International Journal of Pure and Applied Mathematics
Volume 57 No. 4 2009, 575-592

NUMERICAL METHODS OF SOLVING SOME NONLINEAR HEAT TRANSFER PROBLEMS

Harijs Kalis[1 §], Ilmārs Kangro[2], Aigars Gedroics[3]

[1]Institute of Mathematics and Informatics
University of Latvia
29, Raina Blvd., Riga, LV 1459, LATVIA
e-mail: kalis@lanet.lv

[2]Department of Engineering Science
Rezekne Higher Education Institute
90, Atbrivosanas Aleja, Rezekne, LV 4601, LATVIA
e-mail: kangro@cs.ru.lv

[3]Faculty of Physics and Mathematics
University of Latvia
8, Zelluiela, Riga, LV 1002, LATVIA
e-mail: aigars.gedroics@lu.lv

Abstract: We study the numerical methods for solving the initial-boundary value problems of some nonlinear heat transfer equations in multi-layer domain. The approximation of corresponding initial boundary value problems is based on the finite volume method (FVM), on the boundary element method (BEM), and on the finite-difference scheme (FDS). These methods enable to reduce the nonlinear heat transfer problem described by nonlinear partial differential equations (PDEs) to initial value problem for system of nonlinear ordinary differential equations (ODEs) of the first order. An example of the initial-boundary problem for PDEs (with power functions)

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 \lambda (u(x,t))^{\sigma+1}}{\partial x^2} + a(u(x,t))^{\beta}, \quad x \in [0,l], t > 0, \tag{1}$$

by $\sigma \geq 0, \beta > 0, \lambda > 0, a \geq 0$ and with conditions $u(0,t) = u(l,t) = 0, u(x,0) = u_0(x) \geq 0$ is considered.

Received: November 24, 2009

§Correspondence author

A large number of papers in the time period of 1970-1990 are devoted to the quasilinear parabolic equations with the blow-up solutions. In papers [4], [5], [6], [10],[3], [11], [7], [8], [9], [21], [12] the theoretical investigations (self-similar solutions and a'priori estimations for solving Cauchy and boundary-value problems) and numerical experiments (see [13], [14], [18], [19], [1]) by $\lambda = a = 1$ were done. In this paper we study the behaviour of the solutions at the time and also when $t \to \infty$, depending on the parameters $\sigma, \beta, \lambda, a$.

Depending on the parameters two type of solutions are obtained:

1) for large value of the time t the solution is stationary or tends to zero,

2) in the fixed time moment the solution has "blow up" phenomena – the solution is unbounded and tends to infinity in a small interval or in all domains by a fixed time moment.

AMS Subject Classification: 65M06, 65N06, 65M08, 65M20, 65M22, 65N22,65N40
Key Words: nonlinear heat transfer, multi-layer domain, the finite volume method, finite-difference schemes, "blow up" solutions

1. The Mathematical Model

The 1D domain with thickness l is multilayer media Ω of N layers $\Omega = \{x : x \in \Omega_k, k = \overline{1,N}\}$, where each layer is in the form $\Omega_k = \{x : x_{k-1} \le x \le x_k\}$, $x_0 = 0$, $x_N = l$; $x_k (k = \overline{1,N-1})$ are the joint of the layers (the interior grid points). We shall consider the initial-boundary value problem for solving of function $u = u(x,t)$ from the following nonlinear heat transfer PDEs:

$$\frac{\partial u}{\partial t} = \frac{\partial^2(\lambda g(u))}{\partial x^2} + f(u), \quad x \in \Omega, t > 0, \tag{2}$$

where in every layer $\lambda > 0$ is the constants coefficient of heat conductivity, $g(u)$ is nonlinear continuously differentiable function with $\partial g/\partial u = g'(u) > 0$, $f(u)$ is nonlinear continuous source function.

In every layer Ω_k the functions g, f, u and λ are in the form $g(u_k), f(u_k), u_k$, λ_k, $k = \overline{1,N}$, and the PDEs (2) can be rewritten in the following form:

$$\frac{\partial}{\partial x}(\lambda_k g'(u_k)\frac{\partial u_k(x,t)}{\partial x}) = F_k, \quad k = \overline{1,N}, \tag{3}$$

where $F_k = \dot{u}_k(x,t) - f(u_k(x,t))$, $\dot{u}_k = \frac{\partial u_k(x,t)}{\partial t}$, $u_k = u_k(x,t)$ is the unknown function in the layer $\Omega_k (x \in \Omega_k)$, t is the time.

We have the continuity conditions on the interior surfaces $x = x_k, k =$

$\overline{1, N-1}$

$$\begin{cases} u_k(x_k, t) = u_{k+1}(x_k, t), \\ \lambda_k g_x(u_k(x_k, t)) = \lambda_{k+1} g_x(u_{k+1}(x_k, t)), \end{cases} \tag{4}$$

and boundary conditions on the exterior surfaces $x = x_0 = 0,\ x = x_N = l$

$$\begin{cases} \lambda_1 g_x(u_1(0, t)) = \alpha_0(u_1(0, t) - T_0(t)), \\ \lambda_N g_x(u_N(l, t)) = \alpha_l(T_l(t) - u_N(l, t)), \end{cases} \tag{5}$$

where α_0, α_l are heat transfer coefficients, $g_x = \partial g/\partial x = g'(u)\partial u/\partial x$, T_0, T_l are known functions depending on t. If $\alpha_l = 0$, then we have the boundary conditions of the second kind (the conditions of symmetry by $x = l/2$).

For the initial condition by $t = 0$ we give

$$u_k(x, 0) = \phi_k(x), \quad k = \overline{1, N}, \tag{6}$$

where $\phi_k(x)$ is continuous function in every layer.

If the coefficients α_0, α_l equal infinity ($\alpha_0 = \infty$ and $\alpha_l = \infty$), we have the first kind boundary conditions in the form

$$u_1(0, t) = T_0(t), \quad u_N(l, t) = T_l(t). \tag{7}$$

The corresponding linear transfer problems are considered in [15], [17], [16].

2. Some Theoretical Aspects in One Layer ($N = 1$)

Similarly [10], [21] for the homogeneous boundary conditions (7), $T_0 = T_l = 0, N = 1, x_1 = l$), multiplying the equation (2) by function $\psi(x) = \frac{\pi}{2l}\sin(\frac{\pi x}{l}) \geq 0$ (this is the first normed eigenfunction for operator $(-\frac{\partial^2}{\partial x^2})$ with first kind of homogeneous boundary conditions: $\psi''(x) = -\mu_1\psi(x)$, $\psi(0) = \psi(l) = 0, \int_0^l \psi(x)dx = 1$, $\mu_1 = (\frac{\pi}{l})^2$ is the first eigenvalue) and integrating it by parts twice we get

$$\frac{dE}{dt} = \int_0^l \psi(x)(f(u) - \lambda\mu_1 g(u))dx, \tag{8}$$

where $E(t) = \int_0^l \psi(x)u(x, t)dx \geq 0$, if $u(x, t) \geq 0, E(0) = \int_0^l \psi(x)\phi(x)dx$.

