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POSTULATION OF GENERAL UNIONS OF LINES AND A FEW PLANES IN $\mathbb{P}^4$

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Abstract: Let $X \subset \mathbb{P}^4$ be a general union of x planes and y lines. Let $k \geq 1$ be the minimal integer such that $x\binom{k+2}{2} - x(x-1)/2 + y(k+1) \leq \binom{k+4}{4}$. Assume $x \leq k-1$. Then X has the expected postulation.

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1. Introduction

We will say that a union $A \subset \mathbb{P}^4$ of finitely many planes is *maximally disjoint* if no line of $\mathbb{P}^4$ is contained in at least 2 irreducible components of A and no point of $\mathbb{P}^4$ is contained in at least 3 irreducible components of A. Thus if A is a maximally disjoint union of x planes, then $\chi(\mathcal{O}_A(t)) = x\binom{t+2}{2} - x(x-1)/2$ for all $t \in \mathbb{N}$. Moreover $h^0(A, \mathcal{O}_A(t)) = x\binom{t+2}{2} - x(x-1)/2$ and $h^1(A, \mathcal{O}_A(t)) = 0$ for all $t \geq x$ (if $x \geq 2$ use $x-1$ Mayer-Vietoris exact sequences, starting with a plane and adding at each step a new plane). Notice that a general union $A \subset \mathbb{P}^4$ of finitely many planes is maximally disjoint. Notice that $h^0(B, \mathcal{O}_B(t)) = \chi(\mathcal{O}_B(t)) = y(t+1)$ for any disjoint union $B \subset \mathbb{P}^4$ of y lines and any integer $t \geq 0$. These numbers explain the following statement.

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Theorem 1. *Fix non-negative integers x, y. Let $X \subset \mathbb{P}^4$ be a general union of x planes and y lines. Let k be the minimal integer such that*

$$x\binom{k+2}{2} - x(x-1)/2 + y(k+1) \le \binom{k+4}{4} \tag{1}$$

Assume $x \le k-1$. Then $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge k$.

A scheme X with the cohomology claimed in the statement of Theorem 1 is usually said to have *maximal rank* or *good postulation* or *the expected postulation*. The condition "$h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge k$" is equivalent to the condition "$h^0(\mathcal{I}_X(t)) = \binom{t+4}{4} - x\binom{t+2}{2} - y(t+1)$ for all $t \ge k$". Of course, to check the condition "$h^1(\mathcal{I}_X(t)) = 0$ for all $t \ge k$" it is sufficient to prove $h^1(\mathcal{I}_X(k)) = 0$ (e.g. use Castelnuovo-Mumford's Lemma). The integer k appearing in the statement of Theorem 1 is often called the *critical value of X* or the *critical value of the pair (x, y)*.

The case $x = 0$ of the theorem is the case $\mathbb{P}^4$ of [3]. We obviously owe much to the authors of [3]. The case $x = 1$ is true even in higher dimensional projective spaces (see [2]). This paper was stimulated by [1].

2. The Proof

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We use the characteristic zero only in the proofs of Lemmas 1 and the case $c > 0$ of Lemma 2. We expect that it could be removed (with weaker lemmas we would be forced to do by a case by case analysis the proof of H_k for $k \le 9$).

Remark 1. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D, D} \otimes (L|D) \to 0. \tag{2}$$

From (2) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \le h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D, D} \otimes (L|D))$$

for every integer $i \ge 0$.

Fix a hyperplane H of $\mathbb{P}^4$ and a smooth quadric surface $Q \subset H$. For any $P \in \mathbb{P}^4$ and every integral variety $T \subset \mathbb{P}^4$ such that $P \in T_{reg}$ let $\chi_T(P)$ denote the first infinitesimal neighborhood of P in T, i.e. the closed subscheme of T with $(\mathcal{I}_{P,T})^2$ as its ideal sheaf. We have $\chi_T(P)_{red} = \{P\}$ and $\mathrm{length}(\chi_T(P)) =$

$\dim(T)+1$. If $\dim(T)=1$, then we say that $\chi_T(P)$ is a *tangent vector*. If $\dim(T)=2$, then we say that $\chi_T(P)$ is a *planar length* 3 *scheme*. For any scheme $U \subseteq \mathbb{P}^4$ and any $P \in U_{reg}$ let $T_P U \subseteq \mathbb{P}^4$ denote its tangent space.

Remark 2. Fix a reducible conic $D \subset \mathbb{P}^4$ and let P be its singular point. Let M be a hyperplane of $\mathbb{P}^4$ containing D. The scheme $D \cup \chi_M(P)$ is a flat degeneration inside $\mathbb{P}^4$ of a flat family whose general element is the disjoint union of 2 lines (see [3]). Fix a plane A, a line L such that $D \cap L = \{P\}$ and a hyperplane M such that $L \subset M$, but $A \nsubseteq M$. We claim that $A \cup L \cup \chi_M(P)$ is a flat degeneration inside $\mathbb{P}^4$ of a flat family whose general element is the disjoint union of a plane and a line. Take homogeneous coordinates $z_0, \dots, z_4$ such that $P=(1;0;0;0;0)$, $A=\{z_3=z_4=0\}$, $L=\{z_1=z_2=z_3=0\}$ and $M=\{z_1=0\}$. The flat family may be parametrized by $\mathbb{K}$, with $A \cup L \cup \chi_M(P)$ for $t=0$ and with $A \cup L_t$ for $t \neq 0$, where L_t, $t \neq 0$, is the line $L_t=\{z_1=z_2=z_3-tz_0=0\} \subset M$.

