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POSTULATION OF DISJOINT UNIONS OF LINES AND A FIXED SUBSCHEME

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Abstract: Let $Z \subset \mathbb{P}^n$, $n \geq 5$, be any scheme such that $\dim(Z) \leq n-5$. Here we prove the existence of an integer $\alpha_n(Z)$ such that for every integer $y \geq \alpha_n(Z)$ a general union of Z and y lines of $\mathbb{P}^n$ has maximal rank.

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1. Introduction

Here we prove the following asymptotic result.

Theorem 1. *Fix an integer $n \geq 5$ and a scheme $Z \subset \mathbb{P}^n$ such that $\dim(Z) \leq n-5$. Then there exists an integer $\alpha_n(Z) \geq 0$ such that for every integer $y \geq \alpha_n(Z)$ a general union $X = Z \sqcup Y$ of Z and y disjoint lines has maximal rank, i.e. for every integer $t \geq 0$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$.*

The case $Z = \emptyset$ with $\alpha(\emptyset) = 0$ is [2]. The case Z a plane with $\alpha(Z, 0) = 0$ is true ([1], Theorem 3.2). We will use the strategy and the ideas of [2]. A similar result may be proved taking general degree y curves with low genus instead of y disjoint lines.

The first integer $k > 0$ such that $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$ is called the *critical value of the triple* (n, Z, y). Quite often (if $y \gg 0$) this is the first positive integer $x = x_{Z,n,y}$ such that $h^0(X, \mathcal{O}_X(x)) + y(x+1) \leq \binom{n+x}{n}$. This is the case if $h^i(Z, \mathcal{O}_Z(x_{Z,n,y} - i)) = 0$ for all $1 \leq i \leq \dim(Z)$ (Castelnuovo-Mumford's Lemma). Notice that this is the case if Z is a disjoint union of linear

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spaces and/or non-special curves and either $x > 0$ or all curves are smooth and rational. We work over an algebraically closed base field with characteristic zero.

2. The Proof

For any integral scheme M and any $P \in M_{reg}$ let $\chi_M(P)$ denote the first infinitesimal neighborhood of P in M, i.e. the closed subscheme of M with $(\mathcal{I}_{P,M})^2$ as its ideal sheaf. Thus $\chi_M(P)_{red} = \{P\}$, $\mathrm{length}(\chi_M(P)) = \dim_P(M) + 1$ and $\chi_M(P) = \chi_{T_PM}(P)$. If $\dim_P(M) = 1$ (resp. $\dim_P(M) = 2$) we often call $\chi_M(P)$ a tangent vector (resp. a length 3 planar scheme) of $\mathbb{P}^n$ supported by P and associated to the curve (resp. surface) M.

Remark 1. Fix a reducible conic $D \subset \mathbb{P}^n$, $n \geq 3$ and let P be its singular point. Let M be any 3-dimensional linear space containing D. The scheme $D \cup \chi_M(P)$ is a flat degeneration inside M and hence $\mathbb{P}^n$ of a flat family whose general element is the disjoint union of 2 lines (see [2]).

Remark 2. Fix any scheme $W \subset \mathbb{P}^m$ and any integer $t \geq 0$. Then (after fixing W and t) fix a general $P \in \mathbb{P}^m$ and a general tangent vector v such that $v_{red} = \{P\}$. Since P is general, $h^0(\mathcal{I}_{W\cup\{P\}}(t)) = \max\{0, h^0(\mathcal{I}_W(t)) - 1\}$. Thus $h^1(\mathcal{I}_{W\cup\{P\}}(t)) = h^1(\mathcal{I}_W(t))$ if $h^0(\mathcal{I}_W(t)) > 0$ and $h^1(\mathcal{I}_{W\cup\{P\}}(t)) = h^1(\mathcal{I}_W(t)) + 1$ if $h^0(\mathcal{I}_W(t)) = 0$. Since $v_{red} = \{P\}$ and $\mathrm{length}(v) = 2$, we have $h^0(\mathcal{I}_{W\cup\{P\}}(t)) - 1 \leq h^0(\mathcal{I}_{W\cup v}(t)) \leq h^0(\mathcal{I}_{W\cup\{P\}}(t))$. Hence $h^0(\mathcal{I}_{W\cup v}(t)) = 0$ if $h^0(\mathcal{I}_W(t)) \leq 1$. Now assume $h^0(\mathcal{I}_W(t)) \geq 2$. Thus the rational map η from $\mathbb{P}^m$ into $\mathbb{P}^N$, $N := h^0(\mathcal{I}_W(t)) - 1$, is not constant. Since we chosed P general after fixing W and t, P is not a base point of η and at P the differential of η has positive rank (here we use the characteristic zero assumption). Since v is a general element of $T_P\mathbb{P}^m$, the kernel of differential of η at P does not contain v. Thus $h^0(\mathcal{I}_{W\cup v}(t)) = h^0(\mathcal{I}_W(t)) - 2$, i.e. $h^1(\mathcal{I}_{W\cup v}(t)) = h^1(\mathcal{I}_W(t))$.