For the power functions $g(u) = u^{\sigma+1}, g'(u) = (\sigma + 1)u^\sigma, f(u) = au^\beta$, $\beta \geq 1, \sigma \geq 0, a =$ const. > 0 (in [21] $\lambda = a = 1$), $u(x, t) \geq 0$ for all $t \geq 0$, if $\phi(x) = \phi_1(x) \geq 0$ and $E(0) \geq 0, E(t) \geq 0$.

In this case equation (8) has the following form

$$\frac{dE}{dt} = a\int_0^l \psi(x)(u^\beta - Qu^{\sigma+1})dx, \tag{9}$$

where $\boldsymbol{Q = \frac{\lambda\mu_1}{a} > 0}$ is essential parameter.

If $u(0,t) = u(l,t) = 0$, then $g'(u) = 0$ by $x = 0; x = l$ and the solution of the problem (2)-(6) is not classical [21].

For the generalized solution when $\boldsymbol{\beta < \sigma + 1}$ we can obtain from (9) the estimation, using the Jang's inequality $vw \leq \frac{1}{p}\epsilon_1^p v^p + \frac{1}{q}\epsilon_1^{-q} w^q, v = u^\beta, w = 1, p = \frac{\sigma+1}{\beta} > 1, q = \frac{\sigma+1}{\beta+1-\beta} > 1, \frac{1}{p} + \frac{1}{q} = 1$. Then $u^\beta \leq \frac{\beta}{\sigma+1}(\epsilon_1)^{(\sigma+1)/\beta} u^{\sigma+1} + \frac{\sigma+1-\beta}{\sigma+1}(\epsilon_1)^{(\sigma+1)/(\beta-\sigma-1)}$.

If $\epsilon_1 = (\frac{(\sigma+1)Q}{\beta})^{\beta/(\sigma+1)}$, then from (9) the inequality $\frac{dE}{dt} \leq ac\int_0^l \psi(x)dx = ac$ follows, where $c = \frac{\sigma+1-\beta}{\sigma+1}(\frac{(\sigma+1)Q}{\beta})^{\beta/(\beta-\sigma-1)}$.

After integrating this inequality with respect to t we obtain a priori estimation in the form

$$E(t) \leq E(0) + cat,$$

and *the solution is bounded for finite value of* $\boldsymbol{t < \infty}$.

For $\boldsymbol{\beta = \sigma + 1}$ from (9) we get

$$\frac{dE}{dt} = a(1 - Q)I(t), \tag{10}$$

where $I(t) = \int_0^l \psi(x)u^{\sigma+1}dx$.

We can obtain from (10) three types of solutions:

1) if $\boldsymbol{Q > 1}$, then $\frac{dE}{dt} < 0$, $E(t) < E(0)$ for $t > 0$ and $E(t) \to 0$ or $u(x,t) \to 0$ uniformly for all $x \in (0,l)$,if $t \to \infty$, *the stationary solution* $u_{st}(x)$ *is equal to zero,*

2) if $\boldsymbol{Q = 1}$, then $\frac{dE}{dt} = 0$, $E(t) = E(0)$ for $t \geq 0$ and $u(x,t) \to u_{st}(x)$, *the solution tends to stationary solutions* u_{st}, if $t \to \infty$,

3) if $\boldsymbol{Q < 1}$, then $\frac{dE}{dt} > 0$, $E(t) > E(0)$ for $t > 0$ and *the solution is increasing in the time (unbounded).*

In the third case from Jensen's inequality $I(t) \geq E(t)^{\sigma+1}$ for convex functions [21] it follows, that

$$\frac{dE}{dt} \geq a(1 - Q)E^{\sigma+1}.$$

For fixed time moment $T_* < \infty$ when $E^\sigma(T_*) = \infty$ (the "blow up" phenomena) we obtain

$$T_* = \frac{1}{\sigma a(1 - Q}E(0)^{-\sigma}. \tag{11}$$

If $\boldsymbol{\beta > \sigma + 1}$, then similarly [6] from convex function $p(s) = s^{\beta/(\sigma+1)}, s > 0$

and Jensen's inequality follows $(\psi, u^\beta) = \int_0^l \psi(x)u^\beta(x,t)dx \geq I(t)^{\beta/(\sigma+1)}$.

Then from (9) it follows that $\frac{dE}{dt} \geq aI(t)^{\beta/(\sigma+1)}(1 - \frac{Q}{I(t)^{\beta/(\sigma+1)-1}})$.

From Jensen's inequality $I(t) \geq E(t)^{\sigma+1}$ we obtain

$$\frac{dE}{dt} \geq aI(t)^{\beta/(\sigma+1)}(1 - \frac{Q}{E^{\beta-\sigma-1}}). \quad (12)$$

Then from the inequality

$$QE_0^\sigma < E_0^{\beta-1}(E_0 = E(0)) \quad (13)$$

we get $\frac{dE}{dt} \geq 0$ and $E(t) \geq E_0$. From Jensen's inequality and (12) we obtain the inequality

$$\frac{dE}{dt} \geq aE^\beta(1 - \frac{Q}{E_0^{\beta-\sigma-1}}).$$

For fixed time moment $T_* < \infty$ when $E^{\beta-1}(T_*) = \infty$ we obtain

$$T_* = 1/(a(\beta-1)(E_0^{\beta-1} - QE_0^\sigma)). \quad (14)$$

If $\beta = \sigma + 1$, then from (14) follows (11)

For the generalized solution of (1) similarly we can obtain some a priori estimations by $\boldsymbol{\beta \leq \sigma + 1}$.

Multiplying the equation (1) by function $u^{\sigma+1}$, integrating it by parts respect to x from zero to l and integrating with respect to t from zero to $t > 0$ we can easily see that

$$\frac{1}{\sigma+2}||u(t)||_{\sigma+1}^{\sigma+1} + \lambda\int_0^T ||u_x^{\sigma+1}||_2^2 d\tau = a\int_0^t (u^{\sigma+1}, u^\beta)d\tau + \frac{1}{\sigma+2}||u(0)||_{\sigma+1}^{\sigma+1}, \quad (15)$$

where $u_x^{\sigma+1} = (u^{\sigma+1})_x = \frac{\partial u^{\sigma+1}}{\partial x}, (v,w) = \int_0^l v(x,t)w(x,t)dx,\ v = u^\beta, w = u^{\sigma+1}, ||u||_p = (\int_0^l |u(x,t|^p dx)^{1/p}$ is the norm of the functions $u(x,t)$ in the Banach space $L_p(0,l)$ with respect to x by fixed value $t, p > 1$.