Remark 3. Fix a surface $S \subset H$ and $P \in S_{reg}$. Since $\chi_S(P) \subseteq T_P S \cap S$ and $\chi_{T_P S}(P) \subseteq T_P S \cap S$, we have $\chi_S(P)=\chi_{T_P S}(P)$. Thus the length 3 scheme $\chi_S(P)$ depends only upon the choice of P and a plane $A \subset H$ such that $P \in A$, not from the surface smooth at P and with A as its tangent plane at P.

For all integers $k \geq 1$ and $0 \leq x \leq k-1$ define the integers $y_{k,x}$ and $z_{k,x}$ by the relations

$$x\binom{k+2}{2} - x(x-1)/2 + y_{k,x}(k+1) - z_{k,x} = \binom{k+4}{4}, \quad 0 \leq z_{k,x} \leq k. \tag{3}$$

Taking the difference of the equation in (3) with the same equation for the integer $x' := x$ and $k' := k-1$ we get the following equality:

$$(k+1)(y_{k,x} - y_{k-1,x} + x) + y_{k-1,x} + z_{k-1,x} - z_{k,x} = \binom{k+3}{3}. \tag{4}$$

Taking the difference of the equation in (3) for $x=k-1$ with the same equation for the pair $(k',x') := (k-1,k-2)$ we get the following equality:

$$\binom{k+2}{2} + k(k-2) + (k+1)(y_{k,k-1} - y_{k-1,k-2}) + y_{k-1,k-2} + z_{k-1,k-2} - z_{k,k-1} = \binom{k+3}{3}. \tag{5}$$

We say that $H_k(x)$ is satisfied if there is $A \sqcup B \sqcup D \subset \mathbb{P}^4$ such that $h^i(\mathcal{I}_{A \cup B \cup D}(k)) = 0$, $i=0,1$, A is a maximally disjoint union of x planes, B is a disjoint union of $y_{k,x} - 2z_{k,x}$ lines and D is a disjoint union of $z_{k,x}$ reducible conics with singular points in H; moreover, if $x=k-1$ we also make the following assumption; set $u := \lceil (y_{k+1,k} - y_{k,k-1})/(k-1) \rceil$; if $y_{k,k-1} - 2z_{k,k-1} \geq u$, then we assume that at least u of the lines of B are contained in H. We will see in Lemma 12 that $y_{k,k-1} - 2z_{k,k-1} \geq u$ if $k \geq 5$. The assertion $H_k(x)$, $x \leq k-1$, is well-defined, because $2z_{k,x} \leq y_{k,x}$ for all $k \geq 1$ and all $0 \leq x \leq k-1$ (Lemma 11).

Lemma 1. *Fix non-negative integers k, y, b such that $(k+1)y + 2b \leq \binom{k+3}{3}$. Let $Y \subset H$ be a general union of y lines and b tangent vectors. Then $h^1(H, \mathcal{I}_Y(k)) = 0$.*

Proof. Since we are characteristic zero, this follows from the separability of any non-constant map and $h^1(H, \mathcal{I}_A(k)) = 0$, where A is a general union of y lines. □

Lemma 2. *Fix non-negative integers k, y, b, c such that $k \geq 5$, $c + b \leq k$ and*

$$(k+1)y + 2c + 3b \leq \binom{k+3}{3}. \tag{6}$$

Let $Y \subset H$ be a general union of y lines, c tangent vectors and b planar length 3 schemes. Then $h^1(H, \mathcal{I}_Y(k)) = 0$.

Proof. Since we are in characteristic zero, as in Lemma 1 it is sufficient to do the case $c = 0$. We may assume $b > 0$ (see [3]). For each integer $t \geq 0$ set $a_t := (t+3)(t+2)/6$ and $b_t := 0$ if $t \equiv 0, 1 \pmod 3$, $a_t := (t+4)(t+1)/6$ and $b_t := (t+1)/3$ if $t \equiv 2 \pmod 3$. Increasing if necessary y we may assume $y \geq a_{k-2}$. Let z be the minimal integer such that $3b \leq z(k+1)$. Let $A \subset H$ be a general union of a_{k-2} lines.

(a) Here we assume $k \equiv 0 \pmod 3$. Since $b_k = 0$ and (6) holds with $c = 0$, we have $y \leq a_k - z$. We may assume $y = a_k - z$. Since $k - 2 \equiv 1 \pmod 3$, $b_{k-2} = 0$. Thus $h^i(H, \mathcal{I}_A(k-2)) = 0$, $i = 0, 1$. Let $B \subset Q$ be a general union of $b_k - b_{k-2} - z$ lines of type $(0,1)$ and b planar length 3 subschemes of Q. Since $b_k - b_{k-2} - z \leq k-2$ and $b_k - b_{k-2} - z \leq k-3$ if $b \geq \lfloor (k+1)/2 \rfloor$, condition 4°) of [3] is satisfied. Thus $h^1(Q, \mathcal{I}_B(k)) = 0$ (see [3], Lemma 2.3). Since $A \cap Q$ is a general union of $2a_{k-2}$ points and $2a_{k-2} + 3b + (k+1)(b_k - b_{k-2} - z)$ by the definition of z and the assumption $b_{k-2} = b_k = 0$, we have $h^1(Q, \mathcal{I}_{A \cap Q \cup B}(k)) = 0$. Set $Y := A \cup B$ and apply Remark 1.

