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POSTULATION OF GENERAL UNIONS IN $\mathbb{P}^n$, $n \geq 4$, OF A RATIONAL CURVE AND A FEW PLANES

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Abstract: Here we prove the existence of an integer $\alpha \geq 0$ with the following property. Let $X \subset \mathbb{P}^4$ be a general union of x planes and a degree y smooth rational cuve. Let $k \geq 1$ be the minimal integer such that $x\binom{k+2}{2} - x(x-1)/2 + ky + 1 \leq \binom{k+4}{4}$. Assume $x \leq k - \alpha$. Then X has the expected postulation. We extend the result to $\mathbb{P}^n$, $n \geq 5$, when the planes are either disjoint or contained in a 4-dimensional linear subspace.

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1. Introduction

We will say that a union $A \subset \mathbb{P}^4$ of finitely many planes is *maximally disjoint* if no line of $\mathbb{P}^4$ is contained in at least 2 irreducible components of A and no point of $\mathbb{P}^4$ is contained in at least 3 irreducible components of A. Thus if A is a maximally disjoint union of x planes, then $\chi(\mathcal{O}_A(t)) = x\binom{t+2}{2} - x(x-1)/2$ for all $t \in \mathbb{N}$. Moreover $h^0(A, \mathcal{O}_A(t)) = x\binom{t+2}{2} - x(x-1)/2$ and $h^1(A, \mathcal{O}_A(t)) = 0$ for all $t \geq x$ (if $x \geq 2$ use $x-1$ Mayer-Vietoris exact sequences, starting with a plane and adding at each step a new plane). Notice that a general union $A \subset \mathbb{P}^4$ of finite many planes is maximally disjoint. Notice that $h^0(B, \mathcal{O}_B(t)) = \chi(\mathcal{O}_B(t)) = yt + 1$ for any smooth rational curve $B \subset \mathbb{P}^n$, $n \geq 3$, such that $\deg(B) = y$ and any integer $t \geq 0$. These numbers explain the integer k appearing in the following statements.

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Theorem 1. *There is an integer $\alpha \geq 0$ with the following properties. Fix non-negative integers x, y. Let $X \subset \mathbb{P}^4$ be a general union of x planes and a degree y smooth rational curve. Let k be the minimal positive integer such that*

$$x\binom{k+2}{2} - x(x-1)/2 + ky + 1 \leq \binom{k+4}{4}. \tag{1}$$

Assume $x \leq \max\{0, k-\alpha\}$. Then $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$.

Theorem 2. *There is an integer $\alpha_1 \geq 0$ with the following properties. Fix non-negative integers n, y, x such that $n \geq 5$ and a 4-dimensional linear subspace $M \subset \mathbb{P}^n$. Let k be the minimal positive integer such that*

$$x\binom{k+2}{2} - x(x-1)/2 + ky + 1 \leq \binom{k+n}{n}. \tag{2}$$

Assume $x \leq k - \alpha_1$. Let $X \subset \mathbb{P}^n$ be a general union of a maximally disjoint union of x planes of M and a general degree y smooth rational curve of $\mathbb{P}^n$. Then $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$.

Theorem 3. *There is an integer $\alpha_2 \geq 0$ with the following properties. Fix non-negative integers n, y, x, such that $n \geq 5$. Let k be the minimal positive integer such that*

$$x\binom{k+2}{2} + ky + 1 \leq \binom{k+n}{n}. \tag{3}$$

Assume $x \leq k - \alpha_2$. Let $X \subset \mathbb{P}^n$ be a general union of x planes and a general degree y smooth rational curve of $\mathbb{P}^n$. Then $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$.

Notice that in Theorems 2 and 3 the integers α_1 and α_2 are independent from n. Only numerical reasons in $\mathbb{P}^5$ prevented us to check these two theorems with the integers $\alpha_1 = \alpha_2 = 0$.