From Cauchy-Schwarz inequality $((v,w) \leq ||v||_2||w||_2, v = u^\beta, w = u^{\sigma+1})$ and ϵ-inequality $(vw \leq \epsilon|v|^2 + \frac{1}{4\epsilon}|w|^2, v = ||u^\beta||_2, w = ||u^{\sigma+1}||_2, \epsilon > 0)$ follows that the right hand side integral

$$\int_0^t (u^{\sigma+1}, u^\beta)d\tau \leq \epsilon\int_0^t ||u^\beta||_2^2 d\tau + \frac{1}{4\epsilon}\int_0^t ||u^{\sigma+1}||_2^2 d\tau.$$

Using Helder's inequality $((v,w) \leq ||v||_p||w||_q, w = 1, v = u^{2\beta}, \frac{1}{p} + \frac{1}{q} = 1,$

$p = \frac{\sigma+1}{\beta} > 1, q = \frac{\sigma+1}{\sigma+1-\beta} > 1$) we obtain the estimation

$$\int_0^t ||u^\beta||_2^2 d\tau \le l^{(\sigma+1-\beta)/(\sigma+1)} \int_0^t ||u^{\sigma+1}||_2^{2\beta/(\sigma+1)} d\tau.$$

From Poincare's-Friedrichs' inequality ($||v||_2^2 \le \frac{1}{\mu_1}||v_x||_2^2, v = u^{\sigma+1}, \mu_1 = \frac{\pi^2}{l^2}$) it follows that

$$a\int_0^t (u^{\sigma+1}, u^\beta) d\tau \le c_1 \int_0^t ||u_x^{\sigma+1}||_2^{2\beta/(\sigma+1)} d\tau + \frac{a}{4\epsilon\mu_1}\int_0^t ||u_x^{\sigma+1}||_2^2 d\tau, \qquad (16)$$

where $c_1 = a\epsilon l^{(\sigma+1-\beta)/(\sigma+1)}(\mu_1)^{-\beta/(\sigma+1)}$ is a given constant.

To carry out the powers of the same size the Jang's inequality ($w = c_1, v = ||u_x^{\sigma+1}||_2^{2\beta/(\sigma+1)}, \epsilon_1 = (\frac{\lambda(\sigma+1)}{4\beta})^{\beta/(\sigma+1)}, \epsilon = \frac{a}{\lambda\mu_1}$) gives

$$a\int_0^t (u^{\sigma+1}, u^\beta) d\tau \le \frac{\lambda}{2}\int_0^t ||u_x^{\sigma+1}||_2^2 d\tau + c_2 t,$$

where $c_2 = l(\frac{\sigma+1-\beta}{\sigma+1})(\frac{a^2}{\lambda})^{\sigma+1)/(\sigma+1-\beta)}(\frac{4\beta}{\lambda(\sigma+1)})^{\beta/(\sigma+1-\beta)}\mu_1^{(\sigma+1+\beta)/(\beta-\sigma-1)} > 0$.

Furthermore from (15) we have the following a priori estimation:

$$\frac{1}{\sigma+2}||u(t)||_{\sigma+1}^{\sigma+1} + \frac{\lambda}{2}\int_0^t ||u_x^{\sigma+1}||_2^2 d\tau \le \frac{1}{\sigma+2}||u(0)||_{\sigma+1}^{\sigma+1} + c_2 t. \qquad (17)$$

From (17) it follows that by $\boldsymbol{\beta < \sigma + 1}$ *the generalized solutions exist and are bounded for finite value of* $t < \infty$.

If $\boldsymbol{\beta = \sigma + 1}$**,** then from (16) we can directly obtained

$$a\int_0^t (u^{\sigma+1}, u^\beta) d\tau \le \frac{a}{\mu_1}(\epsilon + \frac{1}{4\epsilon})\int_0^t ||u_x^{\sigma+1}||_2^2 d\tau,$$

and from (15) and from the inequality $\frac{a}{\mu_1}(\epsilon + \frac{1}{4\epsilon}) \le (\lambda - \epsilon_2), \lambda > \epsilon_2 > 0$ or $2\epsilon \in [Q - \sqrt{Q^2-1}, Q + \sqrt{Q^2-1}]$, $\boldsymbol{Q > 1}$ an a priori estimation follows

$$\frac{1}{\sigma+2}||u(t)||_{\sigma+1}^{\sigma+1} + \epsilon_2 \int_0^t ||u_x^{\sigma+1}||_2^2 d\tau \le \frac{1}{\sigma+2}||u(0)||_{\sigma+1}^{\sigma+1} \qquad (18)$$

and *the solution is bounded for all values of* $t \le \infty$.

Estimation (18) by $\epsilon_2 = \lambda - a/\mu_1 > 0$ we can obtain directly of (15) using Poincare's-Friedrichs' inequality.

When $t \to \infty$ the integral $\int_0^\infty ||u_x^{\sigma+1}||_2^2 d\tau$ is bounded and $||u_x^{\sigma+1}||_2 \to 0$, or *the stationary solutions* $u_{st}(x)$ *are equal to zero.*

If $\boldsymbol{Q = 1}$, $\epsilon_2 = 0$, then $\epsilon = 0.5$ and we have following estimation for $t \in [0, \infty]$:

$$\frac{1}{\sigma+2}||u(t)||_{\sigma+1}^{\sigma+1} \le \frac{1}{\sigma+2}||u(0)||_{\sigma+1}^{\sigma+1}. \qquad (19)$$

In this case *the solutions are bounded,* $||u_{st}||_{\sigma+2} \leq ||u(0)||_{\sigma+2}$ *for all* $t \in [0, \infty]$.

In paper [4] by $a = \lambda = 1$ it is proved that:

1) by $\beta < \sigma+1$ there exists a global bounded solution for all t and $||u(0)||_2 < \infty$;

2) by $\beta = \sigma+1$ and $\mu_1 > 1$ there exists a global bounded solution for all all t and $||u(0)||_2 < \infty$, but by $\mu_1 < 1$ and sufficient large $||u(0)||_2$ there exists a local solution which tends to infinity if $t \to T_* < \infty (T_*$ is the finite value of $t)$;

3) by $\beta > \sigma + 1$ there exists a global bounded solution for sufficient small $||u(0)||_2$, but for larger $||u(0)||_2$, there exists a finite value T_* when $u(x,t) \to \infty$ if $t \to T_*$.