(b) Here we assume $k \equiv 1 \pmod 3$. Hence $b_k = 0$ and $b_{k-2} = (k-1)/3$. Since $b_k = 0$, we have $y \leq a_k - z$. Hence we may assume $y = a_k - z$. We have $h^1(H, \mathcal{I}_A(k-2)) = 0$ and $h^0(H, \mathcal{I}_A(k-2)) = (k-1)/3$. First assume $b \geq (k-1)/3$. Let $S \subset Q$ be a general subset such that $\sharp(S) = (k-1)/3$. Since $h^0(H, \mathcal{I}_A(k-4)) = 0$, $h^0(H, \mathcal{I}_A(k-2)) = (k-1)/3$ and S is general in Q, we get $h^0(H,\mathcal{I}_{A\cup S}(k-2)) = 0$. Thus $h^1(H,\mathcal{I}_{A\cup S}(k-2)) = 0$. For each $P \in S$ fix a plane $A_P \subset H$ such that $P \in A_P$ and $A_P \neq T_PA$. Set $\chi := \cup_{P\in S}\chi_{A_P}(P)$. Let $B \subset Q$ be a general union of $a_k - z - a_{k-2}$ lines of type $(1,0)$ and $b-(k-1)/3$ triple points. Set $X := A\cup B\cup\chi$. We have $h^1(Q,\mathcal{I}_B(k)) = 0$, because $b-(k-1)/3 \leq (k+1)/2$ and $a_k - z - a_{k-2} \leq k-1$ (see [3], Lemma 2.3). Our assumption on the planes A_P gives $\mathrm{Res}_Q(\chi) = S$ and that $\chi\cap Q$ is a union of $(k-1)/2$ tangent vectors of Q with S as their reduction. For general S and planes A_P, $P \in S$, these tangent vectors are $(k-1)/2$ tangent vectors of Q. $A\cap Q$ is a general union of $2a_{k-2}$ points. Since $2a_{k-2}+(3b-k-1)+(k+1)(a_k-a_{k-2}-z) \leq (k+1)^2$, we get $h^1(Q,\mathcal{I}_{X\cap Q}(k)) = 0$. Since $\mathrm{Res}_Q(X) = A\cup S$, we are done in this case. Now assume $b \leq (k-4)/3$. Hence $z = 1$. Since $(k+1)(a_k-1)+3(k-1)/3 \leq \binom{k+3}{3}$, we may apply the case $y' := a_k - 1$ and $b' := (k-1)/3$ just proved and then discard $(k-1)/3 - b$ of the planar length 3 schemes.

(c) Here we assume $k \equiv 2 \pmod 3$. Hence $b_k = (k+1)/3$ and $b_{k-2} = 0$. If $3b \leq (k+1)/3$, then we may take as a solution the scheme $A\cup B$ with $B \subset Q$ a general union of $a_k - a_{k-2}$ lines of type $(1,0)$ and b planar length 3 subschemes of Q. Now assume $3b > (k+1)/3$. Let w be the minimal non-negative integer such that $w(k+1)+(k+1)/3 \geq 3b$. Since (6) holds with $c = 0$, we have $y \leq a_k - w$. Hence we may assume $y = a_k - w$. Take as a solution $A\cup D$, where $D \subset Q$ is a general union of $a_k - a_{k-2} - w$ lines of type $(1,0)$ of Q and b triple points of Q. To apply [3], Lemma 2.3, it is sufficient to notice that $a_k - a_{k-2} - w \leq k-2$ and that $a_k - a_{k-2} - w \leq k-3$ if $w \geq 2$. □

Lemma 3. *Let $Y \subset H$ be a general union of 5 lines and 3 planar length 3 schemes. Then $h^1(H,\mathcal{I}_Y(4)) = 0$.*

Proof. Fix $P \in Q$. Let $E \subset Q$ be general union of 2 lines of type $(1,0)$ and 2 length 3 planar points. Let $F \subset H$ be a general union of 3 lines. Obviously, $h^i(H,\mathcal{I}_{F\cup\{P\}}(2)) = 0$, $i = 0,1$. Fix a plane $A \subset H$ such that $P \in A$ and $A \neq T_PQ$. Set $X := F\cup E\cup\chi_A(P)$. Notice that $X\cap Q$ is a general union of E, 6 points and the tangent vector associated to the line $A\cap T_PQ$. Obviously $h^1(Q,\mathcal{I}_{X\cap Q}(4)) = 0$ (since $h^0(Q\cap X,\mathcal{O}_{Q\cap X}(4)) \leq 25$, it is sufficient to prove it without the 6 general points $F\cap Q$). Since $\mathrm{Res}_Q(X) = F\cup\{P\}$, it is sufficient to apply Remark 1 and semicontinuity. □

Remark 4. $H_k(0)$ is true for all integers $k \geq 1$ (see [3]).

Lemma 4. *Assume $H_k(x)$. Let $X \subset \mathbb{P}^4$ be a general union of x planes and $y_{k,x}$ lines. Then $h^0(\mathcal{I}_X(k)) = 0$.*

Proof. If $z_{k,x} = 0$, then the statement of the lemma is just the definition of $H_k(x)$. Assume $z_{k,x} > 0$. Take a solution $A \cup B \cup D$ of $H_k(x)$. Thus $h^0(\mathcal{I}_{A\cup B\cup D}(k)) = 0$. For each $P \in \mathrm{Sing}(D)$ fix a general hyperplane H_P of $\mathbb{P}^4$ such that $P \in H_P$. Set $Y := A \cup B \cup D \cup (\bigcup_{P\in\mathrm{Sing}(D)} \chi_{H_P}(P))$. Since $A\cup B\cup D \subseteq Y$, $h^0(\mathcal{I}_Y(k)) = 0$. Since Y is a flat degeneration of a flat family of general unions of x planes and $y_{k,x}$ disjoint lines (Remark 2), the lemma follows from semicontinuity. □

Remark 5. H_1 is obviously true.