Remark 3. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0. \tag{1}$$

From (1) we get

$$h^i(X, \mathcal{I}_Z \otimes L(-D)) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every integer $i \geq 0$.

Proof of Theorem 1. Fix a general hyperplane $H \subset \mathbb{P}^n$. Hence $\dim(Z \cap H) \le \dim(H) - 5$. First assume $n = 5$. Hence Z is zero-dimensional. Set $z := \mathrm{length}(Z)$. For all integers $k \ge 0$ define the integer a_k and b_k by the relations

$$z + (k+1)a_k - b_k = \binom{k+5}{5}, \quad 0 \le b_k \le k. \tag{2}$$

Taking the difference of the equation in (2) with the same equation for $k' := k-1$ we get

$$(k+1)(a_k - a_{k-1}) + a_{k-1} - b_k + b_{k-1} = \binom{k+4}{4}. \tag{3}$$

Fix any integer $\beta > 0$ such that $h^1(\mathcal{I}_Z(\beta)) = 0$. Since $\dim(Z) = 0$, $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \ge \beta$. Since (for fixed z) $a_k \sim k^4/120$ and $a_k - a_{k-1} \sim k^3/30$ for $k \gg 0$ (use (2) and (3)), there is an integer $\beta_1 \ge \beta$ such that $a_k - a_{k-1} \ge 3k$ for all $k \ge \beta_1$. For any integer $k \ge \beta_1$ we define the following assertion A_k;

Assertion A_k: Let $B \sqcup D \subset \mathbb{P}^5$ be a general union of $a_k - 2b_k$ lines and b_k reducible conics with their singular points contained in H. Then $h^i(\mathcal{I}_{Z\cup B\cup D}(k)) = 0$, $i = 0, 1$.

In the set-up of A_k B always denote the disjoint union of $a_k - 2b_k$ lines, while D denote the disjoint union b_k reducible conics with their singular points contained in H. Let $B \sqcup D \subset \mathbb{P}^5$ be a disjoint union of $a_k - 2b_k$ lines and b_k reducible conics. Assume $B \cap Z = D \cap Z = \emptyset$. By (2) we have $h^1(\mathcal{I}_{Z\cup B}(k)) = h^0(\mathcal{I}_{Z\cup B}(k))$. Hence to prove A_k it is sufficient to check either the h^0-vanishing or the h^1-vanishing.

(a) Here we prove the existence of an integer $\beta_2 \ge \beta_1$ such that A_k is true for all $k \ge \beta_2$. Set $\gamma := \binom{5+\beta_1}{5} - z$. Since $h^1(\mathcal{I}_Z(\beta_1)) = 0$, we have $\gamma = h^0(\mathcal{I}_Z(\beta_1)) \ge 0$. We will see that we may take $\beta_2 = \beta_1 + \gamma$. For all integers t such that $\beta_1 \le t \le \beta_1 + \gamma$ we define the integers a'_t and b'_t by the relations

$$z + (\gamma + \beta_1 - t) + (k+1)a'_t - b'_t = \binom{t+5}{5}, \quad 0 \le b'_t \le t. \tag{4}$$

Taking the difference of (4) for $t' = t-1$ from it we get

$$-1 + (k+1)(a'_t - a'_{t-1}) + b'_{t-1} - b'_t = \binom{t+4}{4}. \tag{5}$$

Notice that $a'_{\beta_1} = b'_{\beta_1} = 0$. For large β_1 we have $a'_t \ge 2t$ for all $t > \beta_1$. Thus $a'_t \ge 2b'_t$ for all t such that $\beta_1 \le t \le \beta_1 + \gamma$.