A scheme X with the cohomology claimed in the statements of Theorems 1, 2 and 3 is usually said to have *maximal rank* or *good postulation* or *the expected postulation*. The condition "$h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$" is equivalent to the condition "$h^0(\mathcal{I}_X(t)) = \binom{t+n}{n} - x\binom{t+2}{2} - y(t+1)$ for all $t \geq k$" (with $n = 4$ for Theorem 1). Of course, to check the condition "$h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$" it is sufficient to prove $h^1(\mathcal{I}_X(k)) = 0$ (e.g. use Castelnuovo-Mumford's Lemma). The integer k appearing in the statements of Theorems 1 and 2 is often called the *critical value of* X or the *critical value of the pair* (x, y).

This paper was stimulated by [3] and [4].

2. Preliminaries and Proof of Theorem 1

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Remark 1. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0. \tag{4}$$

From (4) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every integer $i \geq 0$.

Fix a hyperplane H of $\mathbb{P}^n$, $n \geq 4$. For any $P \in \mathbb{P}^n$ and every integral $C \subset \mathbb{P}^n$ such that $P \in Creg$ let $\chi_C(P)$ denote the first infinitesimal neighborhood of P in C, i.e. the closed subscheme of C with $(\mathcal{I}_{P,C})^2$ as its ideal sheaf. We have $\chi_C(P)_{red} = \{P\}$ and $\mathrm{length}(\chi_C(P)) = \dim(C) + 1$. Notice that $\chi_T(P) = \chi_{T_PC}(P)$, where $T_PC \subset \mathbb{P}^n$ is the embedded tangent space of C at its smooth point P.

Remark 2. Take $n = 4$. Fix a plane $U \subset H$ and a degree y curve $B \subset H$ intersecting transversally U. For every $P \in U \cap B$ choose a hyperplane $H_P \subset \mathbb{P}^4$ such that H_P contains the tangent line T_PB to B at U. Set $\chi := \cup_{P\in B\cap U}\chi_{H_P}(P)$ and $Y := U \cup B \cup \chi$. Degenerating a general plane of $\mathbb{P}^4$ to U we get the existence of a flat family of closed subschemes of $\mathbb{P}^4$ whose special fiber is Y and whose general fiber is the disjoint union of B and a plane. We may also find a flat degeneration of Y whose general fiber is the disjoint union of a plane and a curve projectively equivalent to B.

We introduce the following integers $a_{n,k,x}$, $b_{n,k,x}$, $u_{n,k,z}$ and $v_{n,k,z}$.

$$x\binom{k+2}{2} - x(x-1)/2 + k\cdot a_{n,k,x} + 1 + b_{n,k,x} = \binom{n+4}{n},\ 0 \leq b_{n,k,x} \leq k-1, \tag{5}$$

$$z\binom{k+2}{2} + k \cdot u_{n,k,z} + v_{n,k,z} = \binom{n+k}{n},\ 0 \leq v_{n,k,z} \leq k-1. \tag{6}$$

Taking the difference of the equation in (5) with the same equation for the integer $x' := x$ and $k' := k-1$ we get the following equality:

$$k(x + a_{n,k,x} - a_{n,k-1,x}) + a_{n,k,x-1} + b_{n,k,x} - b_{n,k-1,x} = \binom{k+n-1}{n-1}. \tag{7}$$

Taking the difference of the equation in (5) for with the same equation for the pair $(k', x') := (k-1, x-1)$ we get the following equality:

$$\binom{k+2}{2} + 1 - x + k(x - 1 + a_{n,k,n} - a_{n,k-1,x-1}) + a_{n,k-1,x-1} + b_{n,k,x} - b_{n,k-1,x-1} = \binom{k+n-1}{n-1}. \quad (8)$$

To prove Theorem 1 we define the following Assertion $U_k(z)$.

Assertion $U_k(z)$. *$k \geq 1$, $z \geq 0$ and $z\binom{k+2}{2} - z(z-1)/2 - k^2 - 1$: For all integers $e \geq 0$ and $y > 0$ such that*

$$z\binom{k+2}{2} - z(z-1)/2 + ky + 1 + (k+1)e \leq \binom{k+4}{4}, \quad 0 \leq e \leq k-1, \quad (9)$$

a general union $X \subset \mathbb{P}^4$ of z planes, a degree y smooth rational curve and e disjoint lines satisfies $h^1(\mathcal{I}_X(k)) = 0$.