Lemma 5. *H_2 is true.*

Proof. It is sufficient to prove $H_2(1)$ (Remark 4). We have $y_{2,1} = 3$ and $z_{2,1} = 0$ (Remark 6). Hence $H_2(1)$ is contained in the general set-up of [1]. However, we give the following detailed proof. Let A be a general union of a plane and a line. Then $h^i(\mathcal{I}_A(1)) = 0$, $i = 0, 1$. Since A is general, $A \cap H$ is a disjoint union of a line and a point. To prove $H_2(1)$ we take $A \cup D$, where D is a general union of 2 lines of H (Remark 1). □

Lemma 6. *Let $A \subset \mathbb{P}^4$ be a general union of 2 planes and one line. Then $h^0(\mathcal{I}_A(2)) = 1$ and $h^1(\mathcal{I}_A(2)) = 0$.*

Proof. Let $E \subset \mathbb{P}^4$ be a general union of a plane and a line. Obviously, $h^i(\mathcal{I}_E(1)) = 0$, $i = 0, 1$. Fix a general plane $B \subset H$. Since $E \cap H$ is the union of a line not contained in B and a general point of H, $h^0(H, \mathcal{I}_{E\cap H\cup B}(2)) = h^0(H, \mathcal{I}_{E\cap H}(1)) = 1$. Set $A := E \cup B$ and apply Remark 1. □

Lemma 7. *Let $E \subset H$ be a general union of 3 lines, a reducible conic with singular point in H and a point. Then $h^i(H, \mathcal{I}_E(3)) = 0$, $i = 0, 1$.*

Proof. Fix a plane $N \subset H$ and a reducible conic $D \subset N$. Let $F \subset H$ be a general union of 3 lines and a point. Obviously $h^i(H, \mathcal{I}_F(2)) = 0$, $i = 0, 1$, $F \cap D = \emptyset$ and $F \cap N$ is formed by 3 non-collinear points. Set $E := F \cup D$ and apply Remark 1 with respect to the divisor N of H. □

Lemma 8. *H_3 is true.*

Proof. By Remark 4 it is sufficient to check $H_3(1)$ and $H_3(2)$. We first check $H_3(1)$. Let $A \subset \mathbb{P}^4$ be a general union of a plane and 3 lines. Since $y_{2,1} = 5$, $z_{2,1} = 0$ (Remark 6) and $H_2(1)$ is true, we have $h^i(\mathcal{I}_A(2)) = 0$, $i = 0, 1$. The scheme $A \cap H$ is a general union of a line and 3 points, say P_1, P_2, P_3. Let $B \subset H$ be a general union of 4 lines D_1, D_2, D_3, D_4 with $P_i \in D_i$ for $i \in \{1, 2, 3\}$. Thus $H \cap (A \cup B)$ is a general union of 5 lines. Hence $h^i(H, \mathcal{I}_{H\cap(A\cup B)}(3)) = 0$, $i = 0, 1$ (see [3]). Remark 1 gives $h^i(\mathcal{I}_{A\cup B}(3)) = 0$,

$i = 0, 1$, proving $H_3(1)$. Now we check $H_3(2)$. Let $A \subset \mathbb{P}^4$ be a general union of 2 planes and a line. Lemma 6 gives $h^0(\mathcal{I}_A(2)) = 1$ and $h^1(\mathcal{I}_A(2)) = 0$. Let B be the union of a line and a reducible conic T with as singular point a general $P \in H$. Take a 3-dimensional linear subspace M of $\mathbb{P}^4$ such that $M \cap H = \langle T \rangle$. Set $W := A \cup B \cup \chi_M(P)$. W is a flat degeneration of the union of the two planes contained in A and of 4 lines (Remark 2). By semicontinuity to prove $H_3(2)$ it is sufficient to prove $h^i(\mathcal{I}_W(3)) = 0$, $i = 0, 1$. We have $\mathrm{Res}_H(W) = A \cup \{P\}$. Notice that H is not contained in the base locus of $|\mathcal{I}_A(2)|$, because $h^0(\mathcal{I}_A(1)) = 0$. Since $h^0(\mathcal{I}_A(2)) = 1$ and we may take as P a general point of H, we get $h^i(\mathcal{I}_{\mathrm{Res}_H(W)}(2)) = 0$. Notice that $W \cap H = B \cup (A \cap H)$ (scheme-theoretically), because $M \neq H$ and $P \in \mathrm{Sing}(B)$. Since $B \cup (A \cap H)$ is a general union of 3 lines, a reducible conic and a point (the intersection of the line of A with H), Lemma 7 gives $h^i(H, \mathcal{I}_{W \cap H}(3)) = 0$, $i = 0, 1$. Remark 1 gives $h^i(\mathcal{I}_W(3)) = 0$, $i = 0, 1$, proving $H_3(2)$. □

Lemma 9. *H_4 is true.*

Proof. By Remark 4 it is sufficient to check $H_4(x)$ for $x \in \{1, 2, 3\}$. Let A be a general union of x planes.