Claim 1. *For all integers t such that $\beta_1 \le t \le \beta_1 + \gamma$ we have $h^1(\mathcal{I}_{Z\cup Y}(t)) = 0$, where $Y \cap Z = \emptyset$ and Y is a general union of $a'_t - 2b'_t$ lines and b'_t reducible conics with their singular points contained in H.*

Proof of Claim 1. Since $h^1(\mathcal{I}_Z(\beta_1)) = 0$ and $a'_{\beta_1} = b'_{\beta_1} = 0$, Claim 1 is true for $t = \beta_1$. Hence we may assume $t > \beta_1$ and use induction on t. Take a general solution $Y = B \sqcup D$ of the Claim 1 for the integer $t-1$. We first check that the base locus Δ of the linear system $|\mathcal{I}_{Z\cup B\cup D}(t-1)|$ does not contain H. This is obvious if $t = \beta_1 + 1$, because $a'_{\beta_1} = b'_{\beta_1} = 0$ and we took H general after fixing Z. Assume $t \geq \beta_1 + 2$ and $H \subseteq \Delta$. Hence $h^0(\mathcal{I}_{Z\cup B\cup D}(t-1)) = h^0(\mathcal{I}_{Z\cup B\cup D}(t-2))$. Since $h^0(Z\cup B\cup D, \mathcal{O}_{Z\cup B\cup D}(t-1)) = z + ta'_{t-1} - b'_{t-1}$ and $h^1(\mathcal{I}_{Z\cup B\cup D}(t-1)) = 0$ by the inductive assumption, we get $h^0(\mathcal{I}_{Z\cup B\cup D}(t-2)) = \binom{t+4}{5} - z - (t-1)a'_{t-1} - b'_{t-1}$. Since $a'_{t-1} - 2b'_{t-1} \geq a_{t-2}$, we may take $Y' \subseteq B \sqcup D$ containing $a'_{t-2} - 2b'_{t-2}$ general lines and one of the two lines of b'_{t-2} reducible conics with singular point in H. Since $\binom{t+4}{5} - \binom{t+3}{5} > (t-1)^2 \geq (t-1)b'_{t-2}$, we get a contradiction. Hence a general $P \in H$ is not in the base locus of $|\mathcal{I}_{Z\cup B\cup D}(t-1)|$. We fix such a point P.

(a1) Here we assume $b'_t \geq b'_{t-1}$. Let $E \subset H$ be a general union of a reducuble conic with P as its singular point and $a'_t - a'_{t-1} - 2$ lines, with the only restriction that $b'_t - b'_{t-1}$ of them contain a different point of $H \cap B$. Here we use $a'_t - 2b'_t \geq b'_t - b'_{t-1}$, which is true for large β_1 and $t > \beta_1$, because $a'_t \geq 3t$. Fix a 3-dimensional linear subspace H_P of $\mathbb{P}^5$ contained the reducible conic of E, but not contained in H. Set $X := Z \cup B \cup D \cup E \cup \chi$. Since $H_P \nsubseteq H$, we have $\mathrm{Res}_H(X) = Z \cup A \cup D \cup \{P\}$. Since $h^1(\mathcal{I}_{Z\cup B\cup D}(t-1)) = 0$ and $P \notin \Delta$, $h^1(\mathcal{I}_{Z\cup B\cup D\cup\{P\}}(t-1)) = 0$. We have $X \cap H = E$. Hence [2] (with a small improvement to allow one reducible conic), (5), b'_{t-1} applications of Remark 2 to the tangent vectors $D \cap Z$ and the generality of the points $B \cap H$ give $h^1(H, \mathcal{I}_{X\cap H}(t)) = 0$, proving Claim 1 in this case,

(a2) Here we assume $b'_t < b'_{t-1}$. Fix $S \subseteq \mathrm{Sing}(D)$ such that $\sharp(S) = b'_{t-1} - b'_t$. For each $Q \in S$ fix a 3-dimensional linear subspace H_Q containing the connected component of D containing Q. Let $E \subset H$ be a general union of a reducuble conic with P as its singular point and $a'_t - a'_{t-1} - 2$ lines. Set $X := Z \cup B \cup D \cup E \cup \chi_{H_P}(P) \cup (\bigcup_{Q\in S} \chi_{H_Q}(Q))$. The only difference with respect to case (a1) is that now $X \cap H$ contains $b'_{t-1} - b'_t$ general planar length 3 schemes. Since $b'_{t-1} - b'_t \leq t-1$, here we may apply $H''_{t,4}$ of [2], concluding the proof of Claim 1. □

(a3) Here we prove A_k for all integers $k \geq \beta_1 + \gamma + 1$. Claim 1 for the integer $\beta_1 + \gamma + 1$ is equivalent to $A_{\beta_1+\gamma+1}$. If $k > \beta_1 + \gamma + 1$ we use induction as in the proof of Claim 1, just ignoring the point P and avoiding the reducible conic in H with P as its singular point. Hence here $E \subset H$ is a general union of $a_k - a_{k-1}$ lines such that $\max\{0, b_k - b_{k-1}\}$ of them intersect a different point of $B \cap H$; here we use (3) instead of (5) and the two cases are according to the sign of $b_k - b_{k-1}$ instead of the sign of $b'_t - b'_{t-1}$.