For any y, e as in (9) call $\Delta_{k,z}(y, e)$ the difference between the right hand side and the left hand side of the equation in (9). Thus $\Delta_{k,z}(y, e) \geq 0$. In the set-up of $U_k(z)$ we usually write $X = A \sqcup B \sqcup D$, where $A \cap B = A \cap D = B \cap D = \emptyset$, A is a maximally disjoint union of planes, B is a smooth rational curve and D is a disjoint union of lines.

Lemma 1. *There is an integer $\delta \geq 0$ with the following property. For all integers k, z such that $k - \delta \geq z \geq 0$ Assertion $U_k(z)$ is true.*

Proof. Fix an integer $\beta \geq 2$. The case $z = 0$ is true (see [2]). Hence we may assume $z > 0$. Thus $k \geq 3$. We assume $U_{k'}(z')$ for all $k' \leq k-1$ and all $z' \leq k' - \beta$ and look at conditions on β such that $U_k(z)$ is true for all $z \leq k - \beta - 1$ (this will always be satisfied) or for all $z \leq k - \beta$ (this will be true at least if β is large, and here “large ” means “large independently of k”). The lemma will follow by induction on k taking $\alpha = \beta$ with β large enough to do the second inductive step for all large k.

(a) Here we assume $z \leq k - \beta - 1$. Hence $U_{k-1}(z)$ is true. If $\Delta_{k-1,z}(y, e) \geq \binom{k+4}{4} - \binom{k+3}{4} = \binom{k+3}{3}$, then we may take a solution coming from $U_{k-1}(z)$ (use Castelnuovo-Mumford's Lemma). Hence we may assume $\Delta_{k-1,z}(y, e) < 0$, i.e. either $y + e > a_{4,k-1,z}$ or $y = a_{4,k-1,z}$ and $e > b_{4,k-1,z}$.

First assume $e \geq b_{4,k-1,z}$ and $y \geq a_{4,k-1,z} - b_{4,k-1,z}$. Let $Y = A \sqcup B \sqcup D$ be a solution of $U_{k-1}(z)$ for the integers $y_1 := a_{4,k-3,z} - b_{4,k-3,z}$ and $e_1 := b_{4,k-3,z}$. Thus $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. Let $E = B_1 \sqcup D_1 \subset H$ be a general union of a smooth rational normal curve B_1 of degree $y - a_{4,k-1,z} + b_{4,k-1,z}$ and $e - b_{4,k-1,z}$ lines, with the only restriction that B_1 contains exactly one point of B. Thus

$B \cup B_1$ is a flat limit of a flat family of smooth rational curves with degree y. Set $X := Y \cup E$. Obviously $\mathrm{Res}_H(X) = H$. The scheme $Y \cap H$ is a general union of $z + y - a_{4,k-1,z} + b_{4,k-1,z}$ lines, and $a_{4,k-1,z} - 1$ points. By (7) and (9) we have $\chi(\mathcal{O}_{X\cap H}(k)) \le \binom{k+3}{3}$. Hence $h^1(H, \mathcal{I}_{X\cap H}(k)) = 0$ (see [1]). Apply Remark 1.

Now assume $e \ge b_{4,k-1,z}$ and $y < a_{4,k-1,z} + b_{4,k-1,z}$. We make the same construction taking the curve $Y' = A \sqcup B' \sqcup D$ instead of the curve $Y = A \sqcup B \cup D$, where B' is a general smooth rational curve of degree y and $B_1 = \emptyset$.

Now assume $e < b_{4,k-1,z}$. Since $\Delta_{k-1,z}(y, e) > 0$, we may assume $y \ge a_{4,k-1,z} - e$. Let $B_2 \subset H$ be a general smooth rational curve containing exactly one point of $B \cap H$ and one point of $b_{4,k-1,z} - e$ lines of D. Set $X := Y \cup B_2$. Since a general union in H of z lines and a general smooth rational curve of any degree has maximal rank (see [1]), then (7) gives $h^1(H, \mathcal{I}_{X\cap H}(k)) = 0$, concluding this case.