We first check $H_4(1)$. Remark 6 gives $y_{3,1} = 7$, $z_{3,1} = 3$, $y_{4,1} = 11$ and $z_{4,1} = 0$. Let U be a general union of A, one line and 3 reducible conics with singular points contained in H. $H_3(1)$ gives $h^i(\mathcal{I}_U(3)) = 0$, $i = 0, 1$. For each $P \in \mathrm{Sing}(U)$ fix a general hyperplane H_P containing the reducible conic of U with P as its singular point. Set $\chi := \cup_{P \in \mathrm{Sing}(U)} \chi_{H_P}(P)$. Let $E \subset H$ be a general union of 4 lines. Set $X := U \cup E \cup \chi$. Since $\mathrm{Res}_H(X) = U$, we have $h^i(H, \mathcal{I}_{\mathrm{Res}_H(X)}(3)) = 0$, $i = 0, 1$. $X \cap H$ is a general union of 5 lines, one point and 3 planar length 3 schemes. Lemma 2 gives $h^i(H, \mathcal{I}_{X \cap H}(4)) = 0$, $i = 0, 1$. Apply Remark 1.

Now we check $H_4(2)$. Remark 6 gives $y_{3,2} = 4$, $z_{3,2} = 0$, $y_{4,2} = 9$ and $z_{4,2} = 4$. Let B' be a general union of 4 lines. Set $U := A \cup B'$. $H_3(2)$ gives $h^i(\mathcal{I}_U(3)) = 0$, $i = 0, 1$. Let $E \subset H$ be a general union of 5 lines L_i, $1 \le i \le 5$, with the only restriction that each of lines L_i, $1 \le i \le 4$, intersects a different point of $B' \cap H$. Set $X := U \cup E$. X is a disjoint union of a plane, a line and 4 reducible conics with a component contained in H (and hence with singular point contained in H). Since $\mathrm{Res}_H(X) = U$, we have $h^i(H, \mathcal{I}_{\mathrm{Res}_H(X)}(3)) = 0$, $i = 0, 1$. Since $X \cap E$ is a general union of 7 lines, $h^i(H, \mathcal{I}_{H \cap X}(4)) = 0$, $i = 0, 1$ (see [3]), concluding the proof.

Now we check $H_4(3)$. Remark 6 gives $y_{4,3} = 6$ and $z_{4,3} = 2$. Let A' be a general plane of $\mathbb{P}^4$. Every quadric hypersurface containing $A \cup A'$ has as its singular locus at least the 3 non-collinear points $\mathrm{Sing}(A \cap A')$. Thus $h^0(\mathcal{I}_{A\cup A'}(2)) = 0$. Hence $h^1(\mathcal{I}_{A\cup A'}(2)) = 0$. Notice that $h^0(A\cup A', \mathcal{O}_{A\cup A'}(3)) = 27$, $\binom{6}{3} = 20$ and $\binom{7}{3} = 35$. Let $M \subset \mathbb{P}^4$ be a general hyperplane. Let $E \subset M$ be a general union of two lines. Since $E\cup(A\cup A')\cap M$ is a general union of 5 lines, [3] gives $h^i(M, \mathcal{I}_{E\cup,(A\cup A')\cap M}(3)) = 0$, $i = 0, 1$. Hence $h^i(\mathcal{I}_{A\cup A'\cup E}(3)) = 0$, $i = 0, 1$. Let $F \subset H$ be a general union of 2 reducible conics. Since $(A \cup A' \cup E \cup F) \cap H$ is a general union of 2 reducible conics, 3 lines and 2 points, it is easy to get $h^i(H, \mathcal{I}_{F\cup(A\cup A'\cup E)\cap H}(4)) = 0$, $i = 0, 1$. Hence we may take $A \cup A' \cup E \cup F$ as a solution of $H_4(3)$. □

Lemma 10. *H_k is true for all integers $k \geq 5$.*

Proof. By Lemmas 8 and 9 and induction on k we may assume that H_t is true for all integers t such that $3 \leq t \leq k - 1$.

(a) Here we assume $0 \leq x \leq k - 2$ and prove $H_k(x)$. Take $A \sqcup B \sqcup D \subset \mathbb{P}^4$ satisfying $H_{k-1}(x)$ with A (resp. B, resp. D) a general union of x planes (resp. $y_{k-1,x} - 2z_{k-1,x}$ lines, resp. $z_{k-1,x}$ reducible conics with singular points contained in H); here we apply semicontinuity to find $B \sqcup D$ such that none of its lines is contained in H. Hence $h^i(\mathcal{I}_{A\cup B\cup D}(k - 1)) = 0$, $i = 0, 1$.

(a1) Here we assume $z_{k,x} \geq z_{k-1,x}$. Let $N \subset H$ be a general union of $y_{k,x} - y_{k-1,x}$ lines with the only restriction that $z_{k,x} - z_{k-1,x}$ of these lines contain a point of $B \cap H$. To prove $H_k(x)$ in this case it is sufficient to prove $h^i(\mathcal{I}_{A\cup B\cup D\cup N}(k)) = 0$, $i = 0, 1$. Remark 1 shows that it is sufficient to prove

$$h^i(H, \mathcal{I}_{N\cup(A\cup B\cup D)\cap H}(k)) = 0.$$

The scheme $N \cup (A \cup B \cup D)$ is a general union of $x + y_{k,x} - y_{k-1,x}$ lines, $y_{k-1,x} - z_{k,x} + z_{k-1,x}$ points and $z_{k-1,x} \leq k - 1$ tangent vectors. Apply Lemma 1.