(a4) Here and in the next step (a5) we prove that Theorem 1 is true for $n = 5$ and that we may take $\alpha_5(Z) = \beta_1 + \gamma + 2$. Fix integers $k \geq \beta_1 + \gamma + 2$ and y such that the disjoint union in $\mathbb{P}^5$ of Z and y disjoint lines has critical value k. Let $X \subset \mathbb{P}^5$ the union of Z and y general lines. We need to prove $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$. Here we prove $h^0(\mathcal{I}_X(k-1)) = 0$. Take $Z \sqcup B \sqcup D$ satisfying A_{k-1} (step (a3)). Hence $h^0(\mathcal{I}_{Z\cup B\cup D}(k-1)) = 0$. For each $P \in \mathrm{Sing}(D)$ fix a 3-dimensional linear subspace H_P of $\mathbb{P}^5$ containing the connected component of D containing P. Set $\chi := \cup_{P\in\mathrm{Sing}(D)} \chi_{H_P}(P)$. Let $T \subset \mathbb{P}^n$ be a general union of $y - a_{k-1}$ lines; since (Z, y) has critical value k, we have $y > a_{k-1}$ and hence T is well-defined. Since $Z \cup B \cup D \subseteq Z \cup B \cup D \cup T \cup \chi$ and $h^0(\mathcal{I}_{Z\cup B\cup D}(k-1)) = 0$, we have $h^0(\mathcal{I}_{Z\cup B\cup D\cup T\cup\chi}(k-1)) = 0$. Since $B \cup D \cup T \cup \chi$ is a flat limit of a flat family of disjoint unions of y lines and X is general, semicontinuity gives $h^0(\mathcal{I}_X(k-1)) = 0$.

(a5) Here we prove $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$. By Castelnuovo-Mumford's Lemma it is sufficient to prove $h^1(\mathcal{I}_X(k)) = 0$. By (2) and the definition of critical value we have $y \leq a_k$ and $y = a_k$ only if $b_k = 0$. Hence if $y = a_k$ we have $h^i(\mathcal{I}_X(k)) = 0$, $i = 0, 1$, by A_k. Thus we may assume $y < a_k$. Adding lines we reduce to the case $y = a_k - 1$. We take a general $Z \sqcup B \sqcup D$ satisfying A_{k-1}. Then we adapt step (a2) in the following way. For each $Q \in \mathrm{Sing}(D)$ we fix a 3-dimensional linear subspace H_Q containing the connected component of D containing Q. Set $\chi := \cup_{Q\in\mathrm{Sing}(D)} \chi_{H_Q}(Q)$. Let $E \subset H$ be a general union of $y - a_{k-1}$ lines. As in step (a2) we prove $h^1(\mathcal{I}_{Z\cup B\cup D\cup E\cup\chi}(k)) = 0$. Since $Z \cap (B \cup D \cup \chi \cup E) = \emptyset$ and $B \cup D \cup \chi \cup E$ is a flat limit of a flat family of disjoint unions of y lines, we are done, concluding the proof of the case $n = 5$.

(b) Now assume $n \geq 5$ and use induction on n. Since H is general, $\dim(H \cap Z) \leq \dim(H) - 5$ (with the convention $\dim(W) < 0$ only if $W = \emptyset$) and the multiplication by the equation of H induces an injective map $\mathcal{O}_Z \to \mathcal{O}_Z(1)$. We fix an integer $\beta'(Z, n) \geq n$ such that $h^i(\mathcal{I}_Z(t-i)) = 0$ for all $t \geq \beta'(Z, n)$ and all $i > 0$. A standard Castelnuovo-Mumford sequence and the generality of H gives $h^i(H, \mathcal{I}_{Z\cap H}(t-i)) = 0$ for all $t \geq \beta'(Z, n)$ and all $i > 0$. Set $\beta(Z, n) := \max\{\beta'(Z, n), \alpha_{n-1}(Z \cap H)\}$. For all integers $t \geq \beta(Z, n)$ set $z_{n,t,Z} := h^0(Z, \mathcal{O}_Z(t))$. The choice of $\beta(Z, n)$ gives $h^0(\mathcal{I}_Z(t)) = \binom{n+t}{n} - z_{n,t,Z}$ for all $t \geq \beta(Z, n)$. Define the integers $u_{n,t}$ and $v_{n,t}$ by the relations

$$z_{n,t,Z} + (t+1)u_{n,t} - v_{n,t} = \binom{n+t}{n}, \quad 0 \leq v_{n,t} \leq t. \tag{6}$$

