3 | $(1,0.5,1)^t$ | **Alg0** | 6 | 0.109375 |
| | | **Alg1** | 4 | 0.093750 |
| | | **Alg2** | 3 | 0.093750 |
| | | **Alg3** | 6 | 0.109375 |
| 4 | $(1.5,1.5,1.5)^t$ | **Alg0** | 6 | 0.125000 |
| | | **Alg1** | 4 | 0.093750 |
| | | **Alg2** | 3 | 0.093750 |
| | | **Alg3** | 6 | 0.125000 |
| 5 | $(1.5,1.5,1.75)^t$ | **Alg0** | 6 | 0.109375 |
| | | **Alg1** | 4 | 0.093750 |
| | | **Alg2** | 3 | 0.093750 |
| | | **Alg3** | 6 | 0.109375 |
| 6 | $(0.5,0.5,0.5,-0.3)^t$ | **Alg0** | 4 | 0.093750 |
| | | **Alg1** | 2 | 0.031250 |
| | | **Alg2** | 2 | 0.078125 |
| | | **Alg3** | 4 | 0.109375 |

Table 1: The numerical results for Examples 1-6

Its approximate solution is $X^* = (-2.0902946, 2.1402581, -0.2235251)^t$, given in [5].

**Example 3.** (see [9, Example 2])

$$F(x_1,x_2,x_3) = (\cos x_2 - \sin x_1,\ x_3^{x_1} - \frac{1}{x_2},\ e^{x_1} - x_3^2),$$

$X^* = (0.90956949450, 0.6612268323, 1.5758341440)^t$.

**Example 4.**

$F(x_1,x_2,x_3) = (x_1^4 + x_2^4 + x_3^4 - 3,\ x_1^3 - x_2^3 + x_3^3 - 1,\ x_1^2 + x_2^2 - x_3^2 - 1),$

$X^* = (1., 1., 1.)^t$.

| $n$ | Algorithm | iter# | CPU-time(S) |
|---|---|---|---|
| 5 | **Alg0** | 4 | 0.109375 |
| | **Alg1** | 2 | 0.062500 |
| | **Alg2** | 2 | 0.062500 |
| | **Alg3** | 4 | 0.109375 |
| 20 | **Alg0** | 4 | 0.265625 |
| | **Alg1** | 2 | 0.203125 |
| | **Alg2** | 2 | 0.375000 |
| | **Alg3** | 4 | 0.390625 |
| 50 | **Alg0** | 4 | 1.500000 |
| | **Alg1** | 2 | 0.906250 |
| | **Alg2** | 2 | 2.187500 |
| | **Alg3** | 4 | 1.828125 |
| 100 | **Alg0** | 4 | 9.078125 |
| | **Alg1** | 2 | 4.937500 |
| | **Alg2** | 2 | 13.437500 |
| | **Alg3** | 4 | 9.875000 |
| 200 | **Alg0** | 4 | 9.078125 |
| | **Alg1** | 2 | 4.937500 |
| | **Alg2** | 2 | 13.437500 |
| | **Alg3** | 4 | 9.875000 |

Table 2: The numerical results for Example 7 with $c = 0.5$

**Example 5.**

$F(x_1, x_2, x_3) = (x_1^3 + x_2^3 + x_3^3 - 3,\ x_1^2 + x_2^2 - x_3^2 - 1,\ x_1^3 - x_2^4 + x_3^4 - 1),$

$X^* = (1., 1., 1.)^t$.

**Example 6.** (see [9, Example 2])

$$\begin{aligned} f_1 &= x_1x_3 + x_1x_4 + x_3x_4, \\ f_2 &= x_2x_3 + x_2x_4 + x_3x_4, \\ f_3 &= x_1x_2 + x_1x_3 + x_2x_3 - 1, \\ f_4 &= x_1x_2 + x_1x_4 + x_2x_4, \end{aligned}$$

Its approximate solution is given in [9] as $X^* = (x_1, x_2, x_3, x_4)^t$ where

$$\begin{aligned} x_1 &= 0.577350269189625764509148, \\ x_2 &= 0.577350269189625764509148, \\ x_3 &= 0.577350269189625764509148, \\ x_4 &= -0.288675134594812882254574. \end{aligned}$$

In the last example we apply the Newton and three-step methods to solve a real problem. This problem is a nonlinear integral equation. Discretizing the equation by the mid-point integration rule obtains a system of nonlinear equations.

**Example 7.** (Chandrasekhar $H$-Equation, see [18]) The Chandrasekhar integral equation which arises from radiative transfer theory is a nonlinear integral equation which gives a full nonlinear system of equations if discretized. The Chandrasekhar integral equation is given

$$F(P,c)=0, \quad P:[0,1]\to\mathbb{R} \tag{19}$$

with parameter $c$ and the operator $F$ as

$$F(P,c)(y)=P(y)-\left(1-\frac{c}{2}\int_0^1 \frac{y\ P(v)}{y+v}dv\right)^{-1}. \tag{20}$$

If we discretize the integral equation (20) using the mid-point integration rule with $n$ grid points

$$\int_0^1 f(x)\ dx=\frac{1}{n}\sum_{j=1}^{n} f(x_j), \quad x_j=(j-0.5)h, \quad h=\frac{1}{n}, \quad j=1,\cdots,n,$$

we obtain the following system of nonlinear equations:

$$F_i(\mathbf{y},c)=y_i-\left(1-\frac{c}{2n}\sum_{j=1}^{n}\frac{x_i y_i}{x_i+x_j}\right)^{-1}, \quad i=1,\cdots,n. \tag{21}$$

When starting with $(1,1,.....,1)^t$ vector, system (21) has a solution for all $c\in(0,1)$. The $c$ were equally spaced with $\Delta c=0.02$ in the interval $c\in[0,0.7]$ and we choose $n=5,20,50,100$. We note that in these cases the Jacobian is a full matrix. We observe that the **Alg1** gives better results in terms of number of iterations and CPU time than the other methods.

## 5. Conclusions

In this paper, we presented some three-step high-order iterative methods to solve systems of nonlinear equations. The numerical results and figures of residual falls show that our new methods can find zeros of nonlinear systems only by some iterations. As reported results show the number of iterations for the Newton's method is more than number of iterations of the three-step methods. For small test problems, among three-step methods the **Alg1** is the best, even though the number of iterations for that method is sometimes more than **Alg2** but its run time is always better than **Alg2**. As we can see from residual falls of errors the order of convergency of **Alg3** is as same as the Newton's method, this means that the algorithm converges quadratically. Hence, we can claim that Algorithm 1 and Algorithm 2 have faster convergency, e.g., third order convergency. All methods work very well for Chandrasekhar integral $H$-equation. In Chandrasekhar integral equation the Jacobian matrix in any iteration is a full matrix. As Table 2 shows, for this example the **Alg1** is the best, while sometimes **Alg2** or **Alg3** are worse than the Newton's method.

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