3. The Finite Volume Method and the Nonlinear Finite-Difference Scheme

Using the method of finite volumes [15] for the boundary value problem (3)-(5) we obtain the following nonlinear finite-difference scheme (FDS) with respect to the grid points $x_k,\ k = \overline{0, N}$ (for given function F_k) [17]:

$$\lambda_1 h_1^{-1}(g(u_1) - g(u_0)) - \alpha_0(u_0 - T_0) = \bar{R}_0^+, \tag{20}$$

$$\lambda_{k+1} h_{k+1}^{-1}(g(u_{k+1}) - g(u_k)) - \lambda_k h_k^{-1}(g(u_k) - g(u_{k-1})) = \bar{R}_k,\ k = \overline{1, N-1}, \tag{21}$$

$$\alpha_l(T_l - u_N) - \lambda_N h_N^{-1}(g(u_N) - g(u_{N-1})) = \bar{R}_N^-, \tag{22}$$

where:

$$\bar{R}_k = \bar{R}_k^+ + \bar{R}_k^-, \quad \bar{R}_k^+ = I_k^+ + R_k^+, \quad \bar{R}_k^- = I_k^- + R_k^-,$$

$$u_k = u_k(x_k, t) = u_{k+1}(x_k, t),\ k = \overline{1, N-1},\ u_0 = u_1(0, t),\ u_N = u_N(l, t),$$

$$R_k^- = \frac{1}{h_k}\int_{x_{k-1}}^{x_k}(x - x_{k-1})\dot{u}_k(x,t)dx,\ R_k^+ = \frac{1}{h_{k+1}}\int_{x_k}^{x_{k+1}}(x_{k+1} - x)\dot{u}_{k+1}(x,t)dx,$$

$$I_k^- = -\frac{1}{h_k}\int_{x_{k-1}}^{x_k}(x - x_{k-1})f(u_k(x,t))dx,$$

$$I_k^+ = \frac{1}{h_{k+1}}\int_{x_k}^{x_{k+1}}(x_{k+1} - x)f(u_{k+1}(x,t))dx,$$

$h_k = x_k - x_{k-1}, k = \overline{1, N}$.

In the non-stationary problem ($\dot{u}_k \neq 0$), to approximate the right hand side integrals $R_k^{\pm}$ we consider different quadrature formulas [17].

In the stationary case for given function $f = f(x)$ exactly calculating integrals I_k^-, I_k^+ we obtain the exact finite-difference scheme.

4. The Boundary Element Method and the Finite Difference Scheme

The FDS (20)-(22) can be obtained by using the boundary element method (BEM). Multiplying the equation (3) by function $w(x,\xi) = |x-\xi|, \xi \in [x_{k-1}, x_k]$ and integrating it by parts twice we get

$$\lambda_k \int_{x_{k-1}}^{x_k} g(u_k) w'' dx = \int_{x_{k-1}}^{x_k} F_k w dx + \lambda_k P_k, \tag{23}$$

where $P_k = (g(u_k)w' - g_x(u_k)w)|_{xk-1}^{x_k}, w' = \partial w/\partial x$.

Due to equalities $w' = \text{sign}\,(x-\xi), w'' = 2\delta(x-\xi)$ ($\delta(x-\xi)$ is the Dirac-delta function) we obtain the third Green formula for the 1D case:

$$2\lambda_k g(u_k(\xi,t)) = \int_{x_{k-1}}^{x_k} |x-\xi| F_k dx + \lambda_k P_k, \tag{24}$$

where

$$\begin{aligned} P_k = {} & g(u_k(x_k,t))\text{sign}\,(x_k-\xi) - g_x(u_k(x_k,t))|x_k-\xi| \\ & - g(u_k(x_{k-1},t))\text{sign}\,(x_{k-1}-\xi) + g_x(u_k(x_{k-1},t))|x_{k-1}-\xi|. \end{aligned}$$

From (24) for given values $g(u_k(x_k,t))$, $g_x(u_k(x_k,t))$, $g(u_k(x_{k-1},t))$, $g_x(u_k(x_{k-1},t))$, F_k it is possible to find $g(u_k(\xi,t))$ for all $\xi \in [x_{k-1}, x_k]$.

Let us consider two limit cases, when $\xi \to x_{k-1}$ and $\xi \to x_k$. Then we have two equations in the following form:

$$\begin{cases} \lambda_k g(u_k(x_{k-1},t)) = \lambda_k(g(u_k(x_k,t)) - h_k g_x(u_k(x_k,t))) + h_k \bar{R}_k^-, \\ \lambda_k g(u_k(x_k,t)) = \lambda_k(g(u_k(x_{k-1},t)) + h_k g_x(u_k(x_{k-1},t))) + h_k \bar{R}_{k-1}^+. \end{cases} \tag{25}$$

Substituting k by $k+1$ in the second equation of (25), then dividing these expressions by h_k, h_{k+1} respectively and applying the continuity conditions (4) we obtain the difference equations (21).

The equation (20) is obtained from the second equation of (25), if $k = 1$, and the equation (22) – from the first equation of (25), if $k = N$ (the boundary conditions (5) must be used).

From the FDS (20)-(22) we obtain the values $g(u_{k-1}), g(u_k), g(u_{k+1})$, u_{k-1}, $u_k, u_{k+1}, k = \overline{1,N}$ and from (25) – the derivatives $g_x(u_k(x_{k-1},t)), g_x(u_k(x_k,t))$.

5. Approximation of Integrals by Quadrature Formulas

In the non-stationary case one must do integrals $\bar{R}_k^-, \bar{R}_k^+$ approximately with the simpler interpolation quadrature formula that contains only two fixed grid points in following way [16]:

$$\begin{cases} \bar{R}_k^- = h_k[\frac{1}{6}F_k(x_{k-1}) + \frac{1}{3}F_k(x_k))] + O(h_k^3), \\ \bar{R}_k^+ = h_{k+1}[\frac{1}{6}F_{k+1}(x_{k+1}) + \frac{1}{3}F_{k+1}(x_k))] + O(h_{k+1}^3), \end{cases} \tag{26}$$

where $F_k(x) = F_k(u_k(x,t))$.

Using the difference equations (21) and the right hand side integrals approximations with neglected error terms we have the following system of nonlinear ordinary differential equations (ODEs) of the first order

$$h_k[\frac{1}{6}\dot{u}_k(x_{k-1},t) + \frac{1}{3}\dot{u}_k(x_k,t)] + h_{k+1}[\frac{1}{3}\dot{u}_{k+1}(x_k,t) + \frac{1}{6}\dot{u}_{k+1}(x_{k+1},t)]$$
$$= \frac{\lambda_{k+1}}{h_{k+1}}[g_{k+1}(x_{k+1},t) - g_k(x_k,t)] - \frac{\lambda_k}{h_k}[g_k(x_k,t) - g_{k-1}(x_{k-1},t)]$$
$$+ h_k[\frac{1}{6}f_k(x_{k-1},t) + \frac{1}{3}f_k(x_k,t)] + h_{k+1}[\frac{1}{3}f_{k+1}(x_k,t) + \frac{1}{6}f_{k+1}(x_{k+1},t)], \tag{27}$$

where $k = \overline{1, N-1}, f_k(x,t) = f(u_k(x,t)), g_k(x,t) = g(u_k(x,t)),\ g_k(x_{k-1},t) = g_{k-1}(x_{k-1}), \dot{u}_k(x_{k-1},t) = \dot{u}_{k-1}(x_{k-1},t)$.