(b) Here we assume $z = k - \beta$. Fix y, e satisfying (9) for $z = k - \beta$. Fix a general union $A \subset \mathbb{P}^4$ of $k - \beta - 1$ planes and a general plane $A' \subset H$. Thus $A' \cup A$ is a maximally disjoint union of $k - \beta$ planes.

(b1) Here we assume $y + e \ge a_{4,k-1,k-\beta-1}$. Let u be the minimal integer such that

$$(k-\beta-1)\binom{k+1}{2} - (k-\beta-1)(k-\beta-2)/2 + u + k(y+e-u) + 1 \le \binom{k+3}{4}. \quad (10)$$

Thus $u = \lceil ((\binom{k+3}{4} - 1 - k(y+e) - (k-\beta-1)\binom{k+3}{4} + (k-\beta-1)(k-\beta-2)/2)/(k-1) \rfloor$. Let f be the difference between the right hand and the left hand side of (10). The maximality of u gives $0 \le f \le k-2$. First assume $y + e - u \ge f + 1$. Let $Y \subset \mathbb{P}^4$ be a general union of A, a smooth rational curve of degree $y + e - u - f$ and f lines. Set $Y_1 := Y \backslash A$. Since $u \ge 0$, the inductive assumption gives $h^1(\mathcal{I}_Y(k-1)) = 0$. Hence (10) gives $h^0(\mathcal{I}_Y(k-1)) = u$.

Claim. *For general Y we have $h^0(\mathcal{I}_{Y\cup A'}(k-1)) = 0$.*

Proof of the Claim. We specialize $A \cup A'$ to $A'' \cup A' \cup A_1 \cup \eta$, where A'' is a general union of $k - \beta - 2$ planes, A_1 is a general plane in H and η is some nilpotent structure supported by the line $A' \cap A_1$. By semicontinuity it is sufficient to prove $h^0(\mathcal{I}_{Y_1 \cup A'' \cup A' \cup A'' \cup \eta}(k-1)) = 0$. Thus it is sufficient to prove $h^0(\mathcal{I}_{Y_1 \cup A'' \cup A' \cup A'}(k-1)) = 0$. We specialize Y_1 to a curve $Y_2 = E_1 \cup E_2$ with each E_i union of a smooth rational curve and disjoint lines, $E_2 \subset H$, no irreducible component of E_1 is contained in H, $h^0(H, \mathcal{I}_{A'\cup A_1\cup E_2}(k-1)) = 0$ and $h^0(\mathcal{I}_{A''\cup E_1}(k-2)) = 0$. Since $\mathrm{Res}_H(A'' \cup A' \cup A_2 \cup E_1 \cup E_2) = A'' \cup E_1$, if we may find such a degeneration, then the claim is true. $\square$

To check the existence of the degeneration we use that β is sufficiently large. Since $a_{4,k,w}$ has order $k^3/24$ for $k \gg 0$ and all $w \le k$, we have $a_{4,k,w} - a_{4,k-1,w} \sim k^2/8$ for all $w \le k$ and $k \gg 0$. Thus we get (for any $\beta \ge 0$) $u \le\sim k^2/8$. For small $\Delta_{k,k-\beta-1}(y,e)$ (say, $\Delta_{k,k-\beta-1}(y,e) \le 10k$) the integer u depends essentially (up to lower terms in k) only from k. Hence we write it has u_k. In any case we have $u \le\sim u^2/8$ and $u_{k-1} - u_{k-2} \le\sim k/4$. Since $h^0(H, \mathcal{I}_{A' \cup A_1 \cup E_2}(k-1)) = h^0(H, \mathcal{I}_{E_2}(k-3))$, we may take as E_2 a general smooth rational curve of degree $\ge \lceil (\binom{k}{3} - 1)/(k-3) \rfloor \sim k^2/6$. We use the inductive assumption to get E_1. We use that $\lfloor \binom{k+1}{3}/(k-1) \rfloor - \lceil (\binom{k}{3} - 1)/(k-3) \rfloor \sim 2k/3 > u_{k-1} - u_{k-2} + 1$.