(a2) Here we assume $z_{k,x} < z_{k-1,x}$. Fix $S \subseteq \mathrm{Sing}(D)$ such that $\sharp(S) = z_{k-1,x} - z_{k,x}$. For each $P \in S$ choose a general hyperplane $H_P \subseteq \mathbb{P}^4$ such that $P \in H_P$. Let $E \subset H$ be a general union of $y_{k,x} - y_{k-1,x}$ lines. Set $\chi := \cup_{P\in S}\chi_{H_P}(P)$. The first part of Remark 2 shows that it is sufficient to prove $h^i(\mathcal{I}_{A\cup B\cup D\cup E\cup\chi}(k)) = 0$. Hence it is sufficient to prove

$$h^i(H, \mathcal{I}_{(A\cup B\cup D\cup E\cup\chi)\cap H}(k)) = 0, \quad i = 0, 1.$$

The scheme $(A \cup B \cup D \cup E \cup \chi) \cap H$ is a general union of $x + y_{k,x} - y_{k-1,x}$ lines, $z_{k,x}$ tangent vectors, $z_{k-1,x} - z_{k,x}$ planar length 3 schemes and $y_{k-1,x} - z_{k-1,x}$ points. Apply Lemma 2.

(b) Here we prove $H_k(k-1)$. Set $u := \lceil (y_{k,k-1} - y_{k-1,k-2})/(k-1) \rceil$. Notice that u is the minimal integer such that

$$y_{k,k-1} - y_{k-1,k-2} - u \leq ku. \tag{7}$$

Since $y_{k,k-1} > y_{k-1,k-2}$, we have $u > 0$. Lemma 12 gives $u \leq y_{k-1,k-2} - 2z_{k-1,k-2}$. Take $A \sqcup B \sqcup D$ satisfying $H_{k-1}(k-2)$ and a general plane $A' \subset H$, with $B = B' \sqcup J$, B' a general union of $y_{k-1,k-2} - 2z_{k-1,k-2} - u$ lines of $\mathbb{P}^4$ and J a general union of u lines (say $J_1, \dots, J_u$)) of H. Fix a general $S_i \subset J_i$ such that $\sharp(S_i) = k$ for all $1 \leq i \leq u-1$, and $\sharp(S_u) = y_{k-1,k-2} - 2z_{k-1,k-2} - u(k-2)$. The definition of u gives $1 \leq \sharp(S_u) \leq k$. Since $h^1(\mathcal{I}_{A \cup B \cup D}(k-1)) = 0$, the generality of each S_i gives $h^1(\mathcal{I}_{A \cup B' \cup D \cup S}(k-1)) = 0$. Hence $h^0(\mathcal{I}_{A \cup B' \cup D \cup S}(k-1)) = y_{k,k-1} - y_{k-1,k-2} + u$.

Claim 1. *$h^0(\mathcal{I}_{A \cup B' \cup D \cup A'}(k-1)) \leq \sharp(S)$ and $h^0(\mathcal{I}_{A \cup B' \cup D \cup A' \cup S}(k-1)) = 0$.*

Proof of Claim 1. The second assertion implies the first one. Since $\sharp(S) \cap J_i = k$ for all $i < u$, every element of $H^0(\mathcal{I}_{A \cup B' \cup D \cup A' \cup (S \backslash S_u)}(k-1))$ vanishes on $J \backslash J_u$. Thus it is sufficient to prove

$$h^0(\mathcal{I}_{A \cup B' \cup D \cup A' \cup (J \backslash J_u) \cup S_u}(k-1)) = 0.$$

We will even prove that $h^0(\mathcal{I}_{A \cup B' \cup D \cup A' \cup (J \backslash J_u)}(k-1)) = 0$. Indeed, we may fix the general plane $A' \subset H$ after fixing J. Thus we may specialize A' to a general plane containing J_u. Hence semicontinuity gives

$$h^0(\mathcal{I}_{A \cup B' \cup D \cup A' \cup (J \backslash J_u)}(k-1)) \leq h^0(\mathcal{I}_{A \cup B' \cup D \cup J}(k-1)) = 0. \qquad \square$$

(b1) Here we assume $z_{k,k-1} \geq z_{k-1,k-2}$. Fix a general union $E \subset H$ of $y_{k,k-1} - y_{k-1,k-2}$ lines with the only restriction that each point of S is contained in one of these curves and $z_{k,k-1} - z_{k-1,k-2}$ of them contain a point of $B' \cap H$. Notice that $B \cup E \cup D \cup \chi = B' \cup J \cup E \cup D$ is a flat degeneration of a family of disjoint unions of $y_{k,k-1} - 2z_{k,k-1}$ lines and $z_{k,k-1}$ reducible conics with singular points in H. Set $S' := A' \cap (E \cup J)$. Since A' is general, we may assume $\sharp(S') = \deg(E \cup J)$. Varying first J, then varying S inside J, then varying the $z_{k,k-1} - z_{k-1,k-2}$ lines of B' which will meet some line of E and then taking E general for a fixed A', we may also assume that S' is formed by $\deg(E \cup J)$ general points of A'. For each $P \in S'$ take a hyperplane $H_P \subset \mathbb{P}^4$ containing the irreducible component of $E \cup J$ containing P, but not containing A'. Set $\chi' := \cup_{P \in S'} \chi_{H_P}(P)$ and $X := A \cup A' \cup B \cup D \cup E \cup \chi \cup \chi'$. The scheme X is a flat degeneration of a disjoint union of k maximally disjoint planes (none of them contained in H), $z_{k,k-1}$ reducible conics with singular point contained in H and $y_{k,k-1} - 2z_{k,k-1}$ such that $u + y_{k,k-1} - y_{k-1,k-2}$ of these lines are contained in H. We have $u + y_{k,k-1} - y_{k-1,k-2} \geq \lceil (y_{k+1,k} - y_{k,k-1})/k \rceil$ (Lemma 12). Thus to prove $H_k(k-1)$ in this case it is sufficient to prove $h^i(\mathcal{I}_X(k)) = 0$. We have $\mathrm{Res}_H(X) = A \cup B' \cup D \cup S' \cup S$. We have $h^0(\mathcal{I}_{A \cup B' \cup D \cup S}(k-1)) =$