From (20) follows

$$h_1[\frac{1}{6}\dot{u}_1(x_1,t) + \frac{1}{3}\dot{u}_1(x_0,t)] = \frac{\lambda_1}{h_1}[g_1(x_1,t) - g_1(x_0,t)]$$
$$- \alpha_0(u_0 - T_0) + h_1[\frac{1}{6}f_1(x_1,t) + \frac{1}{3}f_1(x_0,t)]. \tag{28}$$

Similarly from (22) we obtain

$$h_N[\frac{1}{6}\dot{u}_{N-1}(x_{N-1},t) + \frac{1}{3}\dot{u}_N(x_N,t)] = -\alpha_l(u_N - T_l)$$
$$- \frac{\lambda_N}{h_N}[g_N(x_N,t) - g_{N-1}(x_{N-1},t)] + h_N[\frac{1}{6}f_N(x_{N-1},t) + \frac{1}{3}f_N(x_N,t)]. \tag{29}$$

The condition of symmetry (29) by $\alpha_l = 0$ follows from (27) by $k = N, h_N = h_{N+1},\ u_{N+1} = u_{N-1}, \dot{u}_{N+1} = \dot{u}_{N-1}, g_{N+1} = g_{N-1}, f_{N+1} = f_{N-1}$.

6. Methods of Lines and FDS in the Case of Uniform Grid

For numerical experiments we consider the homogeneous domain ($\lambda_k = \lambda$), uniform grid ($x_k = kh, k = \overline{0, N}, Nh = l$) and homogeneous boundary conditions of first kind (5) by $x_0 = 0$ ($\alpha_0 = \infty, u_0 = T_0 = \dot{u}_0 = 0$) and the conditions of second kind by $x_N = l$ ($\alpha_l = 0$).

Then from (27), (29) we obtain the nonlinear system of ODEs in the following form

$$\begin{cases} \frac{1}{6}\dot{u}_{k-1} + \frac{2}{3}\dot{u}_k + \frac{1}{6}\dot{u}_{k+1} \\ = \frac{\lambda}{h^2}(g_{k+1} - 2g_k + g_{k-1}) + \frac{1}{6}f_{k-1} + \frac{2}{3}f_k + \frac{1}{6}f_{k+1}, k = \overline{1, N-1}, \\ \frac{1}{3}\dot{u}_{N-1} + \frac{2}{3}\dot{u}_N = -\frac{\lambda}{h^2}(g_N - g_{N-1}) + \frac{1}{3}f_{N-1} + \frac{2}{3}f_N. \end{cases} \tag{30}$$

The expression (30) follows also from (3) by integrating respect to x in segment $[x_{k-1}, x_{k+1}]$, using the finite differences of second order for the approximation of derivatives $g_{xx}(u)$ and using Simpson's quadrature formulae for the approximation of the right hand side integrals.

From (2) we can be directly obtain the system of nonlinear ODEs with the second order of approximation in the space

$$\begin{cases} \dot{u}_k(x_k, t) = \frac{\lambda}{h^2}(g_{k+1}(x_{k+1}, t) - 2g_k(x_k, t) + g_{k-1}(x_{k-1}, t)) + f_k(x_k, t), \\ \dot{u}_N(x_N, t) = -\frac{2\lambda}{h^2}(g_N(x_N, t) - g_{N-1}(x_{N-1}, t)) + f_N(x_N, t), \end{cases} \tag{31}$$

where $k = \overline{1, N-1}, u_{N+1}(x_{N+1}, t) = u_N(x_{N-1}, t), g_{N+1}(x_{N+1}, t) = g_N(x_{N-1}, t) = g_{N-1}(x_{N-1}, t)$.

Using uniform grid of time $t_n = n\tau, n = 0, 1, 2, ...$ we have the following nonlinear FDS with the second order approximation in the space

$$\begin{cases} (y_k^{n+1} - y_k^n)/\tau = (1-\theta)[\frac{\lambda}{h^2}(g_{k+1}^n - 2g_k^n + g_{k-1}^n) + f_k^n] \\ +\theta[\frac{\lambda}{h^2}(g_{k+1}^{n+1} - 2g_k^{n+1} + g_{k-1}^{n+1}) + f_k^{n+1}], k = \overline{1, N-1}, \\ (y_N^{n+1} - y_N^n)/\tau = (1-\theta)[\frac{2\lambda}{h^2}(g_{N-1}^n - g_N^n) + f_N^n] \\ +\theta[\frac{2\lambda}{h^2}(g_{N-1}^{n+1} - g_N^{n+1}) + f_N^{n+1}], \end{cases} \tag{32}$$

where $y_k^n \approx u_k(x_k, t_n), g_k^n \approx g(u_k(x_k, t_n)), f_k^n \approx f(u_k(x_k, t_n)), \theta \in [0, 1]$ is the parameter of FDS (if $\theta = 0.5$, then we have also the second order approximation in the time).

We can rewrite the system (30) in the following matrix form

$$B\dot{U} = \frac{\lambda}{h^2}AG + BF, \tag{33}$$

where A is the standard 3-diagonal matrix of N order with elements $\{1; -2; 1\}$ and with $a_{N,N-1} = 2$, B is a 3-diagonal matrix of N order with elements $\{1/6; 2/3; 1/6\}$ and with element $b_{N,N-1} = 1/3$, $G, F, \dot{U}$ are the vectors-column of N order with elements $g_k = g(u(x_k, t)), f_k = f(u(x_k, t))$, $\dot{u}_k = \dot{u}_k(x_k, t), k = \overline{1, N}$.

The matrix A and B have the eigenvalues $\mu_k(A) = -4\sin^2(\pi(2k-1)/(4N))$ and $\mu_k(B) = \frac{1}{3} + \frac{2}{3}\cos^2(\pi(2k-1)/(4N)), k = \overline{1, N}$.

The system (31) is in a similar form

$$\dot{U} = \frac{\lambda}{h^2} AG + F. \tag{34}$$

The system of ODEs (33) can be rewritten in a normal form

$$\dot{U} = \frac{\lambda}{h^2} B^{-1} AG + F, \tag{35}$$

where matrix $C = B^{-1}A$ have the eigenvalues $\mu_k(A)/\mu_k(B), k = \overline{1, N}$.

For the *linearized case* ($\max(u^\sigma) = M_\sigma = const > 0, \max(u^{\beta-1}) = M_\beta = const > 0$ for $\beta \geq 2$ and $M_\beta = 1$ if $\beta = 1$) the systems of ODEs (33)-(35) are stable because the eigenvalues $\mu_k(A) < 0, \mu_k(B^{-1}A) < 0$. The system (35) is more stable than system (34) because $\mu_k(B^{-1}A) = \mu_k(A)/\mu_k(B) < \mu_k(A)$.

The explicit FDS (32) with $\theta = 0$ has the following matrix form

$$U^{n+1} = U^n + k_0 AG^n + \tau F^n, \tag{36}$$

where the vector columns U^n, G^n, F^n of N orders have the elements u_k^n, g_k^n, f_k^n, $k = \overline{1, N}$, $\boldsymbol{k_0 = \frac{\tau\lambda}{h^2}}$.

The FDS (32) with $\theta = 0.5$ is realized with second order Runge-Kutta method or explicit trapezoid method

$$\begin{cases} \bar{U}^{n+1} = U^n + k_0 AG^n + \tau F^n, \\ U^{n+1} = U^n + \frac{1}{2}[k_0 AG^n + \tau F^n + k_0 A\bar{G}^{n+1} + \tau \bar{F}^{n+1}], \end{cases} \tag{37}$$

where the vectors $\bar{G}^{n+1}, \bar{F}^{n+1}$ are calculated with elements of $\bar{U}^{n+1}$.