Here we also assume $u \ge k$. Hence $u \ge \max\{e-f, f-e\}$. First assume $e \ge f$. Let $B_4 \cup D_4 \subset H$ be a general union of a smooth rational curve B_4 of degree $u - e + f$ and $e - f$ lines. Since $B_4 \cup D_4$ is general in H, it intersects transversally A'. Set $S := A' \cap (B_4 \cup D_4)$. For each $P \in S$ take a 3-dimensional linear subspace $H_P \ne H$ containing the tangent line in P of the connected component of $B_4 \cup D$. Set $\chi := \cup_{P \in S} \chi_{H_P}(P)$ and $X := Y \cup A' \cup B_4 \cup D_4 \cup \chi$. Remark 2 gives that X is a flat limit of a flat family of disjoint unions of $A \cup A'$, a smooth rational curve of degree y and e disjoint lines. Since $X \cap H$ is a general union of A', the $k - \beta - 1$ general lines $A \cap H$, $B_4 \cup D_4$ and the general points $(H \backslash (B_4 \cup D_4)) \cap Y_1$, $h^1(H, \mathcal{I}_{X \cap H}(k)) = 0$. Hence it is sufficient to prove $h^1(H, \mathcal{I}_{\mathrm{Res}_H(X)}(k-1)) = 0$. We have $\mathrm{Res}_H(X) = Y \cup S$. Since $B_4 \cup D_4$ is general, S is a general subset of A' with cardinality u. Since $h^1(\mathcal{I}_Y(k-1)) = 0$, $h^0(\mathcal{I}_Y(k-1)) = u$ and S is general in A', the claim gives $h^i(\mathcal{I}_{Y \cup S}(k-1)) = 0$, $i = 0, 1$, concluding the proof in this case. If $e < f$ we take $D_4 = \emptyset$ and take as B_4 a general smooth rational curve of H with degree u with the only restriction that it contains exactly one point of each connected component of Y_1.

(b2) Here we assume that either $y + e \ge a_{4,k-1,k-\beta-1}$ or $u < k$. In both cases $\Delta_{k,k-\beta-2}(y,e)$ is very large (of order k^3) and hence we may use a solution of $U_{k-1}(k-\beta-2)$, say associated to $e' = \min\{e, k-2\}$ and $y' = a_{4,k-1,k-\beta-2} - 3k$ with $\Delta_{k,k-\beta-2}(y,e) - k^2 \le \Delta_{k-1,k-\beta-2}(y',e') \le \Delta_{k,k-\beta-2}(y,e) - k$. The new integer u' satisfies $u' \ge k$. □

Proof of Theorem 1. Set $\alpha := \delta + 2$, where δ is the non-negative integer whose existence was proved in Lemma 1. Fix x, y with critical value k and assume $x \le k - \alpha$. Let $X \subset \mathbb{P}^4$ be a general union of x planes and of a general smooth rational curve of degree y. Since X (i.e. the triple $(4, z, y)$) has critical value k, we have $a_{4,k-1,x} < y \le a_{4,k,x}$. To prove Theorem 1 it is sufficient to prove $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(k)) = 0$.

(a) Here we prove $h^0(\mathcal{I}_X(k-1)) = 0$. Take a solution $Y = A \sqcup B \sqcup D$ of $U_{k-2}(x)$ for the integers $(y', e') := (a_{4,k-2,x} - b_{4,k-2,x}, b_{4,k-2,x})$. Hence $h^i(\mathcal{I}_Y(k-$

$2)) = 0$, $i = 0, 1$. Then we adapt part (a) of the proof of Lemma 1, i.e. the easy part of the proof, taking as target the triple $(k', y', e') = (k-1, y, 0)$ and with h^0 instead of h^1.