Taking the difference of the equation in (6) with the same equation for the integer $t' = t-1$ we get the equation

$$z_{n,t,Z} - z_{n,t-1,Z} + (t+1)(u_{n,t} - u_{n,t-1}) + v_{n,t-1} - v_{n,t} = \binom{n+t-1}{n-1}. \quad (7)$$

Recall that for $t \geq \beta(Z,n)$ we have $z_{n,t,Z} - z_{n-1,t,Z} = h^0(Z \cap H, \mathcal{O}_{H \cap Z}(t))$. Set $\gamma_n := \binom{\beta(Z,n)+n}{n} - z_{\beta(Z,n),n,Z}$. Hence $\gamma_n = h^0(\mathcal{I}_Z(\beta(Z,n)))$. For all integers t such that $\beta(Z,n) \leq t \leq \beta(Z,n) + \gamma_n$ define the integers $e_{n,t}$ and $f_{n,t}$ by the relations

$$z_{n,t,Z} + (\gamma_n + \beta(Z,n) - t) + (t+1)e_{n,t} - f_{n,t} = \binom{n+t}{t}, \quad 0 \leq f_{n,t} \leq t. \quad (8)$$

For each integer t such that $\beta(Z,n) \leq t \leq \beta(Z,n) + \gamma_n$ we define the following assertion $B_{n,t}$:

Assertion $B_{n,t}$. *Let $Y \subset \mathbb{P}^n$ be a general union of $e_{n,t} - 2f_{n,t}$ lines and $f_{n,t}$ reducible conics with there singular points contained in H. Then $Z \cap Y = \emptyset$ and $h^1(\mathcal{I}_{Z \cup Y}(t)) = 0$.*

Of course, we need that $e_{n,t} \geq 2f_{n,t}$ and this is true, because $e_{f,t} \geq 2t$. By (8) we have $h^1(\mathcal{I}_{Z \cup Y}(t)) = 0$ if and only if $h^0(\mathcal{I}_{Z \cup Y}(t)) = \gamma_n + \beta(Z,n) - t$. For all integers $k \geq \beta(Z,n) + \gamma_n$ we define the following Assertion $A_{n.k}$:

Assertion $A_{n,k}$. *Let $B \subset \mathbb{P}^n$ be a general union of $u_{n,k} - 2v_{n,k}$ lines. Let $D \subset \mathbb{P}^n$ be a general union of $v_{n,k}$ reducible conics with their singular points contained in H.*

As in the case $n = 5$ we first prove $B_{n,t}$ by induction on t. Then we prove $A_{n,k}$ by induction on k. Then we use $A_{n,k-1}$ and $A_{n,k}$ to prove Theorem 1 for the critical value k. The only difference is that if $\dim(Z) > 0$, i.e. if $Z \cap H \neq \emptyset$ every quotation of [2] must be substituted with something proved inductively. Fix an integer $t > \alpha_{n-1}(Z)$ and suppose you know that $h^i(H, \mathcal{I}_{(Z\cap H)\cup B \cup D}(t) = 0$, $i = 0,1$, where $(Z \cap H) \cap B = (Z \cap H) \cap D = \emptyset$, B is a disjoint union of lines and D is a disjoint union of at most t conics. Thus $h^1(H, \mathcal{I}_{(Z\cap H)\cup B}(t)) = 0$ and $h^0(H, \mathcal{I}_{(Z\cap H)\cup B}(t)) \leq t(2t+1)$. Fix an integer c such that $0 \leq c \leq t+1$. Taking a general hyperplane of H we prove the existence of a disjoint union $B' \subset H$ of lines and of f length 3 planar subschemes of H such that $(Z \cap H) \cap B' = (Z \cap H) \cap W = \emptyset$, $h^1(H, \mathcal{I}_{(Z\cap H)\cup B' \cup W}(t+1)) = 0$ and $h^0 1(H, \mathcal{I}_{(Z\cap H)\cup B' \cup W}(t+1)) \leq t(2t+1) + (t+1)$. To use this in each inductive step, say in the step $A_{n,k} \Longrightarrow A_{n,k+1}$ starting with $Z \sqcup B \sqcup D$ we need $\sharp((B \cap H)) \geq \max\{0, v_{n,k+1} - v_{n,k}\} + k(2k+1) + (k+1)$. Thus it is sufficient to have $u_{n,k} \geq k + k(2k+1) + (k+1)$. We arrange this taking if necessary a larger $\alpha_n(Z)$. □

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