Using the Charlie algorithm [2] we can rewrite the FDS (37) in the following form

$$\begin{cases} \bar{U}^{n+1} = U^n + k_0 AG^n + \tau F^n, \\ U^{n+1} = (1-\gamma)\bar{U}^{n+1} + \gamma[U^n + k_0 A\bar{G}^{n+1} + \tau \bar{F}^{n+1}], \end{cases} \tag{38}$$

where $0 < \gamma < 1$ is the parameter.

We can see the Charlie algorithm (38) in the following form

$$\begin{cases} \bar{U}^{n+1} = U^n + k_0 AG^n + \tau F^n, \\ U^{n+1} = U^n + (1-\gamma)[k_0 AG^n + \tau F^n] + \gamma[k_0 A\bar{G}^{n+1} + \tau \bar{F}^{n+1}]. \end{cases} \tag{39}$$

From (39) we obtain (37) by $\gamma = 0.5$ and (36) by $\gamma = 0$.

By investigating the stability of FDS (39) for the linearized case, we obtain the eigenvalues $\mu(\rho)$ of transition matrix $\rho(U^{n+1} = \rho U^n)$ in the following form:

$$\mu(\rho) = 1 - \delta + \gamma\delta^2,$$

where $\delta = k_0|\mu(A)|M_\sigma - \tau a M_\beta, \mu(A)$ is the eigenvalue of matrix A.

The stability condition $|\mu(\rho)| \leq 1$ is satisfied when $0 \leq \delta - \gamma\delta^2 \leq 2$ or $0 \leq \delta \leq \gamma_0 = \gamma^{-1}$ for $\gamma \geq \frac{1}{8}$ and $\delta \leq \gamma_0 = 4/(1+\sqrt{1-8\gamma})$ for $\gamma \leq \frac{1}{8}$. From $\delta > 0$ the inequality $a \le \frac{2\lambda M_\sigma}{l^2 M_\beta}$ follows, because $2/l^2 \le h^{-2}|\mu_1(A)| \le h^{-2}|\mu_N(A)| \le 4h^{-2}$. From others inequality respect to δ it follows that

$$k_0 \le \frac{\gamma_0}{2M_\sigma - h^2 a\lambda^{-1}M_\beta}. \tag{40}$$

If $\gamma = 0$ or $\gamma = 0.5$, then $\gamma_0 = 2$.

Using the monotonous conditions for FDS (32) we can obtain that (40) follows from inequality $a \le \frac{2\lambda M_\sigma}{h^2 M_\beta}$, with $\gamma_0 = 1/(1-\theta)$.

In the case of first kind homogeneous boundary conditions in the segment $[0, L], L = 2l$ by uniform grid $x_k = kh, k = \overline{0, M}, Mh = L, M = 2N$ the matrices A, B are of $(M-1)$ order with eigenvalues $\tilde{\mu}_k(A) = -4\sin^2(\pi k/(2M))$, $\tilde{\mu}_k(B) = \frac{1}{3} + \frac{2}{3}\cos^2(\pi k/(2M)), k = \overline{1, M-1}$. We can see, that $\tilde{\mu}_{2k-1}(A) = \mu_k(A), \tilde{\mu}_{2k-1}(B) = \mu_k(B), k = \overline{1, N}$.

N	$u_5(33)$	$u_5(34)$	$u_2(33)$	$u_2(34)$	u_{2*}	u_{5*}
3	.5191	.5432	.5196	.5429	.5306	.5310
5	.5275	.5362	.5275	.5363	.5310	.5318
10	.5309	.5331	.5313	.5328	.5310	.5320
20	.5320	.5323	.5308	.5317	.5318	.5321
40	.5320	.5322	.5308	.5331	.5330	.5321
50	.5320	.5321	.5322	.5318	.5324	.5321

Table 1: The values of $u(0.5, 10)$ by different space step N

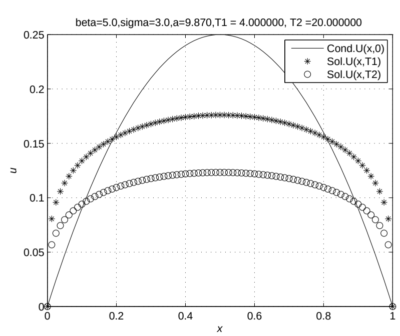

Figure 1: $U_{st} = 0$for $x \in [0, 1]$, $\beta = 4, \sigma = 3, a = \pi^2$

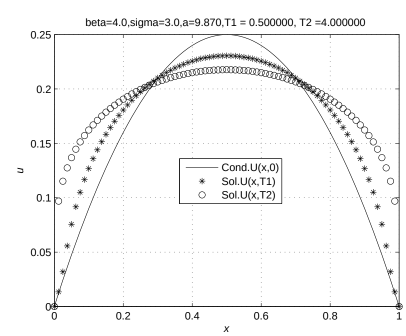

Figure 2: U_{st} for $x \in [0, 1]$, $\beta = 4, \sigma = 3, a = \pi^2$

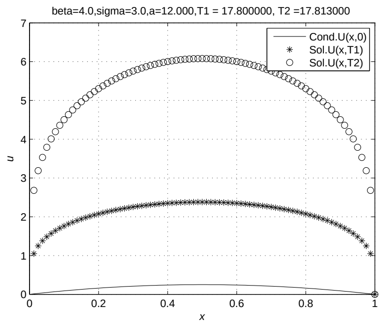

Figure 3: $U \to \infty$ for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 12$

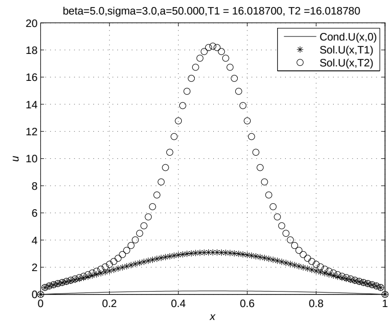

Figure 4: $U \to \infty$ for $x = 0.5$, $\beta = 5, \sigma = 3, a = 50$

In this case we can consider improved spectral FDS [20]. For this scheme in (34) the matrix $A_1 = A/h^2 = PDP^{-1}(P^{-1} = P)$ of order $(M-1)$ is the finite spectral approximation of partial differential operator $(\frac{\partial^2}{\partial x^2})$, where P is the symmetrical orthogonal matrix with elements $p_{i,j} = \sqrt{\frac{2}{M}} \sin \frac{\pi i j}{M}, i, j = \overline{1, M-1}$. The column of the matrix P contains the orthonormal eigenvectors of operator $(\frac{\partial^2}{\partial x^2})$ and D is the diagonal matrix with elements $d_k = -(\frac{k\pi}{L})^2, k = \overline{1, M-1}$ of the first $M-1$ numbers eigenvalues from the differential operator $(\frac{\partial^2}{\partial x^2})$.