(b) Here we prove $h^1(\mathcal{I}_X(k)) = 0$. Apply Assertion $U_k(x)$ with respect to the pair $(y', e') = (y, 0)$. □

Proof of Theorem 2. Fix a hyperplane H of $\mathbb{P}^n$ such that $M \subseteq H$. For all integers n, k, x such that $n \geq 5$ and $k \geq x \geq 0$ we define the following Assertion $V_{n,k}(x)$.

Assertion $V_{n,k}(x)$. *Let $Y \subset \mathbb{P}^n$ be a general union of x planes of M, a smooth rational curve of degree $a_{n,k,x} - b_{n,k,x}$ and $b_{n,k,x}$ disjoint lines. Then $h^i(\mathcal{I}_Y(k)) = 0$, $i = 0, 1$.*

In the case $n = 4$ we required more: Assertion $V_{4,k}(x)$ is just the part of Assertion $U_k(x)$ concerning the pair $(y, e) := (a_{n,k,x} - b_{n,k,x}, b_{n,k,x})$. The proof of Lemma 1 gives the existence of an integer δ_1 such that $V_{n,k}(x)$ is true for all n, k, x such that $0 \leq x \leq k - \delta_1$; we need both parts (a) and (b) of the proof of Lemma 1, but in the modification of part (b) no claim is needed, because now $A' \cap (B_4 \cup D_4) = \emptyset$; the integer δ_1 substitute several numerical lemmas concerning the integers $a_{n,k,x}$, $a_{n,k,x} - a_{n,k-1,x}$, $a_{n,k,x} - a_{n,k-1,x-1}$, $a_{n-1,k-1,x}$ and $a_{n-1,k-1,x-1}$ with asymptotic extimates for the same integers. We are unable to do the numerical lemmas. Set $\alpha_1 := \delta_1 + 2$. Fix non-negative integers n, k, x, y such that $n \geq 5$, (n, x, y) has critical value k and $x \leq k - \alpha_1$. Let $X \subset \mathbb{P}^n$ be a general union of x planes of M and of a general smooth rational curve of degree y. To prove Theorem 2 it is sufficient to prove $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(k)) = 0$.

(a) Here we prove $h^0(\mathcal{I}_X(k-1)) = 0$. Since (n, x, y) has critical value k, we have $y > a_{n,k-1,x}$. Since $x \leq k - \alpha_1 = k - \delta_1 - 2$, we may use $V_{n,k-1}(x)$. Let $T = A \sqcup B_1 \sqcup D_1$ be a solution of $V_{n,k-2}(x)$. Hence $h^i(\mathcal{I}_T(k-2)) = 0$, $i = 0, 1$. For large δ_1 the integer k is large and hence $a_{n,k-1,x} - a_{n,k-2,x} \geq k$ for all $x \leq k$ (use (5) and (7). Hence we may add to T a general smooth rational curve B_4 of H, with the only restriction that it contains $D_1 \cap H$ and exactly one point of $B_1 \cap H$. With this modification we copy part (a) of the proof of Lemma 1.

(b) Here we prove $h^1(\mathcal{I}_X(k)) = 0$. For this part it is sufficient to do the case $y = a_{n,k,x}$. Let $A \sqcup B \sqcup D$ be a solution of $V_{n,k-1}(x)$. Hence $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. Modify the proof of $V_{n,k-1}(x) \Longrightarrow V_{n,k-1}(x)$ to arrive at the end with a smooth rational curve and no line, i.e. add in H a general smooth rational curve B_4 of degree $a_{n,k,x} - a_{n-1,k,x}$ containing all points of $D_4 \cap H$ and exactly one point of $B_4 \cap H$. Here we use the inequality $a_{n,k,x} - a_{n,k-1,x} \geq b_{n,k-1,x}$ to get B_4 with this property. This inequality is true if $a_{n,k,x} - a_{n,k-1,x} \geq k - 2$.

The latter inequality is true for large k by (5) and (7), but only because the integer α allowed us to omit finitely many critical value k. $\square$

Proof of Theorem 3. In the proof of Theorem 2 add at each step a general plane of H, instead of a general plane of M. $\square$

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