$k - \sharp(S_u)$. Since $\sharp(S') \ge k - \sharp(S_u)$ and S' is general in A', Claim 1 implies $h^0(\mathcal{I}_{\mathrm{Res}_H(X)}(k-1)) = 0$. Notice that $X \cap H$ (scheme-theoretic intersection) is the union of $J \cup E$, $\deg(B') - z_{k,k-1} + z_{k-1,k-2}$ general points of H and $z_{k-1,k-2}$ tangent vectors of H. To conclude the proof of $H_k(k-1)$ in this case using Remark 1 it is sufficient to prove $h^1(H, \mathcal{I}_{J\cup E\cup W}(k-1))$, where W is a general union of $z_{k-1,k-2} \le k-1$ tangent vectors. Apply the case $\mathbb{P}^3$ of [3] works even for not completely general union of lines.

(b2) Here we assume $z_{k,k-1} < z_{k-1,k-2}$. Fix $M \subseteq \mathrm{Sing}(D)$ such that $\sharp(M) = z_{k-1,k-2} - z_{k,k-1}$. For each $P \in M$ fix a general hyperplane $H_P \subset \mathbb{P}^4$ containing the connected component of D containing P. Set $\chi_1 := \cup_{P\in M}\chi_{H_P}(P)$. Let $F \subset H$ be a general union of $y_{k,k-1} - y_{k-1,k-2}$ lines of H with the only restriction that each of them contains a point of X. Set $S_2 := A' \cap (J \cup F)$. For general J, F we may assume that S_2 is a general subset of A' with cardinality $y_{k,k-1} - y_{k-1,k-2} + u$. For each $P \in S_2$ fix a hyperplane $H_P \subset \mathbb{P}^4$ containing the irreducible component of $J \cup F$ containing P. Set $\chi_2 := \cup_{P\in S_2}\chi_{H_P}(P)$ and $X' := A \cup A' \cup B \cup D \cup F \cup \chi \cup \chi_1 \cup \chi_2$. The scheme X' a flat degeneration of a disjoint union of k maximally disjoint planes (none of them contained in H), $z_{k,k-1}$ reducible conics with singular point contained in H and $y_{k,k-1} - 2z_{k,k-1}$ such that $u + y_{k,k-1} - y_{k-1,k-2}$ of these lines are contained in H. Up to the identification of S' with S_2 (both are general subsets of A' with the same cardinality) we have $\mathrm{Res}_H(X') = \mathrm{Res}_H(X)$. Hence by part (b1) it is sufficient to prove $h^i(H, \mathcal{I}_{X'\cap H}(k)) = 0$, $i = 0, 1$. Since $X' \cap H$ (scheme-theoretic intersection) is a general union of A', $\deg(B')$ points, $J \cup F$, $z_{k,k-1}$ tangent vectors and $z_{k-1,k-2}$ planar length 3 subschemes. Hence it is sufficient to prove $h^1(H, \mathcal{I}_{J\cup F\cup W_1\cup W_2}(k-1)) = 0$, where W_1 is a general union of $z_{k,k-1}$ tangent vectors and W_2 is a general union of $z_{k-1,k-2} - z_{k,k-1} \le k-1$ length 3 planar schemes. Apply Lemma 2 (with the modifications needed for not completely general lines). □

Proof of Theorem 1. Fix (x, y) and let k be its critical value. First assume $k = 3$. By [3] and [2] we may assume $x = 2$. We have $y_{3,2} = 4$ and $z_{3,2} = 0$. Since $z_{3,2} = 0$, we have $y \le y_{3,2}$. Hence $H_3(2)$ covers all cases with $(k, x) = (3, 2)$. From now on we assume $k \ge 4$.

(a) Here we prove $h^0(\mathcal{I}_X(k-1)) = 0$. Since $\chi(\mathcal{O}_X(k-1)) > \binom{k+3}{4}$ by the definition of k and $0 \le z_{k,x} \le k$, we have $y \ge y_{k-1,x}$. Let $Y \subseteq X$ be the union of the x planes of X and of $y_{k-1,x}$ of its lines. Lemma 4 gives $h^0(\mathcal{I}_Y(k-1)) = 0$. Since $Y \subseteq X$, we get $h^0(\mathcal{I}_X(k-1)) = 0$.

(b) Here we prove $h^1(\mathcal{I}_X(k)) = 0$. Notice that either $y < y_{k,x}$ or $z_{k,x} = 0$ and $y = y_{k,x}$. If $z_{k,x} = 0$ and $y = y_{k,x}$, then $h^i(\mathcal{I}_X(k)) = 0$, $i = 0, 1$ by $H_k(x)$.