The spectral FDS in the linearized case is more stable as the methods (34, 35), because the eigenvalues of matrix A_1 are larger on modulus $(|d_k| > |\mu_k(B^{-1}A)h^{-2}| > |\mu_k(A)h^{-2}|)$.

If $d_k = -\frac{4}{h^2} \sin^2 \frac{k\pi}{2M}$, then we have the method (34).

7. Some Examples and Numerical Results

The numerical experiment for the equation (1) with $\lambda = l = 1$ and $\phi = A_0 x(1-x) \geq 0, a > 0, A_0 \geq 1$, $u(0,t) = u(1,t) = 0$ is produced by different values of σ and β. In ([13], [14], [18], [21]) are developed some numerical experiments for $a = 1$. In paper [19] the difficulties of finite difference schemes by obtaining solutions of regimes with aggravation in the problems of quasilinear parabolic equations are considered.

The first results of calculations of the system ODEs (33), (34) are obtained

		$a = 10$		$a = 20$		$a = 50$		$a = 0.1$	
β	σ	u_{st}	T_{st}	u_{st}	T_{st}	u_{st}	T_{st}	u_{st}	T_{st}
1	1	1.12	1.20	2.24	0.50	5.60	0.21	0.01	31.0
2	2	1.08	1.40	2.17	0.50	5.41	0.15	0.01	2.e3
1	2	1.08	0.70	1.53	0.30	2.41	0.10	0.01	2.e3
3	3	1.06	2.50	2.13	0.80	5.32	0.25	0.01	3.e4
2	3	1.06	0.80	1.50	0.40	2.37	0.12	0.11	300.
1	3	1.06	0.40	1.33	0.30	1.81	0.10	0.23	20.0
4	4	1.05	5.00	2.11	2.00	5.27	0.70	0.02	4.e5
3	4	1.05	1.50	1.48	0.70	2.34	0.25	0.10	2.e3
2	4	1.05	0.70	1.32	0.40	0.79	0.18	0.23	85.0
1	4	1.05	0.30	1.24	0.20	1.56	0.10	0.33	20.0

Table 2: The values of $T_{st}, u_{st}(0.5)$ by $A_0 = 1, a = 0.1; 10; 20; 50, \beta < \sigma + 1$

by *MATLAB 7.4* by the following parameters $\sigma = \beta = 3, a = 5, A_0 = 1(\beta < \sigma + 1)$ are seen in Table 1, where $t = 10$ and the values are approximations of $u(0.5, 10) : u_5(0.5, 10)$ and $u_2(0.5, 10)$ by the relative errors 10^{-5} and 10^{-2}. The values u_{2*}, u_{5*} with the spectral FDS are obtained. We can see, that the sequences of values $u_5(0.5, 10)$ from (33) and u_{5*} depending on N is monotonically increasing sequence, but the sequence $u_5(0.5, 10)$ from (34) is monotonically decreasing. The sequences, obtained by relative error $\epsilon = 10^{-2}$ are oscillating. The values of u_5, obtained from (33) and with the spectral FDS, are already constant by $N \geq 20$ (the solutions is stationary).

These stiff systems of ODEs can by solved with the *MATLAB* solvers "ode15s", "ode23s" and "ode23tb". Comparing different *MATLAB* there was located the low precision of solvers "ode23", "ode 113", "ode 23t" contrary "ode45" with good precision, but with slow time of calculations.

For the test example $\beta = 0, u_{st} = (\frac{a}{2\lambda} x(L - x))^{1/(\sigma+1)}$ the numerical results for different σ, a are agreed with 4 decimal signs with respect to analytical solutions.

The corresponding results from FDS (36)-(39) are obtained for different values of $\tau = h^2 k_0$. For the FDS (37) the convergence to solutions by $t = 10$ is for the parameter $k_0 \leq 0.8$. If $k_0 = 0.85$, then we can consider oscillation of the function $u(x, 10)$, but by $k_0 = 0.9$ this process is divergent.

Charlie algorithm (38) is convergent by $k_0 \leq 2$ and $\gamma = (4k_0)^{-1}$ (the oscillation are by $k_0 = 2.1$).

$\sigma = 3, \beta = 4(\beta = \sigma + 1)$				$\sigma = 3, \beta = 5(\beta > \sigma + 1)$		
A_0	a	$u_{st}(0.5)$	T_{st} or T_*	A_0	a	T_*
1	π^2	0.21765630	$T_{st} = 3.500$	1	49	19.64424
5	π^2	1.08828150	$T_{st} = 0.0033$	1	50	16.01878
1	12	∞	$T_* = 17.80533$	10	5	0.016020
1	20	∞	$T_* = 3.625468$	5	π^2	0.145274
1	15	∞	$T_* =7.306243$	10	π^2	0.001984
1	10	∞	$T_* = 293.4587$	3	20	0.237788
2	20	∞	$T_* = 0.453190$	4	20	0.047108
4	20	∞	$T_* =0.056649$	4	15	0.100255
10	20	∞	$T_* = 0.003626$	5	15	0.027848

Table 3: The values of $T_{st}, T_*, u_{st}(0.5)$ by different values of $A_0, a,$ $\beta \geq \sigma + 1$

The calculation with explicit FDS (36) is possible only for $k_0 \leq 0.25$.

In Figures 1-4 we can see four type solutions for three time moments $(t = 0, t = T1, t = T2 > T1)$, depending on the parameters $\sigma, \beta, a, (A_0 = 1)$, obtained with the spectral FDS by $M = 80, \epsilon = 10^{-7}$:

1) $\sigma = 3, \beta = 5(\beta > \sigma + 1), Q = 1$, the stationary solution $u_{st}(x)$ is zero (for $Q < 1$ the solution is unbounded, see Table 3);

2) $\sigma = 3, \beta = 4(\beta = \sigma + 1), Q = 1$, the stationary solution $u_{st}(x) \neq 0$ is by finite value T_{st}, obtained by $t \to T_{st} < \infty$;

3) $\sigma = 3, \beta = 4(\beta = \sigma + 1), Q = \frac{\pi^2}{12} < 1$, the solution $u(x, t) \to \infty$ globally for all $x \in (0, 1)$, when $t \to T_* < \infty$ (for finite value of T_*, this is "blow up" solution);

4) $\sigma = 3, \beta = 5(\beta > \sigma + 1), Q = \frac{\pi^2}{50} < 1$, the solution $u(x, t) \to \infty$ locally neighbourhood of point $x = 0.5$, when $t \to T_* < \infty$ (for finite value of T_*, this is "blow up" solution).

With the others methods of lines and FDS we have similar results using greater values of N, M.

If $\boldsymbol{\beta < \sigma + 1}$, then for all $a > 0, Q > 0$ we have by $t \to T_{st} < \infty$ the stationary solution $u_{st}(x)$ with maximal value $u_{st} = u_{st}(0.5)$ (Table 2). If $a = 0$, then $\frac{dE}{dt} < 0$ and for all σ the stationary solution $u_{st}(x) = 0$.

If $A_0 > 1$, then the value of u_{st} is similar, but the behaviour in the time is different. For example, by $\beta = \sigma = 3, a = 10, A_0 = 1$ the convergence in the time to value $u_{st}(0.5) = 1.0636$ $(T_{st} = 2.5)$ is monotonically increasing, but by

$A_0 = 10$ – monotonically decreasing with $u_{st}(0.5) = 1.0636, T_{st} = 0.5$.

If $\boldsymbol{\beta = \sigma + 1}$, then for all $a < \pi^2, Q > 1, A_0 \geq 1$ we have the monotonically decreasing sequence in the time ($\frac{dE}{dt} < 0$) and the stationary solution $u_{st}(x) = 0$. The behaviour of the solutions for $Q \leq 1, \sigma = 3, \beta = 4$ we can see in Table 3. If $a = \pi^2, Q = 1$, then $u_{st}(0.5) = A_0 * 0.21765630$ and the convergence to stationary solution is very fast in the time ($A_0 > 5$). If $a > \pi^2, Q < 1$, then the solutions is unbounded in the time $t \geq T_*$ in all interval $x \in (0,1)$ (T_* is finite value). If $a = 12 > \pi^2, A_0 = 1$, then we have from (11) the estimation $T_* = 18.803$ ($E(0) = \frac{2}{\pi^2}$), but from numerical experiment $T_* \approx 17.805$.

If $\boldsymbol{\beta > \sigma + 1}$, then we have "blow up" phenomena by sufficiently large value of $A_a = A_0 * a$. The behaviour of the solution for $\sigma = 3, \beta = 5$ we can see in Table 3 ($A_a \approx 50$). The solution is unbounded in the time $t \geq T_* < \infty$. The solution tends to infinity locally in small neighbourhood of the middle point $x = 0.5$ in segment [0,1]. If $A_a < 50$, then the stationary solution $u_{st}(x)$ is zero. From the theoretical estimations (13), (14) by $A_0 = 10, a = \pi^2$ it follows that $A_a > \frac{\pi^3}{2}(E_0 = A_0 \frac{2}{\pi})$ and $T_* = 0.0030$.

8. Conclusion

The 2D transfer problem described by a nonlinear initial boundary value problem of the PDEs is approximate on the nonlinear initial value problem of a system of ODEs of the first order. Such a procedure enables us to obtain a simple engineering algorithm for solving nonlinear mass transfer equations in multilayered domain. For example of this algorithm power functions are considered and different types of solutions are obtained.

References

[1] V. Buda, R. Ciegis, M. Sapagovas, Numerical analysis of nonlinear blow-up diffusion process, In: *Seminar Mathematical Modelling and Parameter Identification of Transfer Process in Non Homogeneous Media*, Riga (1992); *Acta universitatis Latviensis, Riga*, **592** (1994), 129-134.

[2] S.K. Dey, A novel explicit finite difference scheme for partial differential equations, *Math. Model. and Analysis*, **4** (1999), 70-78.

[3] G.G. Elenin, S.P.Kurdjumov, A.A.Samarskij, Non stationary dissipative

structure in the nonlinear heat transfer media, *J. Numerical Mathematics and Mathematical Physics*, **23**, No. 2 (1983), 380-390, In Russian.

[4] V.A Galaktinov, About some boundary-value problems for nonlinear parabolic equations, *J. Differential Equations*, **17**, No. 5 (1981), 836-842, In Russian.

[5] V.A Galaktinov, About the conditions of nonexistence of globally solutions for one class of quasilinear parabolic equations, *J. Numerical Mathematics and Mathematical Physics*, **22**, No. 2 (1982), 322-338, In Russian.

[6] V.A Galaktinov, About existence and nonexistence of globally solutions for boundary-value problem of quasilinear parabolic equations, *J. Numerical Mathematics and Mathematical Physics*, **22**, No. 6 (1982), 1369-1385, In Russian.

[7] V.A Galaktinov, Prove for localization of unbounded solutions for nonlinear parabolic equations, *J. Differential Equations*, **21**, No. 1 (1985), 15-23, In Russian.

[8] V.A Galaktinov, Asymptotical behaviour of unbounded solutions for nonlinear parabolic equations, *J. Differential Equations*, **21**, No. 7 (1985), 1126-1134, In Russian.

[9] V.A Galaktinov, Using the news theorems of comparison for the investigation of unbounded solutions for nonlinear parabolic equations, *J. Differential Equations*, **22**, No. 7 (1986), 1165-1173, In Russian.

[10] V.A Galaktinov, S.P.Kurdjumov, A.A.Samarskij, About one system of quasilinear parabolic equations, *J. Differential Equations*, **19** (12) (1983), 2122-2140, In Russian.

[11] V.A Galaktinov, S.P.Kurdjumov, A.A.Samarskij, About asymptotical stability for invariant solutions of nonlinear heat conduction with the source, *J. Differential Equations*, **20**, No. 4 (1984), 614-632, In Russian.

[12] V.A Galaktinov, S.A. Posashkov, About one method of investigation for unbounded solutions behaviour of quasilinear parabolic equations, *J. Numerical Mathematics and Mathematical Physics*, **28**, No. 6 (1988), 842-854, In Russian.

[13] V.A Galaktinov, A.A.Samarskij, About difference solutions for one class of quasilinear parabolic equations I, *J. Numerical Mathematics and Mathematical Physics*, **23**, No. 3 (1983), 646-659, In Russian.

[14] V.A Galaktinov, A.A.Samarskij, About difference solutions for one class of quasilinear parabolic equations II, *J. Numerical Mathematics and Mathematical Physics*, **23**, No. 4 (1983), 831-838, In Russian.

[15] H. Kalis, Efficient finite-difference scheme for solving some heat transfer problems with convection in multilayer media, *Intern. J. of Heat and Mass Transfer*, **43** (2000), 4467-4474.

[16] H. Kalis, I. Kangro, Simple algorithm's for the calculation of heat transport problem in plate, *Math. Model. and Analysis*, **6**, No. 1 (2001), 85-96.

[17] H. Kalis, I. Kangro, Simple methods of engineering calculation for solving heat transfer problems, *Math. Model.and Analysis*, **8**, No. 1 (2003), 33-42.

[18] A.S. Makarenko, About calculations of blow-up solutions in the problems of heat transfer equations, *J. Numerical Mathematics and Mathematical Physics*, **24**, No. 8 (1984), 1189-1195, In Russian.

[19] A.S. Makarenko, About calculation for regimes with aggravation in the problems with non continuously solutions, *J. Differential Equations*, **27** (10) (1991), 1766-1771, In Russian.

[20] V.L. Makarov, I.P. Gavrilyuk, On constructing the best net circuits with exact spectrum, *Dopov. Akad. Nauk Ukr. RSR*, **Ser. A** (1975), 1077-1080, In Russian.

[21] A.A. Samarskij, V.A. Galaktinov, S.P. Kurdjumov, A.P. Mihailov, *Blow-Up Solutions in the Problems of Quasilinear Parabolic Equations*, Moscow, Nauka (1987), In Russian.