Hence we may assume $y < y_{k,x}$. In this case the h^1-part of Theorem 1 is true for the pair (x, y) if it is true for the pair $(x, y_{k,x} - 1)$. For the pair $(x, y_{k,x} - 1)$ we use the following modification of the proof of $H_k(x)$ given in Lemma 10. First assume $x \le k-2$. Take a solution $A \sqcup B \sqcup D$ of $H_{k-1}(x)$. Take the set-up of step (a2) of Lemma 10 with the following modifications. Set $S := \mathrm{Sing}(D)$. Hence here $\sharp(S) = z_{k-1,k-2}$, instead of $z_{k-1,x} - z_{k,x}$. Take H_P and $\chi := \cup_{P \in S} \chi_{H_P}(P)$. Here E is a general union of $y_{k,x} - y_{k-1,x} - 1$ lines. Thus now we need to prove $h^1(H, \mathcal{I}_{J \cup W}(k)) = 0$, where J is a general union of $x + y_{k,x} - y_{k-1,x} - 1$ lines and W is a general union of $z_{k-1,x}$ planar length 3 subschemes. Notice that

$$(k+1)(x + y_{k,x} - y_{k-1,x} - 1) + y_{k-1,k-2} - 2z_{k-1,k-2} + 3z_{k-1,k-2} \le \binom{k+3}{3}$$

by (4) and the inequality $z_{k-1,k-2} \le k-1$. Hence we may apply Lemma 2. Now assume $x = k$. We take the set-up of step (b2) of Lemma 10 with the following modifications. Instead of u we take $u' := \lceil (y_{k,k-1} - y_{k-1,k-2} - 1)/(k-1) \rceil$. Hence $u - 1 \le u' \le u$. We deform $A \sqcup B \sqcup D$ so that exactly u' of the lines of B are contained in H, i.e. we take $B = B' \sqcup J$ with J general union of u' lines of H. We take (with the obvious choices of the hyperplanes H_P) $S := \mathrm{Sing}(D)$, $\chi_1 := \cup_{P \in S} \chi_{H_P}(P)$; E is a general union of $y_{k,k-1} - y_{k-1,k-2} - 1$ lines of H, $S_2 := H \cap (E \cup J)$, $\chi_2 := \cup_{P \in S_2} \chi_{H_P}(P)$ and $X' := A \cup A' \cup B \cup D \cup F \cup \chi \cup \chi_1 \cup \chi_2$. As in the proof of Claim 1 in step (b) of the proof of Lemma 10 the choice of u' gives $h^1(\mathcal{I}_{\mathrm{Res}_H(X')}(k-1)) = 0$. Lemma 2 with $b := 0$ gives $h^1(H, \mathcal{I}_{X' \cap H}(k)) = 0$. □

3. Numerical Lemmas

Remark 6. We have $y_{1,0} = 3$, $z_{1,0} = 1$, $y_{2,0} = 5$, $z_{2,0} = 0$, $y_{2,1} = 3$, $z_{2,1} = 0$, $y_{3,0} = 9$, $z_{3,0} = 1$, $y_{3,1} = 7$, $z_{3,1} = 3$, $y_{3,2} = 4$, $z_{3,2} = 0$, $y_{4,0} = 14$, $z_{4,0} = 0$, $y_{4,1} = 11$, $z_{4,1} = 0$, $y_{4,2} = 9$, $z_{4,2} = 4$, $y_{4,3} = 6$, $z_{4,3} = 2$, $y_{5,0} = 21$, $z_{5,0} = 0$, $y_{5,1} = 18$, $z_{5,1} = 3$, $y_{5,2} = 15$, $z_{5,2} = 5$, $y_{5,3} = 11$, $z_{5,3} = 0$, $y_{5,4} = 8$, $z_{5,4} = 0$, $y_{6,0} = 30$, $z_{6,0} = 0$, $y_{6,1} = 26$, $z_{6,1} = 0$, $y_{6,2} = 23$, $z_{6,2} = 6$ $y_{6,3} = 19$, $z_{6,3} = 4$, $y_{6,4} = 15$, $z_{6,4} = 1$ $y_{6,5} = 12$, $z_{6,5} = 4$, $y_{7,6} = 17$, $z_{7,6} = 7$, $y_{8,7} = 23$, $z_{8,7} = 6$, $y_{9,8} = 31$, $z_{9,8} = 7$.

Lemma 11. *We have $y_{k,x} \ge k+1$ for all $k \ge 4$. We have $y_{k,x} \ge 2z_{k,x}$ for all $k \ge 1$ and all $0 \le x \le k-1$.*

Proof. Since $0 \le z_{k,x} \le k$, we have $y_{k,x} \ge 2z_{k,x}+$ if $y_{k,x} \ge 2k$. Since the function $x \mapsto x\binom{k+2}{2} - x(x-1)/2$ is an increasing function in the interval

$0 \le x \le k$, by (3) we have $y_{k,x} \ge 2k$ if

$$2k(k+1) + k\binom{k+2}{2} - k(k-1)/2 \le \binom{k+4}{4} \tag{8}$$

holds. The inequality (8) is satisfied for all $k \ge 7$. For $k \le 6$ use Remark 6. □

Lemma 12. *For all integers $k \ge 4$ we have*

$$(y_{k,k-1} - y_{k-1,k-2})/(k-1) \le y_{k-1,k-2} - 2z_{k-1,k-2}. \tag{9}$$

Proof. If $4 \le k \le 9$, then we use Remark 6. Hence we may assume $k \ge 10$. Use the inequalities $z_{k-1,k-2} \le k-1$, $ky_{k-1,k-2} \ge \binom{k+3}{4} - (k-2)\binom{k+1}{2} + (k-2)(k-3)/2$ and (2). □

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